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Titel: Note on polyadic groups on sets with at most 7 elements

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NOTE ON POLYADIC GROUPS ON SETS WITH AT MOST 7 ELEMENTS

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This note is a continuation of [5]. We will use the methods of that paper for description of all polyadic groups on some small sets. Notice that the methods of [6] can be used for enumerating all polyadic groupoids on finite sets (e.g. the number of 3-groupoids on 2-element sets and 3-element sets is 130 and 1 270 932 917 454, resp.), but they are not applicable for polyadic groups. The terminology and notation used in this paper are the same as in [5].

A non-empty set G with a multi-ary, say $(n+1)$-ary, operation $f:G^{n+1} \longrightarrow G$ is said (following [1]) to be an $(n+1)$-group if f is associative and for every $k \in \{0,1,\dots,n\}$ and fixed $x_i \in G$ $(i \neq k)$ the mapping $z \longmapsto f(x_0,x_1,\dots,x_{k-1},z,x_{k+1},\dots,x_n)$ is bijective. An $(n+1)$-groupoid $(G;f)$ is said to be H-derived (see also [2],[3]) from a group $(G;\circ)$ if there exist an element $a \in G$ and an automorphism α of $(G;\circ)$ such that $\alpha(a) = a$, $\alpha^n(x) = a\circ x\circ a^{-1}$ and $f(x_0,x_1,\dots,x_n) = x_0\circ\alpha(x_1)\circ\dots\circ\alpha^n(x_n)\circ a$. If $(G;f)$ is H-derived from $(G;\circ)$ for some $a \in G$ and for some automorphism α of $(G;\circ)$, then we will also say that $(G;f)$ is $\langle\alpha,a\rangle$-derived from $(G;\circ)$. Recall that (see [7]) an $(n+1)$-groupoid $(G;f)$ is an $(n+1)$-group iff $(G;f)$ is H-derived from a group $(G;\circ)$. Let an $(n+1)$-group $(A;f)$ be $\langle\alpha,a\rangle$-derived from a group $(A;\circ)$ and let an $(n+1)$-group $(B;g)$ be $\langle\beta,b\rangle$-derived from a group $(B;\circ)$. Then (Theorem 3 of [3]) the $(n+1)$-groups $(A;f)$ and $(B;g)$ are isomorphic iff there exist an isomorphism $\sigma:(A;\circ) \longrightarrow (B;\circ)$ of groups and an element $s \in B$ such that

$$\sigma(a) = s \circ \beta(s) \circ \dots \circ \beta^{n-1}(s) \circ b \tag{1}$$

and

$$\sigma(\alpha(x)) \circ s = s \circ \beta(\sigma(x)) \tag{2}$$

for all $x \in A$.

Let $(G;\circ)$ be an arbitrary group, $c \in G$, and $\varphi \in \mathrm{Aut}(G;\circ)$. Then

we define (n+1)-ary operations $f_c^{(\circ)}$, $g_\varphi^{(\circ)}$, and $g_{\varphi,c}^{(\circ)}$ as follows:

$$f_c^{(\circ)}(x_0,x_1,\dots,x_n) = x_0 \circ x_1 \circ \dots \circ x_n \circ c\,, \tag{3}$$

$$g_\varphi^{(\circ)}(x_0,x_1,x_2,\dots,x_n) = x_0 \circ \varphi(x_1) \circ \varphi^2(x_2) \circ \dots \circ \varphi^n(x_n), \tag{4}$$

and

$$g_{\varphi,c}^{(\circ)}(x_0,x_1,x_2,\dots,x_n) = x_0 \circ \varphi(x_1) \circ \varphi^2(x_2) \circ \dots \circ \varphi^n(x_n) \circ c\,. \tag{5}$$

Let $G_m = \{0,1,\dots,m-1\}$. Denote by $C(m)$ the cyclic group $(G_m; + \pmod m)$. If $(G;\circ) = C(m)$, then any automorphism φ of $C(m)$ is of the form $\varphi(x) = dx$ for $d \in G_m \setminus \{0\}$ and we will use the operations $g_d^{(+)}$ and $g_{d,c}^{(+)}$ instead of $g_\varphi^{(\circ)}$ and $g_{\varphi,c}^{(\circ)}$, where

$$g_d^{(+)}(x_0,x_1,\dots,x_n) \equiv x_0 + dx_1 + \dots + d^{n-1}x_{n-1} + x_n \pmod m \tag{4'}$$

(for $d \in G_m \setminus \{0,1\}$, $d^n \equiv 1 \pmod m$; and so g.c.d.$(d,m) = 1$) and

$$g_{d,c}^{(+)}(x_0,x_1,\dots,x_n) \equiv x_0 + dx_1 + \dots + d^{n-1}x_{n-1} + x_n + c \pmod m \tag{5'}$$

(for $c,d \in G_m \setminus \{0,1\}$, $d^n \equiv 1 \pmod m$, and $(d-1)c \equiv 0 \pmod n$).
Observe that by Lemma 2 of [5]

$$(G_m; f_a^{(+)}) \simeq (G_m; f_0^{(+)}) \tag{6}$$

for $a =$ g.c.d.(m,n).
It is easy to see that if $(G;\circ)$ is commutative, then polyadic groups $(G;f)$, where f is $f_c^{(\circ)}$, $g_\varphi^{(\circ)}$, or $g_{\varphi,c}^{(\circ)}$, are abelian (in the general algebraic sense, see [4]); moreover, $g_{\varphi,c}^{(\circ)}$ is commutative iff φ is the identity automorphism. Consequently, $f_c^{(\circ)}$ is always commutative, and for $m \neq 6$, $m \leqslant 7$, all polyadic groups on sets with m elements are abelian.
For the next propositions we need the following obvious lemmas:

LEMMA 1. Let $(G;\circ)$ be an arbitrary group. Then $f_c^{(\circ)}$ is an idempotent (n+1)-ary operation if and only if c is the neutral element of $(G;\circ)$ and $s \mid n$, where $s =$ l.c.m.$(\mathrm{ord}(a) \mid a \in G)$.

LEMMA 2. Let $g_d^{(+)}$ be an (n+1)-ary operation on the set G_m, where n is even and $d = m-1$. Then $g_d^{(+)}$ is idempotent.

Indeed, Lemma 2 follows from the obvious equality

$$1 + (m-1) + (m-1)^2 + \dots + (m-1)^n \equiv 1 \pmod m.$$

(Observe that $d \neq m-1$ for $m > 2$ and n odd.)

LEMMA 3. Let m be prime and let $g_d^{(+)}$ be an $(n+1)$-ary operation on the set G_m. Then $g_d^{(+)}$ is idempotent.

From Lemmas 1-3 and Theorem 1 of [5] (and (6)) we obtain immediately the following propositions:

PROPOSITION 1 (cf. [8], p.295). Every $(n+1)$-group on a two-element set is isomorphic to exactly one $(n+1)$-group $(G_2;f)$, where f is one of the following commutative $(n+1)$-group operations:
(a) $f_0^{(+)}$ or $f_1^{(+)}$ for even n,
and
(b) $f_0^{(+)}$ for odd n,
where $(G_2;+)$ is the cyclic group $C(2)$. Moreover, only the operation $f_0^{(+)}$ for even n is idempotent.

PROPOSITION 2 (cf. [8], p.295). Let $k = g.c.d.(n,6)$ and let $(G_3;+)$ be the cyclic group $C(3)$. Every $(n+1)$-group on a 3-element set is isomorphic to exactly one $(n+1)$-group $(G_3;f)$, where f is one of the following $(n+1)$-group operations:
(a) $f_0^{(+)}$, $f_1^{(+)}$, or $g_2^{(+)}$ for $k = 6$,
(b) $f_0^{(+)}$ or $f_1^{(+)}$ for $k = 3$,
(c) $f_0^{(+)}$ or $g_2^{(+)}$ for $k = 2$,
(d) $f_0^{(+)}$ for $k = 1$.
Moreover, only $f_0^{(+)}$ for $k = 3$ and 6, and $g_2^{(+)}$ for $k = 2$ and 6 are idempotent.

For the case $m = 4$ we additionally need the following Lemma 4.

LEMMA 4. Let $k = g.c.d.(n,12)$, $(G_4;*)$ be the Klein four-group (with 0 as the neutral element), let $\gamma,\delta \in Aut(G_4;*) \subset S_4$, where γ is of order 2 and δ of order 3, and let $c \in G_4 \setminus \{0\}$ be the fix point of γ. Then every $(n+1)$-group H-derived from $(G_4;*)$ is isomorphic to exactly one $(n+1)$-group $(G_4;f)$, where f is one of the following $(n+1)$-group operations:
(a) $f_0^{(*)}$, $f_1^{(*)}$, $g_\gamma^{(*)}$, $g_{\gamma,c}^{(*)}$, or $g_\delta^{(*)}$ for $k = 12$,
(b) $f_0^{(*)}$, $f_1^{(*)}$, $g_\delta^{(*)}$, or $g_\gamma^{(*)}$ for $k = 6$,
(c) $f_0^{(*)}$, $f_1^{(*)}$, $g_\gamma^{(*)}$, or $g_{\gamma,c}^{(*)}$ for $k = 4$,
(d) $f_0^{(*)}$ or $g_\delta^{(*)}$ for $k = 3$,
(e) $f_0^{(*)}$, $f_1^{(*)}$, or $g_\gamma^{(*)}$ for $k = 2$,

(f) $f_o^{(*)}$ for $k = 1$.

Moreover, $g_\delta^{(*)}$ is idempotent, and $g_\gamma^{(*)}$ is idempotent iff $n \equiv 0 \pmod 4$.

Proof. Let $(G_4;g_{\alpha,a}^{(*)})$ and $(G_4;g_{\beta,b}^{(*)})$ be $(n+1)$-groups $\langle\alpha,a\rangle$-derived and $\langle\beta,b\rangle$-derived, resp., from $(G_4;*)$. By Theorem 3 of [3] (see (2)), if $(G_4;g_{\alpha,a}^{(*)}) \simeq (G_4;g_{\beta,b}^{(*)})$ and α or β is the identity mapping, then both the mappings are the identities. Observe that, for $\alpha = \beta = id$, formula (2) is satisfied for every $\sigma \in \mathrm{Aut}(G_4;*)$ and $s \in G_4$. Moreover, formula (1) takes the form

$$\sigma(a) = \begin{cases} b & \text{for even } n, \\ s * b & \text{for odd } n. \end{cases} \tag{1'}$$

Therefore, $(G_4;f_{c_1}^{(*)}) \simeq (G_4;f_{c_2}^{(*)})$ for arbitrary $c_1, c_2 \neq 0$.

Furthermore, if $c \neq 0$, then $(G_4;f_c^{(*)}) \simeq (G_4;f_o^{(*)})$ only for odd n.

Denote by γ_i $(i = 1,2,3)$ the automorphism of $(G_4;*)$ of order 2 with $\gamma_i(i) = i$ and by δ_1, δ_2 different automorphisms of $(G_4;*)$ of order 3. Note that for every pair of automorphisms $\alpha,\beta \in \mathrm{Aut}(G_4;*)$ there exists an automorphism σ of $(G_4;*)$ satisfying (2) iff α and β are of the same order. Hence $(G_4;g_{\gamma_i}^{(*)}) \simeq (G_4;g_{\gamma_j}^{(*)})$ $(i,j \in \{1,2,3\})$ and $(G_4;g_{\delta_1}^{(*)}) \simeq (G_4;g_{\delta_2}^{(*)})$, because formula (1) holds for $s = 0$. From the definition of an H-derived $(n+1)$-group it follows that $g_{\gamma_i,i}^{(*)}$ $(i = 1,2,3)$ is an $(n+1)$-group operation iff n is even. Similarly, $g_{\delta_i}^{(*)}$ $(i = 1,2)$ is an $(n+1)$-group operation iff $n \equiv 0 \pmod 3$.

Consider $n \equiv 0 \pmod 4$. It is easy to verify that (1) takes the form (1'). Consequently, the $(n+1)$-groups $(G_4;g_{\gamma_i}^{(*)})$ and $(G_4;g_{\gamma_i,i}^{(*)})$ are not isomorphic. Putting $\sigma = \gamma_3$ and $s = 0$ into (1') and (2) we have $(G_4;g_{\gamma_1,1}^{(*)}) \simeq (G_4;g_{\gamma_2,2}^{(*)})$. Similarly, $(G_4;g_{\gamma_1,1}^{(*)}) \simeq (G_4;g_{\gamma_3,3}^{(*)})$.

Let now $n \equiv 2 \pmod 4$. Then (1) takes the form

$$\sigma(a) = s * \gamma_i(s) * b. \tag{1''}$$

Taking $\sigma = \gamma_1$ and $s = 2$ into (1'') and (2) we get

$(G_4;g^{(*)}_{\gamma_1}) \simeq (G_4;g^{(*)}_{\gamma_1,1})$. Similarly, $(G_4;g^{(*)}_{\gamma_i}) \simeq (G_4;g^{(*)}_{\gamma_i,i})$ for $i = 2$ and 3. The rest of our Lemma is obvious.

Therefore we have

PROPOSITION 3. Let $k = g.c.d.(n,12)$, $(G_4;+)$ be the cyclic group $C(4)$, and let $(G_4;*)$ be the Klein four-group. Every $(n+1)$-group on a 4-element set is isomorphic to exactly one $(n+1)$-group $(G_4;f)$, where f is one of the following $(n+1)$-group operations:

(a) $f_0^{(+)}$, $f_1^{(+)}$, $f_2^{(+)}$, $f_0^{(*)}$, $f_1^{(*)}$, $g_3^{(+)}$, $g_{3,2}^{(+)}$, $g_\gamma^{(*)}$, $g_{\gamma,1}^{(*)}$, or $g_\delta^{(*)}$ for $k = 12$,

(b) $f_0^{(+)}$, $f_1^{(+)}$, $f_0^{(*)}$, $f_1^{(*)}$, $g_3^{(+)}$, $g_{3,2}^{(+)}$, $g_\gamma^{(*)}$, or $g_\delta^{(*)}$ for $k = 6$,

(c) $f_0^{(+)}$, $f_1^{(+)}$, $f_2^{(+)}$, $f_0^{(*)}$, $f_1^{(*)}$, $g_3^{(+)}$, $g_{3,2}^{(+)}$, $g_\gamma^{(*)}$, or $g_{\gamma,1}^{(*)}$ for $k = 4$,

(d) $f_0^{(+)}$, $f_0^{(*)}$, or $g_\delta^{(*)}$ for $k = 3$,

(e) $f_0^{(+)}$, $f_1^{(+)}$, $f_0^{(*)}$, $f_1^{(*)}$, $g_3^{(+)}$, $g_{3,2}^{(+)}$, or $g_\gamma^{(*)}$ for $k = 2$,

(f) $f_0^{(+)}$ or $f_0^{(*)}$ for $k = 1$,

where $\gamma,\delta \in \mathrm{Aut}(G_4;*) \subset S_4$ are cyclic permutations: $\gamma = (2,3)$ and $\delta = (1,2,3)$. Moreover, only $g_3^{(+)}$, $g_\delta^{(*)}$, $f_0^{(*)}$ (for n even), $f_0^{(+)}$ (for $n \equiv 0 \pmod 4$) and $g_\gamma^{(*)}$ (for $n \equiv 0 \pmod 4$) are idempotent.

PROPOSITION 4. Let $k = g.c.d.(n,20)$ and let $(G_5;+)$ be the cyclic group $C(5)$. Then every $(n+1)$-group on a 5-element set is isomorphic to exactly one $(n+1)$-group $(G_5;f)$, where f is one of the following $(n+1)$-group operations:

(a) $f_0^{(+)}$, $f_1^{(+)}$, $g_2^{(+)}$, $g_3^{(+)}$, or $g_4^{(+)}$ for $k = 20$,

(b) $f_0^{(+)}$, $f_1^{(+)}$, or $g_4^{(+)}$ for $k = 10$,

(c) $f_0^{(+)}$ or $f_1^{(+)}$ for $k = 5$,

(d) $f_0^{(+)}$, $g_2^{(+)}$, $g_3^{(+)}$, or $g_4^{(+)}$ for $k = 4$,

(e) $f_0^{(+)}$ or $g_4^{(+)}$ for $k = 2$,

(f) $f_0^{(+)}$ for $k = 1$.

Moreover, only $g_2^{(+)}$, $g_3^{(+)}$, $g_4^{(+)}$, and $f_0^{(+)}$ (for $n \equiv 0 \pmod 5$) are idempotent.

Taking additionally into account Corollary 9 of [5] we get

PROPOSITION 5. Let $k = g.c.d.(n,6)$, let $(G_6;+)$ be the cyclic group $C(6)$, and $(G_6;\circ)$ be the symmetric group S_3. Every $(n+1)$-group on a 6-element set is isomorphic to exactly one $(n+1)$-group $(G_6;f)$, where f is one of the following $(n+1)$-group operations:

(a) $f_0^{(+)}$, $f_1^{(+)}$, $f_2^{(+)}$, $f_3^{(+)}$, $f_0^{(\circ)}$, $g_5^{(+)}$, or $g_{5,3}^{(+)}$ for $k = 6$,

(b) $f_0^{(+)}$, $f_1^{(+)}$, or $f_0^{(\circ)}$ for $k = 3$,

(c) $f_0^{(+)}$, $f_1^{(+)}$, $f_0^{(\circ)}$, $g_5^{(+)}$, or $g_{5,3}^{(+)}$ for $k = 2$,

(d) $f_0^{(+)}$ or $f_0^{(\circ)}$ for $k = 1$.

Moreover, only $g_5^{(+)}$, $f_0^{(+)}$ (for $n \equiv 0 \pmod 6$), and $f_0^{(\circ)}$ (for $n \equiv 0 \pmod 6$) are idempotent.

PROPOSITION 6. Let $k = g.c.d.(n,42)$ and let $(G_7;+)$ be the cyclic group $C(7)$. Every $(n+1)$-group on a 7-element set is isomorphic to exactly one $(n+1)$-group $(G_7;f)$, where f is one of the following $(n+1)$-group operations:

(a) $f_0^{(+)}$, $f_1^{(+)}$, $g_2^{(+)}$, $g_3^{(+)}$, $g_4^{(+)}$, $g_5^{(+)}$, or $g_6^{(+)}$ for $k = 42$,

(b) $f_0^{(+)}$, $f_1^{(+)}$, $g_2^{(+)}$, or $g_4^{(+)}$ for $k = 21$,

(c) $f_0^{(+)}$, $f_1^{(+)}$, or $g_6^{(+)}$ for $k = 14$,

(d) $f_0^{(+)}$ or $f_1^{(+)}$ for $k = 7$,

(e) $f_0^{(+)}$, $g_2^{(+)}$, $g_3^{(+)}$, $g_4^{(+)}$, $g_5^{(+)}$, or $g_6^{(+)}$ for $k = 6$,

(f) $f_0^{(+)}$, $g_2^{(+)}$, or $g_4^{(+)}$ for $k = 3$,

(g) $f_0^{(+)}$ or $g_6^{(+)}$ for $k = 2$,

(h) $f_0^{(+)}$ for $k = 1$.

Moreover, only the operations $g_i^{(+)}$ $(i = 2,\ldots,6)$ and $f_0^{(+)}$ (for $n \equiv 0 \pmod 7$) are idempotent.

Finally, we tabularize the numbers of $(n+1)$-groups on m-element sets with $m < 8$ (for $m = 2$ and 3 see also [8], p. 293–295):

m=2;k=g.c.d.(n,2)	k = 2	k = 1
all	2	1
commut. idemp.	1	0
commut. non-idemp.	1	1

m=3;k=g.c.d.(n,6)	k = 6	k = 3	k = 2	k = 1
all	3	2	2	1
commut. idemp.	1	1	0	0
commut. non-idemp.	1	1	1	1
non-commut. idemp.	1	0	1	0

m=4;k=g.c.d.(n,12)	k =12	k = 6	k = 4	k = 3	k = 2	k = 1
all	10	8	9	3	7	2
commut. idemp.	2	1	2	0	1	0
commut. non-idemp.	3	3	3	2	3	2
non-commut. idemp.	3	2	2	1	1	0

m=5;k=g.c.d.(n,20)	k =20	k =10	k = 5	k = 4	k = 2	k = 1
all	5	3	2	4	2	1
commut. idemp.	1	1	1	0	0	0
commut. non-idemp.	1	1	1	1	1	1
non-commut. idemp.	3	1	0	3	1	0

m=6;k=g.c.d.(n,6)	k = 6	k = 3	k = 2	k = 1
all	7	3	5	2
commut. idemp.	1	0	0	0
commut. non-idemp.	3	2	2	1
abe.non-commut.idemp.	1	0	1	0
abelian non-commut. non-idemp.	1	0	1	0
non-abel. non-idemp.	0	1	1	1

m=7;k=g.c.d.(n,42)	k=42	k=21	k=14	k=7	k=6	k=3	k=2	k=1
all	7	4	3	2	6	3	2	1
commut. idemp.	1	1	1	1	0	0	0	0
commut. non-idemp.	1	1	1	1	1	1	1	1
non-commut. idemp.	5	2	1	0	5	2	1	0

REFERENCES

[1] DÖRNTE, W.: Untersuchungen über einen verallgemeinerten Gruppenbegriff. Math.Z. 29(1928), 1-19.

[2] DUDEK, W.A., and J. MICHALSKI: On a generalization of Hosszú theorem. Demonstratio Math. 15(1982), 783-805.

[3] DUDEK, W.A., and J. MICHALSKI: On retracts of polyadic groups. Demonstratio Math. 17(1984), 281-301.

[4] GŁAZEK, K., and B. GLEICHGEWICHT: Abelian n-groups. In: Colloq. Math. Soc. J. Bolyai, v.29 (Universal Algebra, Esztergom (Hungary), 1977), North Holland Publ. Comp., Amsterdam 1981, p. 321-329.

[5] GŁAZEK, K., J. MICHALSKI, and I. SIEROCKI: On evaluation of numbers of some polyadic groups. In: Contributions to General Algebra 3 (Proc. Vienna Conf., 1984), Verlag Hölder-Pichler-Tempsky, Wien 1985, p. 157-171.

[6] HARRISON, M.A.: The number of isomorphism types of finite algebras. Proc. Amer. Math. Soc. 17(1966), 731-737.

[7] HOSSZÚ, M.: On the explicit form of n-group operations. Publ. Math. Debrecen 10(1963), 88-92.

[8] POST, E.L.: Polyadic groups. Trans. Amer. Math. Soc. 48 (1940), 208-350.

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