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On the SOMA-cube-puzzle

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In the wellknown SOMA-cube-puzzle is to solve the following problem:
Given are the 7 nonregular elements consisting of no more than 4, connected by the planes, uniform cubes, namely

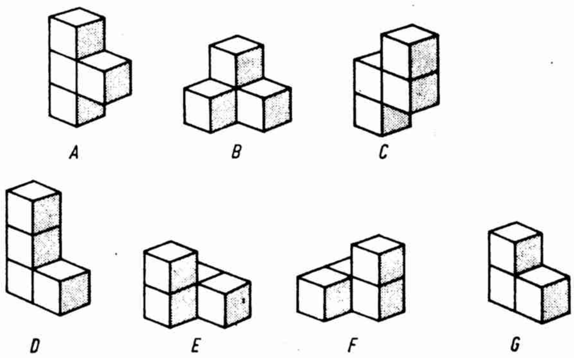

Fig. 1

Now these elements have to arrange to a $3 \times 3 \times 3$-cube.
This problem is due to the Dane Piet Hein.
M. Gardner [1] mentioned that Richard K. Guy had found more than 230 nonisomorphic solutions but the exact number of solutions was unknown. Two solutions S_1 and S_2 are isomorphic if and only if there is a transformation (rotations and reflections) such that S_1 is congruent to the image of S_2.
In this note we will present the following results.

Theorem 1. *There are exactly* 240 *nonisomorphic solutions of the SOMA-cube-puzzle.*

Theorem 2. *There are exactly* 11520 *solutions of the SOMA-cube-puzzle.*
In order to fix the position of an element in the $3 \times 3 \times 3$-cube we number the 27 uniform cubes as follows

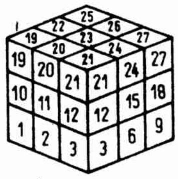

Fig. 2

(The below and the middle level are numbered analogously to the above level.)
Now A (23, 25, 26, 27) means that A contains the cubes with the numbers 23, 25, 26, and 27. The cubes 1, 3, 7, 9, 19, 21, 25, and 27 will be called corners of the $3 \times 3 \times 3$-cube.
Now we assume that we have a solution and we will derive necessary conditions of the positions of some elements.

Lemma 1. *The element A contains 2 corners of the $3 \times 3 \times 3$-cube.*

Proof. As easily can be verified,

i) each of the elements B, C, E, F, G contains at most 1 corner,
ii) D contains 0, 1, or 2 corners,
iii) A contains 0 or 2 corners.

Since the $3 \times 3 \times 3$-cube contains 8 corners the statement follows.
The cube of A or B having 3 neighbour-cubes is called central-cube of A or B, respectively.

Lemma 2. *The central-cube of B has an odd number.*

Proof. Each pair of neighboured cubes in the $3 \times 3 \times 3$-cube has one even and one odd number. Thus,

i) each of the elements C, D, E, F has 2 even and 2 odd numbers,
ii) the element G has 2 even and 1 odd or 1 even and 2 odd numbers,
iii) each of the elements A and B has 3 even and 1 odd or 1 even and 3 odd numbers.

Since the $3 \times 3 \times 3$-cube contains 14 odd and 13 even numbers, the central-cube of A has an even number if and only if the central-cube of B has an odd number. Lemma 1 implies that the central-cube of A has an even number. Hence, the lemma follows..

Lemma 3. *Two different solutions S_1 and S_2 having the same positions of A and B are nonisomorphic.*

Proof. Assume the contrary. Then there are rotations and reflections which transform S_1 in S_2. Clearly these transformations can be expressed as at most one reflection — and then in such way that A remains A — and rotations. Hence, A has the same places in the solutions S_1 and S_2 only if the rotations are the identical rotation. Finally, by the same argument with respect to B there can not be a reflection in the transformation. Thus, the total transformation is the identical one. Thus, the solution are the same in contradiction to our supposition.

Proof of Theorem 2.

i) Every solution can be transformed uniquely in normal form, i.e. A (1, 2, 3, 11) and B contains no of the cubes 3, 6, 9, 12, 15, 18, 21, 24, 27. A (1, 2, 3, 11) can be obtained by rotations only, of course. If B does not have the described property we reflect the solution on the plane determined by the cubes 2, 8, 20, 26. By the argument of the proof of Lemma 3 these rotations and the reflection (if necessary) are unique.

ii) From every solution in normal form we can derive exactly 48 different solutions; 24 positions for A and then always 2 for B (reflection). It is clear that no 2 different solutions in normal form have same derivations.
Now Theorem 1 implies Theorem 2.

To Theorem 1. By the arguments above we may assume that A (1, 2, 3, 11) and B contains no of the cubes 3, 6, 9, 12, 15, 18, 21, 24, 27. The 12 possible positions of B (see Lemma 2) are given in the table below. Now one can discuss — in every combination of A and B — the possible positions of C and D and so on.
The second author [2] has done this laborious work under the direction of the first author. Using methods like those used in the proofs of the lemmas above and many statements which early garantee that a combination can not be completed to a solution made it possible to write the whole discussion down on nearly 100 pages. But we will not give it here. The number of solutions in the positions of B is listed in the table below also.

No.	Position of B	Number of solutions
1	4, 5, 8, 14	4
2	4, 7, 8, 16	53
3	4, 13, 14, 16	6
4	4, 10, 13, 14	11
5	10, 13, 14, 22	4
6	13, 14, 16, 22	2
7	8, 14, 16, 17	5
8	14, 16, 17, 26	0
9	10, 19, 20, 22	93
10	14, 20, 22, 23	3
11	14, 22, 23, 26	2
12	16, 22, 25, 26	57
		240

Note added in proof. Recently M. Gardner and R. K. Guy informed us that J. H. Conway and M. J. T. Guy also found that there are exactly 240 nonisomorphic solutions.

REFERENCES

[1] Gardner, M.: Mathematische Rätsel und Probleme. Vieweg, Braunschweig 1966.
[2] Larisch, H.-H.: Nichtisomorphe Lösungen des SOMA-Würfels, Diplomarbeit, Wilhelm-Pieck-Universität Rostock 1979.

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