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Strong purity in lattices

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1. Introduction

This paper is thought to be a continuation of the papers [15] and [16]. Since the notions of this paper may be useful for a larger class of lattices than ZI-lattices we intend to investigate that larger class.
As we can see in the papers of R. FRITZSCHE [3, 4, 5, 6, 7] und G. RICHTER [7, 13, 14] the notion of purity, which was first given by T. J. HEAD [8] and A. KERTÉSZ [9], plays an important role in the theory of ZM-lattices, i.e., cyclically generated modular lattices. But we cannot get similar results neither in the theory of ZI-lattices nor in the theory of Baer lattices (see [17], [18] and [19]). Therefore we shall define the notion of strong purity. Note that a strongly pure element is always pure in an algebraic lattice.
Further we intend to investigate weakly independent subsets of lattices. In an algebraic lattice each weakly independent subset is independent and in every lattice each independent subset is weakly independent. Moreover, we shall show some connections between the properties of each element of a lattice L being strongly pure and of L being atomistic.

2. Basic notions

Let L be a complete lattice. If $a, b \in L$ and $a \leqq b$ we shall define the quotient

$$b/a := \{x : a \leqq x \leqq b\}.$$

An element $q \in b/a$ is called inaccessible (from below) (see G. BIRKHOFF and O. FRINK [1] and G. GRÄTZER and E. T. SCHMIDT [2]) if

$$q = \bigvee (a_\nu : a_\nu \in b/a, \nu \in N)$$

implies

$$q = \bigvee (a_\nu : \nu \in N' \subseteqq N, |N'| < \infty).$$

Further we define $Q(b/a)$ to be the set of all inaccessible elements, $K(b/a)$ to be the set of all compact elements and $V(b/a)$ to be the set of all join-irreducible elements of the quotient b/a.
Finally we define

$$V_1(b/a) := Q(b/a) \cap V(b/a),$$

i.e., $V_1(b/a)$ is the set of all completely join-irreducible elements of b/a,

$$V_2(b/a) := \{x : x \in V(b/a),\quad x \neq \bigvee (v : v \in V_1^* \subseteq V_1(b/a);\; V_1^* \text{ arbitrary subset of } V_1(b/a))\},$$

$$V_3(b/a) := \{x : x = q \vee \bigvee (v_i : i = 1, \ldots, n;\, v_i \in V_2(b/a)),\, q \in Q(b/a)\},$$

$$V_0(b/a) := V_1(b/a) \cup V_2(b/a),$$

$$Q_0(b/a) := Q(b/a) \cap V_3(b/a).$$

To imagine $V_2(b/a)$ we look, for instance, at the intervall $[0, 1]$ of real numbers. Then $V_2([0, 1]) = (0, 1]$.
For $E = Q, Q_0, K, V, V_0, V_1, V_2, V_3$ we define $E := E(L)$ and L is called an E-lattice if for any element $a \in L$ the condition $a = \bigvee (q_\nu : q_\nu \in E, \nu \in N)$ holds.
A subset $U = \{u_\nu : u_\nu \in b/a, \nu \in N\}$ of b/a is called independent in b/a, if $u_{\nu_0} \wedge \bigvee (u_\nu : \nu \in N \setminus \{\nu_0\}) = a$ holds for each $\nu_0 \in N$. A subset $U \subseteq b/a$ is called weakly independent in b/a, if any finite subset $U' \subseteq U$ is independent in b/a.
If $x = \bigvee (u : u \in U)$ and $U \subseteq b/a$ is independent in b/a, then x is called the direct join of the elements of U in the sublattice b/a.

This fact will be denoted by $x = \dot{\bigvee}_a (u : u \in U)$. $x = \overset{w}{\bigvee}_a (u : u \in U)$ means that $x = \bigvee (u : u \in U)$ and that U is weakly independent in $1/a$ (and in b/a, respectively, if $U \subseteq b/a$). In this case x is called the w-direct join of the elements of U.
If $U = \{u_1, u_2, \ldots, u_n\}$ is independent or weakly independent, then we write

$$u_1 \dot{\vee}_a \cdots \dot{\vee}_a u_n := \dot{\bigvee}_a (u : u \in U)$$

or

$$u_1 \overset{w}{\vee}_a \cdots \overset{w}{\vee}_a u_n := \overset{w}{\bigvee}_a (u : u \in U).$$

Instead of $\dot{\bigvee}_0$ and $\overset{w}{\bigvee}_0$ we simply use $\dot{\bigvee}$ and $\overset{w}{\bigvee}$.
If $U \subseteq M \subseteq b/a$ is either independent or weakly independent in b/a, and $U \cup \{x\}$ is not independent or not weakly independent for each $x \in M \setminus U$, then U is called maximal independent or maximal weakly independent, respectively, in M.
Because any independent subset is also weakly independent, it follows that weak independence is a property of finite character. Therefore any subset $M \subseteq b/a$ contains a maximal weakly independent subset.

If either $b = \dot{\bigvee}_a (q : q \in B \subseteq Q_0(b/a))$ or $b = \overset{w}{\bigvee}_a (q : q \in B \subseteq Q_0(b/a))$, then B is called either a basis or a weak basis of b/a.
An element $s \in b/a$ is called strongly pure in b/a, if for any $q \in Q_0(b/a)$ there exists an element $r \in b/a$ with the property $s \vee q = s \dot{\vee}_a r$.
Let L be a V_1-lattice and denote by $F(b/a)$ the set of all joins of finitely many elements of $V_1(b/a)$.
An element $s \in b/a$ is called strictly pure in b/a, if for each $f \in F(b/a)$ there exists an element $g \in b/a$ with $s \vee f = s \dot{\vee}_a g$.
Now let L be a lattice and denote by T a subset of L. We say T satisfies the Isomorphism Property (I) if for each $t \in T$ and for each $b \in L$ there exists an isomorphism $\varphi : x \mapsto \varphi(x) = x \vee b$ ($x \in t/b \wedge t$) of $t/b \wedge t$ onto $t \vee b/t$ such that $\varphi^{-1} : y \mapsto \varphi^{-1}(y) = y \wedge t$ ($y \in b \vee t/b$) holds (see also [17, Definition 3.1.]).

3. Q_0-lattices

A. KERTÉSZ proved the following theorem in [9] (Satz 1).

Theorem. *Let L be an algebraic modular lattice. A subset B of K is a basis of L if and only if B is maximal independent in K and $\dot{\bigvee} (b:b \in B)$ is pure in L.*

Theorem 1 and Theorem 2 are generalizations of this Theorem for Q_0-lattices.

Theorem 1. *Let L be a complete Q_0-lattice, in which Q_0 satisfies the Isomorphism Property* (I). *B is a weak basis of L if and only if there exists a maximal weakly independent subset B_0 of Q_0 with $B \subseteqq B_0$ and $\overset{w}{\vee} (b: b \in B) = \overset{w}{\vee} (b:b \in B_0)$ is strongly pure in L.*

Proof. Let L be a Q_0-lattice with a weak basis B. Then B is a weakly independent set which can be extended to a maximal weakly independent subset B_0 of Q_0. Therefore $B \subseteqq B_0$ and $1 = \overset{w}{\vee} (b:b \in B) = \overset{w}{\vee} (b:b \in B_0)$ are satisfied and 1 is always strongly pure.

Now we assume that there exist a weakly independent subset B of Q_0 and a maximal weakly independent subset B_0 of Q_0 with $B \subseteqq B_0$ and $s := \overset{w}{\vee} (b:b \in B) = \overset{w}{\vee} (b:b \in B_0)$ is strongly pure. If $q \in Q_0 \setminus B_0$ and $s < s \vee q \leqq 1$, then there exists an element $a \in L$ with $s \vee q = s \dot{\vee} a$, because s is strongly pure. Since L is a Q_0-lattice there is at least one element $p \in Q_0(a/0)$ with $0 < p$. The independence of $\{s, p\}$ follows from the independence of $\{s, a\}$. Let $q_0 \in B_0$ with

$$q_0 \wedge \left(\dot{\bigvee} (q: q \in B^* \subseteqq B_0, q_0 \notin B^*, |B^*| < \infty) \dot{\vee} p\right) = d > 0,$$

then (I) yields that there is an element $p_0 \leqq p$ with

$$\dot{\bigvee} (q: q \in B^*) \vee d = \dot{\bigvee} (q: q \in B^*) \dot{\vee} p_0,$$

i.e., $0 < p_0 \leqq p \wedge s$ in contradiction to the independence of $\{s, p\}$. Consequently, $B_0 \cup \{p\}$ is weakly independent. But this is impossible since B_0 is already maximal weakly independent. Therefore such an element $q \in Q_0 \setminus B_0$ does not exist, i.e., $s = 1$.

Theorem 2. *Let L be a complete Q_0-lattice, in which Q_0 satisfies* (I). *B is a basis of L if and only if B is maximal independent in Q_0 and $\dot{\bigvee} (b:b \in B)$ is strongly pure in L.*

Proof. If B is a basis of L, then $q \leqq 1 = \dot{\bigvee} (b:b \in B)$ holds for each $q \in Q_0$, i.e., B is maximal independent in Q_0. 1 is always strongly pure.
Now we assume that B is maximal independent in Q_0 and $s := \dot{\bigvee} (b:b \in B)$ is strongly pure. As in the proof of Theorem 1 we can also show that $q \leqq s$ is satisfied for any $q \in Q_0$, i.e., $s = 1$.

Because of $V_0 \subseteqq Q_0$ any V_0-lattice is a Q_0-lattice. Theorem 3 asserts that the converse of this fact does not hold.

Theorem 3. *There exists a Q_0-lattice which is not a V_0-lattice.*

Proof. Figure 1 shows an algebraic modular lattice. 1 is not the join of completely join-irreducible elements, since we have

$$1 = x_1 \vee y_i \qquad (i = 1, 2, \ldots)$$

and

$$y_i = x_{i+1} \vee y_j \qquad (j = i + 1, i + 2, \ldots).$$

In the definition of a Baer lattice is said that every element of a Baer lattice is a join of completely join-irreducible elements. Since $V_1 \subseteqq Q \subseteqq Q_0$ in a complete lattice holds by definition of V_1, Q, Q_0, any complete Baer lattice is a V_1-lattice, a Q-lattice and a Q_0-lattice.

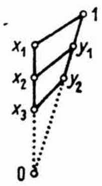

Fig. 1

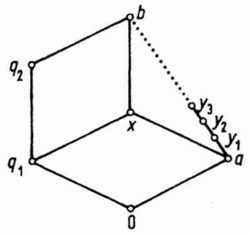

Fig. 2

4. Special connections

In [8] HEAD investigated some connections between the intervals of an algebraic modular lattice L and the whole lattice L. He showed, for instance, that $c \in K(b/a)$ if and only if $c = a \vee c_0$ with $c_0 \in K(b/0)$. In this paragraph we want to investigate some similar connections in Q-lattices or in Q_0-lattices, respectively.

Lemma 4. *Let L be a Q_0-lattice. $a \leqq b$ and $q \in Q_0(b/0)$ do not yield necessarily that $a \vee q \in Q_0(b/a)$.*

Proof. Figure 2 shows a V_1-lattice which is not algebraic and not modular. In any V_1-lattice $Q = Q_0$ holds since $V_2 = \emptyset$ by definition of V_2.
We have $b = a \vee q_2$ with $q_2 \in Q(b/0)$, $b = \bigvee (y_i : i = 1, 2, \ldots)$, and for any natural number n the condition $b > \bigvee (y_i : i = 1, \ldots, n)$ holds, i.e., $b \notin Q(b/a) = Q_0(b/a)$.

Lemma 5. *Let L be a Q-lattice.*
If $q \in Q(b/a)$, then there exists an element $c \in b/0$ where $c = \bigvee \big(q_i : q_i \in Q(b/0),\ i = 1, \ldots, n\big)$ and $q = a \vee c$.

Proof. Since L is a Q-lattice we can conclude that

$$q = \bigvee \big(q_\nu : q_\nu \in Q(b/0),\ \nu \in N\big).$$

$q \in Q(b/a)$ yields

$$q = \bigvee (a \vee q_\nu : \nu \in N' \subseteqq N,\ |N'| < \infty) = a \vee \bigvee (q_\nu : \nu \in N').$$

Now put $c = \bigvee (q_\nu : \nu \in N')$.

Theorem 6. *Let L be a Q-lattice in which Q satisfies* (I). *Then $q \in Q(b/a)$ if and only if $q = a \vee c$ with $c \in Q(b/0)$.*

Proof. Suppose $c \in Q(b/0)$. Then $c \in Q(b/a \wedge c)$ also holds and therefore $a \vee c \in Q(b/a)$ inasmuch as $a \vee c/a \cong c/a \wedge c$. Assume $q \in Q(b/a)$. Applying Lemma 5 we get $q = a \vee c$ where $c = \bigvee \big(q_i : q_i \in Q(b/0),\ i = 1, \ldots, n\big)$.
In the following Lemma 7 we shall show that $c \in Q(b/0)$ and then the proof will be complete.

Lemma 7. *If Q satisfies* (I), *then*

$$c = \bigvee (q_i : q_i \in Q,\ i = 1, \ldots, n) \in Q.$$

Proof. We have only to show that the join of the two elements $q_1, q_2 \in Q$ is an element of Q.
Suppose $q_1 \vee q_2 \notin Q$. Then there exists an infinite set $Q_* \subseteqq Q(q_1 \vee q_2/0)$ where

$$q_1 \vee q_2 = \bigvee (q : q \in Q_*),$$

$$q_1 \vee q_2 \neq \bigvee (q : q \in Q' \subseteqq Q_*, |Q'| < \infty). \tag{a}$$

According to the first part of the proof of Theorem 6 we get $q_1 \vee q_2 \in Q(q_1 \vee q_2/q_1)$, i.e.,

$$q_1 \vee q_2 = q_1 \vee \bigvee (q_i : q_i \in Q_*, i = 3, \ldots, m).$$

Then also $q_1 \vee q_2 \in Q\big(q_1 \vee q_2/\bigvee (q_i : i = 3, \ldots, m)\big)$, i.e.

$$q_1 \vee q_2 = q_3 \vee \cdots \vee q_m \vee q_1 = q_3 \vee \cdots \vee q_m \vee q_{m+1} \vee \cdots \vee q_n$$

where $q_i \in Q_*$ for $i = 3, \ldots, n$ in contradiction to (a).

In [8] T. J. Head showed that a pure element a of an algebraic modular lattice is complemented in L if

$$1 = \dot{\bigvee}_a \big(c_\nu : c_\nu \in K(1/a), \nu \in N\big).$$

In modular Q-lattices we are able to prove a similar theorem. Before doing this we prove

Lemma 8. *Let L be a modular Q_0-lattice, $a, b \in L$, $a \leqq b$, $b = \overset{w}{\bigvee}_a (a \vee y_\nu : y_\nu \in L, \nu \in N)$. If $y_\nu \wedge a = 0$ for each $\nu \in N$, then $b = a \overset{w}{\vee} \overset{w}{\bigvee} (y_\nu : \nu \in N)$.*

Proof. According to the definition weak independence is a property of finite character. We have only to prove that each finite subset of the set $\{a, y_\nu : \nu \in N\}$ is independent. $a \wedge y_\nu = 0$ holds for each index $\nu \in N$.
Suppose that every subset $\{a, y_{\nu_i} : i = 1, \ldots, n, \nu_i \in N, n \leqq k\}$ is independent.
Assume $0 < d = y_{\nu_{k+1}} \wedge \big(a \dot{\vee} \dot{\bigvee} (y_{\nu_i} : i = 1, \ldots, k)\big)$ where $\nu_i \in N$ for $i = 1, \ldots, k + 1$. Then

$$\begin{aligned} a \leqq a \vee d &= a \vee \big(y_{\nu_{k+1}} \wedge (a \dot{\vee} \dot{\bigvee} (y_{\nu_i} : i = 1, \ldots, k))\big) \\ &= (a \vee y_{\nu_{k+1}}) \wedge \dot{\bigvee}_a (a \dot{\vee} y_{\nu_i} : i = 1, \ldots, k) = a, \end{aligned}$$

for L is modular and $\dot{\bigvee}_a (a \vee y_{\nu_i} : i = 1, \ldots, k + 1)$ exists. Because of $a \wedge y_{k+1} = 0$ we get $a \wedge d = 0$ and therefore $d = 0$, i.e., $\{a, y_{\nu_i} : i = 1, \ldots, k + 1\}$ is an independent subset of $\{a, y_\nu : \nu \in N\}$ and $\{a, y_\nu : \nu \in N\}$ is a weakly independent set.

Theorem 9. *Let L be a modular Q-lattice and b a strongly pure element of L. If $1 = \overset{w}{\bigvee}_b (q_\nu : q_\nu \in Q(1/b), \nu \in N)$, then $1 = \overset{w}{\bigvee} (q_\nu^* : q_\nu^* \in Q, \nu \in N) \overset{w}{\vee} b$.*

Proof. According to Theorem 6 there exists to each index $\nu \in N$ an element $p_\nu \in Q$ such that $q_\nu = b \vee p_\nu$.

Since b is strongly pure there further exist elements q_ν^* with $b \vee p_\nu = b \dot{\vee} q_\nu^*$. $q_\nu \in Q(1/b)$ and $q_\nu/b = b \dot{\vee} q_\nu^*/b \cong q_\nu^*/0$ imply $q_\nu^* \in Q$. According to Lemma 8 we get

$$1 = b \overset{w}{\vee} \overset{w}{\bigvee} (q_\nu^* : q_\nu^* \in Q, \nu \in N).$$

5. Q-lattices

T. J. HEAD [8] and A. KERTÉSZ [9] showed that in algebraic modular lattices the following two conditions are quivalent:

(1) L is atomistic,

(2) every element of L is pure.

In this section we shall get a similar result for Q-lattices. Henceforth L denotes a Q-lattice.

Lemma 10. *Let $c \in Q$. Then there exists an element c_0 where $c_0 \prec c$.*

Proof. Let $c_1 < c$. According to the chain axiom (see G. SZÁSZ [20, p. 28]) there exists a maximal chain C between c and c_1, containing c and c_1. Then $e := \bigvee (d : d \in C \setminus \{c\})$ exists since L is complete. If $e = c$ we get $c = d_1 \vee \cdots \vee d_n$ where $d_i \in C \setminus \{c\}$ for $i = 1, \ldots, n$, because $c \in Q$. But then there exists an index l, $1 \leqq l \leqq n$, $d_i \leqq d_l < c$ for $i = 1, \ldots, n$, i.e., $c < c$.
Therefore $e < c$. If $e \not\prec c$ there is an element e_1 such that $e < e_1 < c$ in contradiction to the maximality of C.
Hence $e \prec c$ and the proof is complete.

Lemma 11. *Let the covering property* (C) *hold in L, i.e., if p is an atom and $p \wedge a = 0$, then $a \prec a \vee p$. If the set $\{p_1, \ldots, p_{n-1}, p_n \dot\vee c\}$ is an independent set where p_i are atoms for $i = 1, \ldots, n$ then $\{p_1, \ldots, p_n, c\}$ is an independent set* (*see also* [10, Lemma 6]).

Proof. Assume $p_n \leqq c \dot\vee p_1 \dot\vee \cdots \dot\vee p_{n-1}$. Because $p_n \nleqq c$ there exists an index r where $1 \leqq r \leqq n-1$ and

$$p_n \leqq c \dot\vee p_1 \dot\vee \cdots \dot\vee p_r,\ p_n \nleqq c \dot\vee p_1 \dot\vee \cdots \dot\vee p_{r-1}.$$

The property (C) yields $c \dot\vee p_1 \dot\vee \cdots \dot\vee p_{r-1} \prec c \dot\vee p_1 \dot\vee \cdots \dot\vee p_r$ and

$$c \dot\vee p_1 \dot\vee \cdots \dot\vee p_{r-1} \prec (c \dot\vee p_1 \dot\vee \cdots \dot\vee p_{r-1}) \dot\vee p_n \leqq c \dot\vee p_1 \dot\vee \cdots \dot\vee p_r.$$

On that account $(c \dot\vee p_1 \dot\vee \cdots \dot\vee p_{r-1}) \dot\vee p_n = c \dot\vee p_1 \dot\vee \cdots \dot\vee p_r$, i.e., $p_r \leqq p_1 \dot\vee \cdots \dot\vee p_{r-1} \dot\vee (c \dot\vee p_n)$ in contradiction to the independence of $\{(c \dot\vee p_n), p_1, \ldots, p_{n-1}\}$.

Similarly we can conclude that $c \wedge (p_1 \vee \cdots \vee p_n) = 0$. Therefore $\{p_1, \ldots, p_n, c\}$ is independent.

Lemma 12. *Let L be an atomistic lattice satisfying* (C), *i.e., L is a so-called AC-lattice.*
An element $c \in L$ is inaccessible if and only if $c = p_1 \dot\vee \cdots \dot\vee p_n$ where p_i are atoms for $i = 1, \ldots, n$.

Proof. Since L is atomistic $c = \bigvee (p_\nu : p_\nu \text{ atom}, \nu \in N)$ holds for each element $c \in L$. If c is inaccessible then $c = \bigvee (p_{\nu_i} : i = 1, \ldots, n, \nu_i \in N)$. The finite set $\{p_{\nu_1}, \ldots, p_{\nu_n}\}$ contains an independent subset $\{p_1, \ldots, p_r\}$ where $i \in \{\nu_1, \ldots, \nu_n\}$ for $i = 1, \ldots, r$ and $c = p_1 \dot\vee \cdots \dot\vee p_r$.

Suppose that $c = p_1 \dot\vee \cdots \dot\vee p_n$ where p_i is an atom for $i = 1, \ldots, n$ is not inaccessible. Then there exists a set $\{p_\nu : \nu \in N, |N| = \infty\}$ of atoms where

$$c = \bigvee (p_\nu : \nu \in N), \quad c > \bigvee (p_\nu : \nu \in N' \subseteqq N, |N'| < \infty). \tag{b}$$

It is evident that there exists an independent subset N^* of N with the finite cardinality n.

Then $p_{\nu_i} \leqq c = p_1 \dot{\vee} \cdots \dot{\vee} p_n$ for $i = 1, \ldots, n$; $\nu_i \in N^*$, i.e., there exists an index r with $p_{\nu_i} \leqq p_1 \dot{\vee} \cdots \dot{\vee} p_r$, $p_{\nu_1} \nleqq p_1 \dot{\vee} \cdots \dot{\vee} p_{r-1}$ and therefore $c = p_{\nu_1} \dot{\vee} p_1 \dot{\vee} \cdots \dot{\vee} p_{r-1} \dot{\vee} p_{r+1} \dot{\vee} \cdots \dot{\vee} p_n$. Because $p_{\nu_i} \nleqq p_{\nu_1} \dot{\vee} \cdots \dot{\vee} p_{\nu_{i-1}}$ we get in the same way $c = p_{\nu_1} \dot{\vee} \cdots \dot{\vee} p_{\nu_n}$ in contradiction to (b).

Theorem 13. *Let L possess the covering property* (C). *The following conditions are equivalent:*

(1) *L is atomistic,*

(2) *Q is an ideal and L satisfies* (C*),

(3) *Q is an ideal, every element of L is strongly pure and L satisfies* (C**).

The conditions (C*) *and* (C**) *are given by the following rules:*

(C*) $c_0 \prec c$ *and* $c \in Q$ *imply that an atom* $p \in L$ *exists and* $c_0 \dot{\vee} p = c$ *holds.*

(C**) $c \prec c \dot{\vee} p$ *implies that an atom* $p_0 \leqq p$ *exists where* $c \dot{\vee} p = c \dot{\vee} p_0$ *holds.*

(See also [11, Theorem 2]).

Proof. (1) implies (3): Lemma 12 yields that each independent subset of atoms of the quotient $c/0$ where c is inaccessible, i.e., $c = p_1 \dot{\vee} \cdots \dot{\vee} p_n$, has a cardinality not greater than n. Therefore each element of the quotient $c/0$ is inaccessible. The join of two inaccessible elements is a join of a finite number of atoms and according to Lemma 12 inaccessible, i.e., Q is an ideal.

Let now $c \prec c \dot{\vee} p$. Because L is atomistic there exists an atom $p_0 \leqq p$ and $p_0 \wedge c = 0$ holds.

(C) yields $c \prec c \dot{\vee} p_0 = c \dot{\vee} p$, i.e., L satisfies (C**). Let b be an arbitrary element of L and c an arbitrary inaccessible element of L. According to Lemma 12 $c = p_1 \dot{\vee} \cdots \dot{\vee} p_n$ where p_i is an atom for $i = 1, \ldots, n$. Let $q_0 = 0$ and suppose that $b \dot{\vee} q_r$, $r < n$, is existing. If $p_{r+1} \leqq b \dot{\vee} q_r$ then let $q_{r+1} = q_r$ otherwise let $q_{r+1} = q_r \dot{\vee} p_{r+1}$.

If $q_{r+1} = q_r \vee p_{r+1}$ and $0 < d := b \wedge q_{r+1}$ then there is an atom $p \leqq d \leqq q_{r+1}$, $p \nleqq q_r$ thus $q_{r+1} = q_r \vee p$ and also $p_{r+1} \leqq b \vee q_r$.

Therefore always $b \wedge q_{r+1} = 0$. Finally we get $b \wedge q_n = 0$ and $b \dot{\vee} q_n = b \vee c$, i.e., b is strongly pure.

(3) implies (2):

$c_0 \prec c$, $c \in Q$ and c_0 strongly pure imply that there is an element p such that $c_0 \vee c = c_0 \dot{\vee} p$ holds. According to (C**) there is an atom p_0 where $c_0 \dot{\vee} p_0 = c$, i.e., L satisfies (C*).

(2) implies (1):

In this part of the proof we have only to show that each inaccessible element of L is a join of atoms.

Let c be an arbitrary inaccessible element. According to Lemma 10 there is an element c_1 covered by c and since Q is an ideal $c_1 \in Q$ holds. In the same way we get now a descending chain of inaccessible elements:

$$c \succ c_1 \succ c_2 \succ c_3 \succ \cdots.$$

Lemma 11 and (C*) yield $c = p_1 \dot{\vee} p_2 \dot{\vee} \cdots$ where p_i are atoms for $i = 1, 2, \ldots$

Because c is inaccessible we get $c = p_1 \dot{\vee} \cdots \dot{\vee} p_n$ for a suitable n. Therefore L is atomistic and the proof is complete.

In [1] (Theorem 2) G. Birkhoff and O. Frink showed that a Q-lattice L is algebraic if and only if L is upper continuous.

In this section we shall give another condition for AC-lattices.

Theorem 14. *An AC-lattice L is algebraic if and only if every weakly independent subset of atoms is independent.*

Proof. It is obvious that every weakly independent subset of atoms is independent if L is algebraic.
Let every weakly independent subset of atoms be independent and denote by c an arbitrary inaccessible element of L. The first part of the proof of Lemma 12 yields that $c = p_1 \dot{\vee} \cdots \dot{\vee} p_n$ where p_i are atoms for $i = 1, \ldots, n$.
Let $c \leqq \vee (a_\nu : a_\nu \in L, \nu \in N)$. We have only to show that $p_i \leqq \vee (a_\nu : \nu \in N_i \subseteqq N, |N_i| < \infty)$ for $i = 1, \ldots, n$ holds.
Inasmuch as L is atomistic we have $a_\nu = \vee (p_{\nu_\mu} : p_{\nu_\mu}$ atom, $\mu \in M_\nu)$ for all $\nu \in N$.
Let $\Gamma := \cup (M_\nu : \nu \in N)$. According to our assumption the independence of subsets of atoms is a property of finite character and hence Γ possesses a maximal independent subset Γ^*. If $\dot{\bigvee} (p_\gamma : \gamma \in \Gamma^*) < \vee (p_\gamma : \gamma \in \Gamma)$ then there exists an index $\gamma_0 \in \Gamma \setminus \Gamma^*$ with $p_{\gamma_0} \wedge \dot{\bigvee} (p_\gamma : \gamma \in \Gamma^*) = 0$. If there is an index $\gamma_1 \in \Gamma^*$ with $p_{\gamma_1} \leqq p_{\gamma_0} \dot{\vee} \dot{\bigvee} (p_\gamma : \gamma \in \Gamma^* \setminus \{\gamma_1\})$ then

$$\dot{\bigvee} (p_\gamma : \gamma \in \Gamma^*) = p_{\gamma_0} \dot{\vee} \dot{\bigvee} (p_\gamma : \gamma \in \Gamma^* \setminus \{\gamma_1\})$$

follows from (C) in contradiction to $p_{\gamma_0} \nleqq \dot{\bigvee} (p_\gamma : \gamma \in \Gamma^*)$. Therefore $\dot{\bigvee} (p_\gamma : \gamma \in \Gamma^*) = \vee (p_\gamma : \gamma \in \Gamma)$ holds.
In the same way we can show that the sets $\{p_i, p_\gamma : \gamma \in \Gamma^*\}$, $i = 1, \ldots, n$, are independent if $p_i \nleqq \dot{\bigvee} \{p_\gamma : \gamma \in \Gamma_0^*\}$ for each finite subset Γ_0^* of Γ^*. But that would be a contradiction to $p_i \leqq \dot{\bigvee} (p_\gamma : \gamma \in \Gamma^*)$. Therefore there are finite subsets Γ_i^* with $p_i \leqq \dot{\bigvee} (p_\gamma : \gamma \in \Gamma_i^*)$ for $i = 1, \ldots, n$.
For each $\gamma \in \Gamma_i^*$ $(1 \leqq i \leqq n)$ there exists an index $\nu_\gamma \in N$ with $p_\gamma \leqq a_{\nu_\gamma}$. Let $N_i := \{\nu_\gamma : \gamma \in \Gamma_i^*\}$.
Then $|N_i| < \infty$ and $p_i \leqq \vee (a_\nu : \nu \in N_i)$ hold and the proof is complete.
The lattice in Figure 3 shows that this result does not hold in an arbitrary Q-lattice. That lattice is a V_1-lattice which is not algebraic and every weakly independent subset is independent.
We have $y_1 \dot{\vee} y_2 = y_1 \dot{\vee} v = y_2 \dot{\vee} v$ with $v = \vee (x_i : i = 1, 2, \ldots)$, $\vee (x_i : i = 1, \ldots, n) = x_n$ and $y_1 \nleqq y_2 \vee x_n$ for any $n \geqq 1$, i.e., y_1 is not compact.

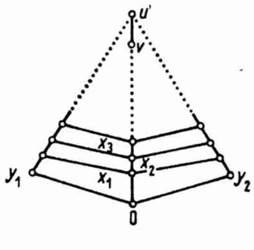

Fig. 3

6. V_1-lattices

This section deals with V_1-lattices that means with lattices which satisfy the property that each element has a representation as a join of completely join-irreducible elements. We shall continue investigations of the preceding section. As we can see in Figure 4 the assumption of Theorem 13 that L satisfies (C) is very strong.
On the other hand Figure 5 shows that there are atomistic lattices in which not all elements are strongly pure, since a is not a strongly pure element.

Theorem 15. *If every element of a V_1-lattice L is strongly pure, then L is atomistic.*

Proof. We have to show that every element of V_1 is an atom. Let v be an arbitrary element of V_1. According to Lemma 10 there is an element $v_0 \prec v$. Because v_0 is strongly pure and v is inaccessible there exists an element $v_1 \in L$ with $v = v_0 \vee v = v_0 \dot{\vee} v_1$. On account of $v \in V_1$ we get $v_0 = 0$ and $v_1 = v$, i.e., v is an atom.

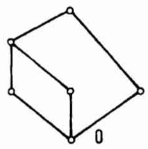

Fig. 4

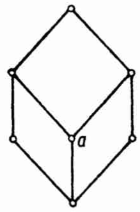

Fig. 5

In this paper we are not able to give necessary and sufficient conditions in order that every element of a V_1-lattice is strongly pure. But we found a necessary and sufficient condition in order that every element of a V_1-lattice is strictly pure. Note in any V_1-lattice in which V_1 satisfies (I) an element is strongly pure if and only if it is strictly pure, and in algebraic V_1-lattices purity, strong purity and strict purity mean the same.

Theorem 16. *L is a V_1-lattice in which every element is strictly pure if and only if L is atomistic and* (X) *is satisfied.*

(X) *$a, b, f, p \in L$, p atom, $f \in F$, $a \wedge b = p \wedge b = 0$, $b \vee a \vee p = b \vee f$ imply that there exists an element c with $b \vee a \vee p = b \vee f = b \dot{\vee} c$.*

Proof. If L is a V_1-lattice in which every element is strictly pure then we can show as in the proof of Theorem 15 that L is atomistic.
If $b \in L$ and $f \in F$ then there always exists an element c with $b \vee f = b \dot{\vee} c$ in particular in that case if $b \vee f = b \vee a \vee p$ with $a \wedge b = p \wedge b = 0$ holds.
Let L be an atomistic lattice in which (X) is satisfied, b an arbitrary element of L and f an arbitrary element of F. Then $f = p_1 \vee \cdots \vee p_n$ with atoms p_i for $i = 1, \ldots, n$ because V_1 is the set of all atoms of L in that case.
Let now $f_i := p_1 \dot{\vee} \cdots \dot{\vee} p_i$. Then $f_i \in F$ for $i = 1, \ldots, n$.
Let $q_0 = 0$ and $f_0 = 0$. Let us assume that we got an element q_l, $0 \leqq l < n$, with $b \dot{\vee} q_l = b \vee f_l$.
If $p_{l+1} \leqq b \dot{\vee} q_l$ then let $q_{l+1} = q_l$.
If $p_{l+1} \nleqq b \dot{\vee} q_l$, i.e., $p_{l+1} \wedge (b \dot{\vee} q_l) = 0$, then on account of (X) there exists an element q_{l+1} with $b \dot{\vee} q_{l+1} = b \vee q_l \vee p_{l+1} = b \vee f_{l+1}$ and also an element q_n with $b \dot{\vee} q_n = b \vee f_n = b \vee f$ exists, i.e., b is strictly pure and the proof is complete.

In a subsequent paper we shall deal with V_1-lattices in which V_1 satisfies (I). Special kinds of this lattices are Baer lattices, ZI-lattices, AC-lattices, cyclically generated modular lattices. We shall define a similar unary operation as we can find it in the papers [3, 4, 5, 6, 7, 12, 13, 14, 15, 16]. With the aid of this operation we shall be able to define terms like order and height of an element of V_1.

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