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Groups and Join Spaces

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1. Introduction

In [4] W. Prenowitz and J. Jantosciak systemize common properties of ordered and partially ordered linear, spherical, and projective geometries by the concept of a join space. The theorems appearing there are similar to ideas in group theory where linear sets in a join space play the role of normal subgroups in a group. This is completed by the hint in [4] that each abelian group gives rise to a join space. First of all however homomorphic and isomorphic theorems given in [4] indicate that the connection between the theory of join spaces and the group theory is explained incompletely by abelian groups, that on the contrary the essential relations between the two theories just exist to not necessarily commutative groups.
A first association between a join space and an arbitrary group G is given by the proof that the set of classes of conjugate elements from G form a join space with a join operation induced by complex multiplication. More general we show that this fact is also valid for those classes of G, which are invariant with reference to a subgroup $\mathfrak{U}$ of the automorphism group of G containing the group of inner automorphisms. We get further join spaces forming the factor space of this join space with respect to a $\mathfrak{U}$-invariant subgroup of G in the sense of [4].
Finally an application to the group theory is described by a construction of normal subgroups, which appear when central series are formed.

2. Join spaces

Let $V = \{a, b, c, \ldots\}$ be an arbitrary nonemty set, subsets of which are denoted $A, B, C, \ldots$ A *join operation* „$\circ$" in V is a mapping of $V \times V$ into the family of subsets of V. If (a, b) is an element of $V \times V$, its image under $\circ$ is denoted $a \circ b$. Such join operation induces an inverse operation / by

$$a/b := \{x \mid a \in b \circ x\}.$$

The join operation and its inverse are extended to subsets of V by defining

$$A \circ B := \bigcup a \circ b, \qquad A/B := \bigcup a/b,$$

where the unions are taken over all $a \in A$ and all $b \in B$. If $A = \emptyset$ or $B = \emptyset$, then $A \circ B = A/B = \emptyset$.

A pair $(V, \circ)$ is said to be a *join space*, if it satisfies the following postulates for all $a, b, c, d, \in V$:

(I) $a \circ b \neq \emptyset, \quad a/b \neq \emptyset,$
(II) $a \circ b = b \circ a,$
(III) $a \circ (b \circ c) = (a \circ b) \circ c,$
(IV) $a/b \cap c/d \neq \emptyset$ implies $a \circ d \cap b \circ c \neq \emptyset.$

Let $(V, \circ)$ be a join space. The following concepts and results out of [4] are used in this paper:

The subset A of V is said to be *convex* if $A \circ A \subseteq A$ and *linear* if $A/A \subseteq A$. Linear sets also are convex.
Let A be a nonempty linear subset of V. Then a is *congruent* to b modulo A, if $a \circ A \cap b \circ B \neq \emptyset$. This is a congruence relation in V, and we let $(a)_A$ denote the congruence class of a in V. It holds $(a)_A = a \circ A/A$. Let $V{:}A$ be the set of congruence classes modulo A, then a join operation $*$ in $V{:}A$ is given by

$$(a)_A * (b)_A := \{(x)_A \mid x \in a \circ b\}.$$

$(V{:}A, *)$ is a join space, the so-called *factor space* V modulo A. Let $(V, \circ)$ be a join space and $e \in V$. Then e is an *identity* of $(V, \circ)$ if $e \circ a = a \circ e = a$ for all $a \in V$. $(V, \circ)$ is called a join space with identity. b is an *inverse* of a if $e \in a \circ b$. Let the inverse of a be denoted by a^{-1}. For sets let $A^{-1} := \{x^{-1} \mid x \in A\}$. Identity e as well as inverse a^{-1} of a are unique. Further $A/B = A \circ B^{-1}$.
Again let $(V, \circ)$ be an arbitrary join space and $A \subseteq V$. Then the *linear access* lin A of A is the set of all $a \in V$, if there exists an element b such that $a \circ b \subseteq A$. This set was first used by KLEE [3] in linear spaces. The properties of linear access in a convexity space, which is a special case of a join space, are given in [1]. If a subset A is linear in a join space with identity, then lin A is linear, too.

3. The set of classes of conjugate elements from a group as join space

Let $G = \{\varepsilon, \alpha, \beta, \gamma, \ldots\}$ be an arbitrary group with the unit ε and $V = \{e, a, b, c, \ldots\}$ the set of classes of conjugates from G. Then two elements α and β are contained in the same class of G if, and only if, there exists an inner automorphism of G such that α is mapped onto β, or expressed otherwise, if there exists an element $\tau \in G$ such that $\tau^{-1}\alpha\tau = \beta$. Hence any element of G permutable with all elements of G, thus contained in the center of G, readily forms a class alone. Let e denote the class containing the unit ε of G.
The two following results concerning classes of conjugates of G are well-known:

The complex product ab of two classes a and b is the union of such classes. (1)

Let a be a class, then the inverse class $a^{-1} := \{\alpha^{-1} \mid \alpha \in a\}$ is a class, too. (2)

Property (1) shows that complex multiplication in G induces a join operation $\circ$ in V. Let

$$x \in a \circ b :\Leftrightarrow x \subseteq ab. \tag{3}$$

We have the following

Theorem 1. *The set V of classes of conjugates from a group G is a join space with identity with respect to the join operation* (3).

Proof. We have to show that the properties (I), ..., (IV) are satisfied.

(I) Clearly $a \circ b \neq \emptyset$.

Now let be α_1 a fixed and β an arbitrary element out of a and b, respectively. Then there exists a $\xi \in G$ such that $\alpha_1 = \beta\xi$. Let x be the class generated by ξ. Then for any $\alpha \in a$ there always exists a $\tau \in G$ such that $\alpha = \tau^{-1}\alpha_1\tau$. Hence $\alpha = (\tau^{-1}\beta\tau)(\tau^{-1}\xi\tau)$ and $a \subseteqq bx$. Using (3) we have $a \in b \circ x$, that $x \in a/b$ and therefore $a/b \neq \emptyset$.

(II) Each class a is mapped onto itself by each inner automorphism of G. Hence a is permutable with each element of G. Moreover $ab = ba$ for an arbitrary class b. Thus using (3) we have the statement $a \circ b = b \circ a$.

(III) This fact is obtained by the associative law for complex multiplication in a group.

(IV) Assume $a/b \cap c/d \neq \emptyset$. Then there exists a class x such that $a \in b \circ x$ and $c \in d \circ x$. Hence there are elements $\beta_1 \in b$ and $\xi_1 \in x$ to each $\alpha_1 \in a$ such that $\alpha_1 = \beta_1\xi_1$ and elements $\delta_1 \in d$ and $\xi_2 \in x$ to each $\gamma_1 \in c$ such that $\gamma_1 = \delta_1\xi_2$. Since ξ_1 and ξ_2 are contained in the same class, there exists a $\tau \in G$ such that $\tau^{-1}\xi_1\tau = \xi_2$. Hence $\tau^{-1}(\beta_1^{-1}\alpha_1)\,\tau = \delta_1^{-1}\gamma_1$ and therefore $\beta_1^{-1}\alpha_1$ and $\delta_1^{-1}\gamma_1$ are in the same class of G. Using (2) and (3) the intersection of $b^{-1} \circ a$ and $d^{-1} \circ c = c \circ d^{-1}$ (because of (I)) contains a class y at least. Let η be any element out of y. Then once there exist elements $\alpha_2 \in a$ and $\beta_2 \in b$ such that $\eta = \beta_2^{-1}\alpha_2$ and on the other hand elements $\gamma_2 \in c$ and $\delta_2 \in d$ such that $\eta = \gamma_2\delta_2^{-1}$. But $\beta_2^{-1}\alpha_2 = \gamma_2\delta_2^{-1}$ implies $\alpha_2\delta_2 = \beta_2\gamma_2$, and we have the desired result $a \circ d \cap b \circ c \neq \emptyset$.

Finally $e = \{\varepsilon\}$ is the identity of the join space, and the proof is complete.

In this join space $(V, \circ)$ there corresponds a subset of G to a subset of V in a natural manner using the elements of G which are contained in the elements of V. Considering all nonempty subsets of V we get the so-called invariant complexes of G. These sets are those nonempty subsets of G exactly mapped onto itself by all inner automorphisms of G. Conversely a nonempty subset of V corresponds to each invariant complex of G. Moreover we have

Theorem 2. a) *An invariant subsemigroup of G corresponds to each nonempty convex subset of V and vice versa.*

b) *A normal subgroup of G corresponds to each nonempty linear subset of V and vice versa.*

Proof. a) If a subset A of V is convex then using $A \circ A \subseteqq A$ and (3), the corresponding invariant complex of G is closed under multiplication.

b) If a subset A of V is even linear, that is $A/A \subseteqq A$ then A is convex more as before. Hence by a) the corresponding invariant complex N in G is a subsemigroup of G. Furthermore let a be any class out of A. $e \in a/a$ implies $e \in A$ and $a^{-1} \in e/a$ implies $a^{-1} \in A$. Hence N is a subgroup of G and therefore a normal subgroup of G, too.

The converse statements can be shown without trouble.

4. An application to the group theory

Let $(V, \circ)$ be the join space of classes of conjugates from the group G. For any normal subgroup N of G let $Z(G \div N)$ the normal subgroup of G such that $Z(G \div N)/N$ is the center of the factor group G/N (Denotion like in [2]). Let A denote the linear set corresponding to N in $(V, \circ)$ according Theorem 2. Then the linear set lin A of V corresponds to the normal subgroup $Z(G \div N)$ of G. This fact shows the following theorem giving an interesting characterization of the normal subgroup $Z(G \div N)$. This subgroup plays an important role in constructing central series for instance.

Theorem 3. *Let N be a normal subgroup of the group G and let $Z(G \div N)/N$ be the center of the factor group G/N. Then $Z(G \div N)$ is the union of classes a of conjugates from G such that there exists a class b with $ab \subseteqq N$.*

Proof. For any $\alpha \in G$ let a be the class of conjugates of G containing α. Since

$$Z(G \div N) = \{\alpha \mid (\alpha N)(\varrho N) = (\varrho N)(\alpha N) \text{ for all } \varrho \in G\}$$

and

$$(\alpha N)(\varrho N) = (\varrho N)(\alpha N) \Leftrightarrow (\varrho^{-1}\alpha\varrho) N = \alpha N$$

we have

$$\alpha \in Z(G \div N) \Leftrightarrow aN = \alpha N. \tag{4}$$

Let T denote the union of classes a of conjugates from G such that there exists a class b with $ab \subseteqq N$. We have to show $T = Z(G \div N)$:
Let $a \subseteqq Z(G \div N)$, hence $\alpha \in Z(G \div N)$ and therefore $\alpha^{-1} \in Z(G \div N)$, too. Using (4) we have $aN = \alpha N$ and $a^{-1}N = \alpha^{-1}N$. Thus $(aN)(a^{-1}N) = (\alpha N)(\alpha^{-1}N)$ and $aa^{-1} \subseteqq N$. Therefore $a \subseteqq T$ and $Z(G \div N) \subseteqq T$.
To prove the reverse inclusion suppose $a \subseteqq T$. Then there exists a class b with $ab \subseteqq N$. This implies $(aN)(bN) = N$ and $(aN)(\alpha bN) = \alpha N$. Since $\alpha b \subseteqq N$ so $\alpha bN = N$. Therefore $aN = \alpha N$ and using (4) $\alpha \in Z(G \div N)$. Thus $a \subseteqq Z(G \div N)$ and $T \subseteqq Z(G \div N)$, and we have our result.

5. Join spaces and automorphisms of groups

Let $\mathfrak{A}$ be the automorphism group of the group G, $\mathfrak{J}$ the group of inner automorphisms of G and $\mathfrak{U}$ a subgroup of $\mathfrak{A}$ containing $\mathfrak{J}$. Let $V_{\mathfrak{U}}$ denote the set of those classes of G satisfying the following property:

Each class is a set of elements of G mapped onto itself by the automorphisms of $\mathfrak{U}$.

Then the complex product of two classes is also a union of such classes. According to (3) a join operation $\circ$ is defined in V. In analogy to the proof of Theorem 1 we can show that $(V_{\mathfrak{U}}, \circ)$ is a join space with identity. But we use hypothesis $\mathfrak{J} \subseteqq \mathfrak{U}$ in order to prove (II) only.
Let N be a subgroup of G being invariant with respect to the automorphisms of $\mathfrak{U}$, called a $\mathfrak{U}$-invariant subgroup. Then in analogy to Theorem 2 we can show, that a $\mathfrak{U}$-invariant subgroup N of G corresponds to a linear subset A of V and vice versa. Each $\mathfrak{U}$-invariant subgroup N of G, in case of $\mathfrak{U} = \mathfrak{J}$ it is a normal subgroup of G, gives rise to another join space, namely the factor space $V_{\mathfrak{U}}: A$. Since

$$(a)_A = a \circ A/A = (a \circ A) \circ A^{-1} = a \circ A$$

the elements of the join space $V_{\mathfrak{U}}:A$ are unions of cosets of the $\mathfrak{U}$-invariant subgroup N and have the form of aN.

6. Problems

a) Suppose $G = A \cup B \cup C \cup \cdots$ is a partitioning of the group G in classes satisfying the following condition:

The complex product of two classes is a union of such classes.

Look for those partitionings in classes $V = \{A, B, C, \ldots\}$ of G such that V is a join space with respect to the join operation (3) induced by complex multiplication.[1])

The class containing the unit of G is not necessarily a subgroup of G:

Let G be the symmetric group of third degree, $A = \{(1), (12), (123)\}$ and $B = \{(13), (23,) (132)\}$, then A and B form a join space of two elements with respect to (3).

Such examples can be stated for abelian groups, too.

b) Realize each join space by a class-partitioning of a group as described in a).

This is true for the 6 isomorphic types of join spaces containing two elements.

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1) This problem was established by G. PAZDERSKI.