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Titel: Variations on a theorem of J. D. DIXON

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Variations on a theorem of J. D. Dixon

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0. Introduction

In [1] J. D. DIXON proves the following

Theorem A. *Let S be a finite set consisting of h elements of $\mathfrak{U}(d)$ and suppose that the unit matrix $I \in S$. We define a linear mapping $\sigma(S)\colon \mathfrak{M}(d) \to \mathfrak{M}(d)$ by*

$$\sigma(S)\, B = \frac{1}{h} \sum_{U \in S} U^* B U.$$

Then for each $A_0 \in \mathfrak{M}(d)$ we can define a sequence (A_n) in $\mathfrak{M}(d)$ by putting $A_n = \sigma(S)^n A_0$ for $n = 1, 2, \ldots$ Then (A_n) converges always in $\mathfrak{M}(d)$ and its limit, say A, has the property $AU = UA$ for all $U \in S$.

In the formulation of the theorem $\mathfrak{U}(d)$ denotes the group of unitary $d \times d$ matrices, while $\mathfrak{M}(d)$ denotes the algebra of all $d \times d$ complex matrices.
The applications of the given procedure DIXON has in mind concern representations of finite groups. In this case it is enough to deal with unitary matrices. But already DIXON mentions in the introduction to his paper a possible application to the study of representations of finitely generated groups. Moreover H. SCHLOSSER proved in [2], that the set of operators intertwining the operators of a representation of a Lie group is determined by the representation operators of a finite number suitably chosen group elements. Since the theorem of MASCHKE fails in these cases the question of determining intertwining operators becomes valid for a finite set of non-unitary regular matrices. On the other side approximation of unitary matrices by non-unitary ones on a computer yield the same question.

1. Variation

At first sight it is evident from the proof given by DIXON, that the following generalization holds

Theorem A′. *Let S_1, S_2 be ordered finite sets consisting of h elements of $\mathfrak{U}(d_1), \mathfrak{U}(d_2)$ respectively. Assume $U_1 = I_1 \in S_1 \subseteq \mathfrak{U}(d_1)$ and $V_1 = I_2 \in S_2 \subseteq \mathfrak{U}(d_2)$. Define a linear mapping $\sigma(S_1, S_2)\colon \mathfrak{M}(d_1, d_2) \to \mathfrak{M}(d_1, d_2)$ by*

$$\sigma(S_1, S_2)\, B = \frac{1}{h} \sum_{i=1}^{h} U_i^* B V_i. \tag{1}'$$

Then for each $A_0 \in \mathfrak{M}(d_1, d_2)$ *we can define a sequence* (A_n) *in* $\mathfrak{M}(d_1, d_2)$ *by putting* $A_n = \sigma(S_1, S_2)^n A_0$ *for* $n = 1, 2, \ldots$ *Then* (A_n) *is always convergent in* $\mathfrak{M}(d_1, d_2)$ *and its limit, say* A, *satisfies the equations* $AV_i = U_iA$ *for* $i = 1, \ldots, h$.

By $\mathfrak{M}(d_1, d_2)$ is denoted the space of $d_1 \times d_2$ complex matrices.

An analysis of the proof shows, that besides the procedure given in the formulation of the theorem there are two interesting characterizations of the algebra of intertwining matrices:

$$\mathfrak{I}(S_1, S_2) = \{X \in \mathfrak{M}(d_1, d_2);\ XV_i = U_iX \text{ for } i = 1, \ldots, h\}$$

by a fixed point property and by a nonlinear equation

I. $\mathfrak{I}(S_1, S_2) = \operatorname{Fix} \sigma(S_1, S_2)$,

II. $\mathfrak{I}(S_1, S_2) = \{X \in \mathfrak{M}(d_1, d_2);\ \|\sigma(S_1, S_2)\, X\| = \|X\|\}$.

$\operatorname{Fix} \sigma(S_1, S_2)$ denotes the set of fixed points of the linear mapping defined by (1'). The norm in II. is the Hilbert-Schmidt norm: $\|X\| = (\operatorname{trace} X^*X)^{1/2}$, i.e. the euclidean norm in $\mathfrak{M}(d_1, d_2)$. Euclidicity of the norm we are dealing with is essential in the proof.

2. Variation

Let T_1, T_2 denote two ordered sets consisting of h elements of $GL(d_1)$, $GL(d_2)$ respectively and assume $A_1 = I_1 \in T_1 \subseteqq GL(d_1)$ and $B_1 = I_2 \in T_2 \subseteqq GL(d_2)$. By $\mathfrak{I}(T_1, T_2) = \{X \in \mathfrak{M}(d_1, d_2);\ XB_i = A_iX \text{ for } i = 1, \ldots, h\}$ we denote as in 1. the algebra of intertwining matrices. Define a linear mapping on $\mathfrak{M}(d_1, d_2)$ by

$$\tau(T_1, T_2)\, X = \frac{1}{h} \sum_{i=1}^{h} A_i^{-1} X B_i. \tag{2}$$

The idea of DIXONS proof yields a characterization of the algebra $\mathfrak{I}(T_1, T_2)$ of intertwining matrices as follows:

Proposition 1. *The elements of the algebra* $\mathfrak{I}(T_1, T_2)$ *are given by any one of the following conditions:*

I. $X \in \mathfrak{I}(T_1, T_2)$ *if and only if*

a) $X \in \operatorname{Fix} \tau(T_1, T_2)$ *and* b) $\max_{i=1}^{h} \|A_i^{-1} X B_i\| \leqq \|X\|$.

II. $\mathfrak{I}(T_1, T_2) = \{X \in \mathfrak{M}(d_1, d_2);\ \|\tau(T_1, T_2)\, X\| = \|X\|$ *and* $\|A_i^{-1} X B_i\| = \|X\|$ *for* $i = 1, \ldots, h\}$.

As above $\|.\|$ *denotes the Hilbert-Schmidt norm.*

Proof. By assumption holds $\|A_i^{-1} X B_i\| \leqq \|X\|$ for $i = 1, \ldots, h$ in both cases. Hence

$$\|\tau(T_1, T_2)\, X\| \leqq \frac{1}{h} \sum_{i=1}^{h} \|A_i^{-1} X B_i\| \leqq \|X\|.$$

Using $X \in \operatorname{Fix} \tau(T_1, T_2)$ or the equation $\|\tau(T_1, T_2)\, X\| = \|X\|$ we get: to every $i = 1, \ldots, h$ exists a real number a_i such that $0 \leqq a_i \leqq 1$ and $A_i^{-1} X B_i = a_i X$ and $\sum_{i=1}^{h} a_i = h$ holds. Then $a_i = 1$ and $XB_i = A_iX$ for $i = 1, \ldots, h$. The necessity of the conditions I and II is evident. Moreover if the matrices of T_1 and T_2 are unitary, the conditions reduce to those mentioned in 1.

The following example shows, that it is impossible to miss an additional condition to $X \in \operatorname{Fix} \tau(T_1, T_2)$ in the general case.

Example. Assume $d_1 = d_2 = 3$, $T_1 = T_2$ and

$$A_1 = I, \quad A_2 = \begin{Vmatrix} 1 & -\frac{1}{2} & -\frac{3}{4} \\ 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{2} \end{Vmatrix}, \quad A_3 = \begin{Vmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

If $X = \begin{Vmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$, then

$$A_2^{-1} X A_2 = \begin{Vmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \quad \text{and} \quad A_3^{-1} X A_3 = \begin{Vmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix},$$

hence $\tau(T_1, T_2)\, X = X$, but $XA_2 \neq A_2 X$ and $XA_3 \neq A_3 X$.

3. Variation

Starting with an element $X_0 \in \mathfrak{M}(d_1, d_2)$ define a sequence (X_n) by putting $X_n = \tau(T_1, T_2)^n X_0$. Since no reasonable condition is known on X_0 that the sequence (X_n) is convergent in $\mathfrak{M}(d_1, d_2)$ not converges to an element of $\mathfrak{F}(T_1, T_2)$ we shall give some almost trivial remarks.

Remark 1. If $\max_{i=1}^{h} \|A^{-1} X_n B_i\| = \|X_n\|$ holds for almost all natural numbers n, then the sequence (X_n) is convergent and its limit, say X', belongs to $\mathfrak{F}(T_1, T_2)$.

Remark 2. If the sequence (X_n) is convergent, then its limit, say X', belongs to $\mathfrak{F}(T_1, T_2)$ if and only if the following estimate is valid $\|A_i^{-1} X_n B_i\| = \|X_n\| + o(1)$ for $i = 1, \ldots, h$ and $n \to \infty$.

Remark 3. If $\|A_i^{-1} X_n B_i\| = \|X_n\| + o(1)$ holds for $i = 1, \ldots, h$ and $n \to \infty$, then the sequence (X_n) is bounded and hence possesses a convergent subsequence. It is easy to verify, that the limit of the subsequence in this case belongs to $\mathfrak{F}(T_1, T_2)$.

Remark 4. If for some $\varepsilon > 0$ holds $\|A_i^{-1} X_n B_i - X_n\| = O(n^{-1-\varepsilon})$ for $i = 1, \ldots, h$ and $n \to \infty$ then the sequence (X_n) is convergent in $\mathfrak{M}(d_1, d_2)$ and its limit belongs to $\mathfrak{F}(T_1, T_2)$.

4. Variation

Let (X_n) denote the sequence defined in 3. We shall prove the following

Lemma. *If the C^k-limitation $(X_n^{(k)})$ of the sequence (X_n) is convergent in $\mathfrak{M}(d_1, d_2)$, then its limit, say X', belongs to* $\operatorname{Fix} \tau(T_1, T_2)$.

The limit X' belongs to $\mathfrak{H}(T_1, T_2)$ if moreover the sequences $(\|A_i^{-1}X_nB_i - X_n\|)$ for $i = 1, \ldots, h$ are C^k-summable to zero.

A well known condition on C^k-summability yields the

Corollary. *A necessary condition for the sequence (X_n) to be C^k-summable to an element of $\mathfrak{H}(T_1, T_2)$ is the estimate*

$$\|A_i^{-1}X_nB_i - X_n\| = o(n^k) \quad \text{for } i = 1, \ldots, h \text{ and } n \to \infty.$$

Proof of the Lemma. The elements of $(X_n^{(k)})$ are defined by

$$X_n^{(k)} = \frac{1}{\binom{n+k}{k}} \sum_{m=0}^{n} \binom{n+k-m-1}{k-1} X_m.$$

We get

$$\tau(T_1, T_2)\, X_{n-1}^{(k)} = \frac{1}{\binom{n+k-1}{k}} \sum_{m=0}^{n-1} \binom{n+k-m-2}{k-1} X_{m+1}$$

$$= \frac{1}{\binom{n+k-1}{k}} \sum_{m=1}^{n} \binom{n+k-m-1}{k-1} X_m.$$

Hence

$$X_n^{(k)} - \frac{\binom{n+k-1}{k}}{\binom{n+k}{k}} \tau(T_1, T_2)\, X_{n-1}^{(k)} = \frac{\binom{n+k-1}{k-1}}{\binom{n+k}{k}} X_0$$

or

$$X_n^{(k)} - \frac{n}{n+k} \tau(T_1, T_2)\, X_{n-1}^{(k)} = \frac{k}{n+k} X_0.$$

This proves the first statement. The second is a consequence of the following inequalities

$$\|A_i^{-1}X_nB_i - X_n\| \leqq \frac{1}{\binom{n+k}{k}} \sum_{m=0}^{n} \binom{n+k-m-1}{k-1} \|A_i^{-1}X_nB_i - X_n\|$$

$$= \|A_i^{-1}X_nB_i - X_n\|^{(k)} \quad \text{for} \quad i = 1, \ldots, h.$$

5. Variation

The necessity to use an euclidean norm on the operator algebra implies essential restrictions to the applicatibility of DIXONS method in the infinite dimensional case. Given a finite set of unitary operators on a Hilbert space SCHLOSSER ([2]) pointed out, that the method of DIXON allows to determine the strong intertwining operators, i.e. the operators of Hilbert-Schmidt type intertwining the given unitary operators. A simple modification of the proof using the euclidicity of the norm in the under-

lying space instead of the norm in the operator algebra yields a characterization of the algebra of intertwining operators which applies also in the infinite dimensional case.
Let H_1, H_2 denote separable Hilbert spaces and let T_1, T_2 be ordered sets consisting of h regular bounded operators on H_1, H_2 respectively. Assume the first operator in each set to be the identity on the corresponding space. Define the mapping $\tau(T_1, T_2)$ as an endomorphism on the space $\mathfrak{L}(H_2, H_1)$ of bounded operators by equation (2). As before $\mathfrak{I}(T_1, T_2)$ denotes the subalgebra of $\mathfrak{L}(H_2, H_1)$ consisting of those bounded operators, which intertwine corresponding operators of the given sets T_1 and T_2.

Proposition 2. *The algebra $\mathfrak{I}(T_1, T_2)$ is characterized by any one of the following conditions:*

I. $X \in \mathfrak{I}(T_1, T_2)$ *if and only if*

a) $X \in \operatorname{Fix} \tau(T_1, T_2)$ *and* b) $\max_{i=1}^{h} \|A_i^{-1} X B_i y\| \leqq \|Xy\|$ *for every* $y \in H_2$.

II. $\mathfrak{I}(T_1, T_2) = \{X \in \mathfrak{L}(H_2, H_1);\ \|\tau(T_1, T_2)\, Xy\| = \|Xy\|$ *and* $\|A_i^{-1} X B_i y\| = \|Xy\|$ *for* $i = 1, \ldots, h$ *and every* $y \in H_2\}$.

If $\mathfrak{B} = (y_j)$ denotes a basis of H_2, the conditions can be reformulated by

I′. $X \in \mathfrak{I}(T_1, T_2)$ *if and only if*

a) $X \in \operatorname{Fix} \tau(T_1, T_2)$ *and* b) $\max_{i=1}^{h} \|A_i^{-1} X B_i y_j\| \leqq \|Xy_j\|$ *for* $j = 1, 2, \ldots$

II′. $\mathfrak{I}(T_1, T_2) = \{X \in \mathfrak{L}(H_2, H_1);\ \|\tau(T_1, T_2)\, Xy_j\| = \|Xy_j\|$ *and* $\|A_i^{-1} X B_i y_j\| = \|Xy_j\|$ *for* $i = 1, \ldots, h$ *and* $j = 1, 2, \ldots\}$.

Proof. For every $y \in H_2$ holds by assumption

$$\|\tau(T_1, T_2)\, Xy\| \leqq \frac{1}{h} \sum_{i=1}^{h} \|A_i^{-1} X B_i y\| \leqq \|Xy\|.$$

The equality $\|\tau(T_1, T_2)\, Xy\| = \|Xy\|$ implies the existence of real numbers $a_i = a_i(y)$ satisfying the properties $0 \leqq a_i \leqq 1$ and $A_i^{-1} X B_i y = a_i(y)\, Xy$ for $i = 1, \ldots, h$ and every $y \in H_2$ and $\sum_{i=1}^{h} a_i = h$. From this we get $a_i(y) = 1$ for $i = 1, \ldots, h$ and every $y \in H_2$, hence $A_i X = X B_i$ for $i = 1, \ldots, h$. Necessity is evident.
If the Hilbert space H_2 is finite dimensional, the statements I′ and II′ contain only a finite number of conditions. Rewriting these conditions for matrices under the assumption that H_1 is finite dimensional too, and using the canonical basis (e_j) of $H_2 = \mathbf{R}^{d_2}$ the additional conditions $\max_{i=1}^{h} \|A_i^{-1} X B_i e_j\| \leqq \|Xe_j\|$ and $\|A_i^{-1} X B_i e_j\| = \|Xe_j\|$ for $j = 1, \ldots, d_2$ are stronger, than the additional conditions $\max_{i=1}^{h} \|A_i^{-1} X B_i\| \leqq \|X\|$ and $\|A_i^{-1} X B_i\| = \|X\|$ of Proposition 1.

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