Beiträge zur Algebra und Geometrie
11 (1981), 29–34

On generalized connections

IVAN KOLÁŘ

Recently several authors pointed out, [1, 2, 6, 7, 9], that it is useful to study the following generalization of the classical concept of a connection. Given any fibered manifold $Y \to X$, a (generalized) connection on Y is a section $\Gamma\colon Y \to J^1Y$ (= the first jet prolongation of Y). The classical connection on a principal fiber bundle $P(X, G)$ is determined by additional assumption that Γ is G-invariant, [3]. In this case, Γ is called *principal connection*, [6]. If E is a vector bundle, one frequently requires $\Gamma\colon E \to J^1E$ to be a vector bundle morphism; such a connection is called *linear*, [2, 6]. Even though the curvature of a (generalized) connection on Y was already studied from different points of view, [1, 2, 6, 10], we start the present paper with another original definition of the curvature by means of a general concept of the intrinsic exterior differential. Using a vertical parallelism on Y, we deduce generalized Bianchi formula, which clarifies from a more general point of view how the classical Bianchi identity depends on the invariance property of principal connections. Then we treat the torsion form of a generalized connection on the first order frame bundle of X. Finally we deduce that any connection on Y is prolonged into a connection on the space of all velocities of any order and dimension on Y. The prolonged connection on the tangent bundle of Y is used for a simple construction of the curvature of the original connection on Y.
Our consideration is in the category C^∞. All morphisms of fibered manifolds are base-preserving.

1. Consider a vector bundle $E \to B$, a linear connection γ on E and a vector bundle morphism $\varphi\colon TB \to E$. The exterior differential $d_\gamma\varphi\colon \wedge^2TB \to E$ of φ with respect to γ is defined by

$$(d_\gamma\varphi)(\xi, \eta) = \nabla_\xi\varphi(\eta) - \nabla_\eta\varphi(\xi) - \varphi([\xi, \eta]) \tag{1}$$

for any vector fields ξ, η on B. If $\varphi \equiv \varphi^\lambda_i dx^i$ is the coordinate expression of φ in some local coordinates x^i on B and some linear fiber coordinates z^λ on E, then (1) implies

$$d_\gamma\varphi \equiv d\varphi^\lambda_i \wedge dx^i + \varphi^\mu_i\, dx^i \wedge \Gamma^\lambda_{\mu j}\, dx^j, \tag{2}$$

where $\Gamma^\lambda_{\mu j}$ are the Christoffel symbols of γ.
Assume that φ has constant rank, so that its kernel K is a vector bundle over B. Formula (2) shows that the restriction of $d_\gamma\varphi$ to K does not depend on γ. Hence we obtain a map $\Phi\colon \wedge^2K \to E$ (determined by φ only), which will be called the *intrinsic exterior differential* of φ. Using (1), we find the following geometric interpretation

of Φ. If ξ, η are two vector fields in K, then

$$\Phi(\xi, \eta) = -\varphi([\xi, \eta]). \tag{3}$$

(We shall show in a next paper that the intrinsic exterior differential plays an important role in the theory of semi-holonomic jets of the second order.)

2. Consider a fibered manifold $\pi: Y \to X$ and a (generalized) connection $\Gamma: Y \to J^1Y$. Let

$$x^i, y^\alpha, \qquad i, j, \ldots = 1, \ldots, n = \dim X,$$
$$\alpha, \beta, \ldots = 1, \ldots, m = \dim Y - \dim X,$$

be some local fiber coordinates on Y. Every $\Gamma(u) \in J^1Y$, $u \in Y$, is identified with an n-dimensional subspace in T_uY of a form

$$dy^\alpha = F_i^\alpha(x, y)\, dx^i. \tag{4}$$

We shall say that (4) are the equations of Γ. Let $T(Y/X)$ be the bundle of all vertical tangent vectors of Y. The connection morphism $\omega = \omega_\Gamma: TY \to T(Y/X)$ assigns to every vector $A \in T_uY$ its projection into $T_u(Y/X)$ in the direction $\Gamma(u)$. In coordinates,

$$\omega \equiv dy^\alpha - F_i^\alpha(x, y)\, dx^i. \tag{5}$$

The kernel K_Γ of ω_Γ is a vector bundle over Y generated by the subspaces $\Gamma(u)$. Obviously, K_Γ can be identified with π^*TX (= the pull-back of TX with respect to π). We now define the curvature morphism $\Omega = \Omega_\Gamma$: $\Lambda^2 K_\Gamma \to T(Y/X)$ as the intrinsic exterior differential of the connection morphism. Using (3), one verifies that our definition is equivalent to that one by LIBERMANN, [6], as well as to the definition of the curvature of an arbitrary Pfaff system by PRADINES, [8], and to the construction of the difference tensor of the prolonged section Γ' by DEKRÉT, [1]. In coordinates,

$$\Omega \equiv (\partial_j F_i^\alpha + F_j^\beta \partial_\beta F_i^\alpha)\, dx^i \wedge dx^j. \tag{6}$$

Taking into account the projection $h_\Gamma: TY \to K_\Gamma$, $h_\Gamma(A) = A - \omega_\Gamma(A)$, we can also consider Ω as a morphism (denoted by the same symbol of $\Lambda^2 TY$ into $T(Y/X)$.

Let $p: E \to B$ be a vector bundle, the standard fiber of which is a vector space E_0. The space PE of all linear isomorphisms of E_0 into the individual fibers of E is a principal fiber bundle over B with structure group $GL(E_0)$. A parallelism on E is a section $Q: B \to PE$. Every $A \in E_0$ defines a section $\tilde{A}: B \to E$, $\tilde{A}(x) = Q(x)(A)$, which will be called *fundamental Q-section*. If A_i is a basis of E_0, then Q is determined by the fundamental Q-sections $\tilde{A}_i$. Any map $f: M \to E$ is transformed by Q into a map $f_Q: M \to E_0$, $f_Q(a) = Q^{-1}\big(pf(a)\big)\big(f(a)\big)$, $a \in M$.

A vertical parallelism on a fibered manifold Y is a parallelism on the vertical tangent bundle $T(Y/X)$. Denote by V the standard fiber of $T(Y/X)$, i.e. V is an m-dimensional vector space. If Q is a vertical parallelism on Y, then ω_Q and Ω_Q are V-valued forms. Given a principal fiber bundle $P(X, G)$, we have a canonical vertical parallelism N on P such that the fundamental N-section determined by $A \in \mathfrak{g}$ (= the Lie algebra of G) is the classical fundamental vector field on P determined by A. If Γ is a principal connection on P, then $\omega_N: TP \to \mathfrak{g}$ is the classical connection form of Γ.

Let W be a vector space and φ a W-valued k-form on Y. The absolute differential $D\varphi = D_\Gamma\varphi$ of φ with respect to a connection $\Gamma: Y \to J^1Y$ is a W-valued $(k+1)$-form on Y defined by

$$(D\varphi)(A_1, \ldots, A_{k+1}) = (d\varphi)(h_\Gamma A_1, \ldots, h_\Gamma A_{k+1}).$$

Proposition 1. *For any vertical parallelism Q on Y, it holds*

$$\Omega_Q = D\omega_Q.$$

Proof. Let Q be determined by vector fields $a^\beta_\alpha(x, y)\frac{\partial}{\partial y^\beta}$. Then $\omega_Q \equiv \bar{a}^\alpha_\beta(dy^\beta - F^\beta_i dx^i)$, provided $a^\alpha_\beta \bar{a}^\beta_\gamma = \delta^\alpha_\gamma$. We find directly $D\omega_Q \equiv \bar{a}^\alpha_\beta(\partial_j F^\beta_i + F^\gamma_j \partial_\gamma F^\beta_i)\, dx^i \wedge dx^j = \Omega_Q$, QED.

3. To deduce the structure equations of ω_Q, we need

Lemma 1. *Let η be a vertical vector field on Y and ξ a vector field in K_Γ. Then the vector $\omega_\Gamma([\eta, \xi]_u) \in T_u(Y/X)$ depends only on the value ξ_u of ξ at $u \in Y$.*

Proof. If $\eta \equiv \eta^\alpha(x, y)\frac{\partial}{\partial y^\alpha}$ and $\xi \equiv \xi^i(x, y)\frac{\partial}{\partial x^i} + F^\alpha_i \xi^i \frac{\partial}{\partial y^\alpha}$, then

$$\omega_\Gamma([\eta, \xi]) \equiv \xi^i\left(\eta^\beta \frac{\partial F^\alpha_i}{\partial y^\beta} - F^\beta_i \frac{\partial \eta^\alpha}{\partial y^\beta} - \frac{\partial \eta^\alpha}{\partial x^i}\right)\frac{\partial}{\partial y^\alpha}, \tag{7}$$

which proves Lemma 1.

We construct a mapping $\delta(Q) = \delta(Q, \Gamma)\colon V \times K_\Gamma \to V$ as follows. Let $A \in V$, $\xi_u \in \Gamma(u)$ and ξ be a vector field in K_Γ extending ξ_u. By Lemma 1, $\omega_\Gamma([\tilde{A}, \xi]_u)$ depends only on ξ_u and there is a unique vector $\delta(Q)\,(A, \xi_u) \in V$ such that $\omega_\Gamma([A, \xi]_u)$ belongs to the fundamental Q-section determined by $\delta(Q)\,(A, \xi_u)$. Using (7), we deduce the coordinate form of $\delta(Q)$

$$\delta(Q, \Gamma) \equiv a^\gamma_\beta(\partial_i \bar{a}^\alpha_\gamma + F^\varepsilon_i\, \partial_\varepsilon \bar{a}^\alpha_\gamma + \bar{a}^\alpha_\varepsilon\, \partial_\gamma F^\varepsilon_i). \tag{8}$$

Since $\delta(Q, \Gamma)$ is bilinear, it can be considered as a $V \otimes V^*$-valued 1-form on K_Γ, which will be called the *deviation form* of the pair (Q, Γ). By construction, $\delta(Q, \Gamma)$ vanishes iff $[\tilde{A}, \xi]$ belongs to K_Γ for any $A \in V$ and any vector field ξ in K_Γ. This means that Γ (as a distribution on Y) is invariant with respect to every fundamental Q-field on Y. For the canonical vertical parallelism N on a principal fiber bundle, $\delta(N, \Gamma) = 0$ iff Γ is a principal connection. Taking into account h_Γ, we can also consider $\delta(Q)$ as a 1-form on TY.

On the other hand, for any $A_1, A_2 \in V$ and $u \in Y$, there is a unique vector $q(u, A_1, A_2) \in V$ such that $[\tilde{A}_1, \tilde{A}_2]_u$ belongs to the fundamental Q-section determined by $q(u, A_1, A_2)$. In coordinates, $q \equiv \bar{a}^\alpha_\delta a^\varepsilon_{[\beta}\, \partial_{|\varepsilon|} a^\delta_{\gamma]}$, provided the square bracket denotes antisymmetrization. Hence q is a mapping $q\colon Y \to V \otimes \Lambda^2 V^*$ called the *structure function* of Q. Consider further $\omega_Q\colon TY \to V$. Then the natural composition of q and ω_Q defines a V-valued 2-form $q(\omega_Q, \omega_Q)$ on Y. We recall that, for a W_1-valued r-form $\varphi \equiv f^a_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$ and a $W_2 \otimes W_1^*$-valued s-form $\psi \equiv g^z_{a i_1 \dots i_s} dx^{i_1} \wedge \dots \wedge dx^{i_s}$ (W_1 and W_2 being vector spaces), the exterior multiplication and the tensor contraction determine a W_2-valued $(r + s)$-form $\varphi \barwedge \psi$,

$$\varphi \barwedge \psi \equiv f^a_{i_1 \dots i_r} g^z_{a i_{r+1} \dots i_{r+s}} dx^{i_1} \wedge \dots \wedge dx^{i_{r+s}}. \tag{9}$$

By direct evaluation, we now deduce

Proposition 2 (Structure equations). *It is*

$$d\omega_Q = -q(\omega_Q, \omega_Q) - \omega_Q \barwedge \delta(Q) + \Omega_Q. \tag{10}$$

In the case of a principal connection Γ on P, we have $\delta(\Gamma, N) = 0$ and the structure function of N coincides with the Lie algebra multiplication in $\mathfrak{g} = V$. Then (10) are the classical structure equations of a principal connection.

Proposition 3 (Generalized Bianchi formula). *It is*

$$D\Omega_Q = \Omega_Q \barwedge \delta(Q). \tag{11}$$

Proof. As $q(\omega_Q, \omega_Q)$ is bilinear in ω_Q, we have $D\big(q(\omega_Q, \omega_Q)\big) = 0$. For similar reasons, $D\big(\omega_Q \barwedge \delta(Q)\big) = D\omega_Q \barwedge \delta(Q)$. Hence we obtain (11) by absolute differentiating (10), QED.

In particular, there are two simple cases in which $D\Omega_Q$ vanishes. If $\Omega_Q = 0$, we have the trivial case of an integrable connection. On the other hand, $\delta(Q, \Gamma) = 0$ means that Γ is invariant with respect to Q. The latter case gives a generalization of the classical Bianchi identity (the canonical parallelism N on a principal fiber bundle is a very special kind of a vertical parallelism, as the induced parallelism on each fiber is a group parallelism).

4. Let φ be a horizontal W-valued k-form on Y, i.e. $\varphi(A_1, \ldots, A_k) = 0$ whenever at least one of the vectors $A_1, \ldots, A_k$ is vertical. We find remarkable to deduce a formula for the second absolute differential $D^2\varphi$ of φ. Obviously, φ can be interpreted as a morphism $Y \to W \otimes \Lambda^k\pi^* T^*X$. We recall that, for any vector bundle $E \to X$ and any morphism ψ: $Y \to E$, the fiber differential $d_{Y/X}\psi$: $Y \to T^*(Y/X) \otimes \pi^*E$ is defined by differentiating ψ on each fiber of Y separately. In particular, we have $d_{Y/X}\varphi$: $Y \to T^*(Y/X) \otimes W \otimes \Lambda^k\pi^* T^*X$, while Ω: $\Lambda^2\pi^* TX \to T(Y/X)$. Similarly to (9), we obtain a well-defined product $\Omega \barwedge d_{Y/X}\varphi$: $Y \to W \otimes \Lambda^{k+2}\pi^* T^*X$, i.e. a horizontal W-valued $(k + 2)$-form on Y. By simple evaluation, we prove

Proposition 4. *It holds*

$$D^2\varphi = -\Omega \barwedge d_{Y/X}\varphi. \tag{12}$$

5. Consider further a special fibered manifold $H^1X \to X$ of all first order frames on X. Local coordinates x^i on X are prolonged into fiber coordinates x^i, x^i_j on H^1X. Let Γ be a generalized connection on H^1X with equations

$$dx^i_j = F^i_{jk}(x^l, x^l_m)\, dx^k. \tag{13}$$

There is a canonical $\mathbf{R}^n$-valued form Θ on H^1X, the absolute differential $D\Theta$ of which will be called the torsion form of Γ. Let N be the canonical vertical parallelism on H^1X.

Proposition 5 (Structure equations). *It holds*

$$d\Theta = \Theta \barwedge \omega_N + D\Theta. \tag{14}$$

Proof. The coordinate expression of Θ or ω_N is $\bar{x}^i_j dx^j$ or $\bar{x}^i_j(dx^j_k - F^j_{kl}dx^l)$ respectively, provided $x^i_j\bar{x}^j_k = \delta^i_k$. Proposition 5 is then proved by direct evaluation.

There is a canonical identification $J^1H^1X \approx \bar{H}^2X$ (= the bundle of all semi-holonomic 2-frames on X). The following assertion shows that the classical Kobayashi's result on principal connections on H^1X remains to be true even for generalized connections on H^1X.

Proposition 6. *The values of Γ: $H^1X \to J^1H^1X \approx \bar{H}^2X$ are holonomic 2-frames iff the torsion form of Γ vanishes.*

Proof. This follows from the structure equations and Proposition 5 of [5].

6. The space T^r_kY of all k-dimensional velocities of order r on fibered manifold π: $Y \to X$ is a fibered manifold $T^r_k\pi$: $T^r_kY \to T^r_kX$. Any connection Γ on Y is prolonged into a connection on T^r_kY as follows. For $A \in T_xX$ and $u \in Y_x$, denote by

$L(u, A)$ the Γ-lift of A at u, i.e. the vector in $\Gamma(u)$ over A. Hence L is a map of $Y \oplus TX$ (= the fiber product over X) into TY and it is prolonged into $T_k^rL: T_k^rY \oplus T_k^r(TX) \to T_k^r(TY)$, the latter fiber product being over T_k^rX. Let $\varkappa_X: T(T_k^rX) \to T_k^r(TX)$ and $\varkappa_Y: T(T_k^rY) \to T_k^r(TY)$ be the canonical diffeomorphisms, [4]. (For $r = k = 1$, $\varkappa_X$ or $\varkappa_Y$ is the canonical involution of TTX or TTY, respectively.) We define a mapping $\lambda: T_k^rY \oplus T(T_k^rX) \to T(T_k^rY)$ by

$$\lambda(U, S) = \varkappa_Y^{-1}\big(T_k^rL(U, \varkappa_X(S))\big), \tag{15}$$

$U \in T_k^rY$, $S \in T_u(T_k^rX)$, $u = T_k^r\pi(U)$. One verifies by induction with respect to r that $S \mapsto \lambda(U, S)$ is a linear map for every $U \in T_k^rY$. Hence λ determines a connection $T_k^r\Gamma: T_k^rY \to J^1T_k^rY$.

In the special case $k = r = 1$, we get a connection $T\Gamma$ on $TY \to TX$. We shall show that $T\Gamma$ can be used for a simple construction of the curvature morphism of Γ. Let (4) be the equations of Γ and let x^i, y^a, $\xi^i = dx^i$, $\eta^a = dy^a$ be the induced local coordinates on TY. We deduce by (15) that the equations of $T\Gamma$ are (4) and

$$d\eta^a = (\xi^j\,\partial_jF_i^a + \eta^\beta\,\partial_\beta F_i^a)dx^i + F_i^a d\xi^i. \tag{16}$$

The $T\Gamma$-lift of a vector $A = (x^i, \xi^i, dx^i, d\xi^i) \in TTX$ at $U = (x^i, y^a, \xi^i, \eta^a = F_i^a\xi^i) \in TY$ has additional coordinates $dy^a = F_i^a dx^i$ and

$$d\eta^a = (\partial_jF_i^a + F_j^\beta\partial_\beta F_i^a)\,\xi^j dx^i + F_i^a d\xi^i. \tag{17}$$

On the other hand, construct $T\Gamma$-lift of $\varkappa_XA$ at $(x^i, y^a, \xi^i = dx^i, \eta^a = F_i^a dx^i)$ and apply then $\varkappa_Y$. We get a vector at U with coordinates dx^i, $d\xi^i$, $dy^a = F_i^a dx^i$ and

$$d\eta^a = (\partial_jF_i^a + F_j^\beta\,\partial_\beta F_i^a)\,\xi^i dx^j + F_i^a d\xi^i. \tag{18}$$

Subtracting the second vector from the first one, we obtain a vector Z with coordinates $dx^i = 0$, $dy^a = 0$, $d\xi^i = 0$ and

$$d\eta^a = (\partial_jF_i^a + F_j^\beta\,\partial_\beta F_i^a)\,(\xi^j dx^i - \xi^i dx^j). \tag{19}$$

Since $dx^i = 0 = dy^a$, Z belongs to the tangent space $T_U(T_uY)$ of vector space T_uY, $u \equiv (x^i, y^a)$. Hence Z is canonically identified with an element of T_uY. As $d\xi^i = 0$, the latter vector belongs to $T_u(Y/X)$. After this identification, (19) is just the curvature morphism of Γ.

REFERENCES

[1] Dekrét, A.: On a horizontal structure in fibered manifolds, to appear.
[2] Tong Van Duc; Sur la géométrie différentielle des fibrés vectoriels, Kódai Math. Sem. Rep. 26 (1975), 349—408.
[3] Kolář, I.: On the torsion of spaces with connection, Czechoslovak Math. J. 21 (96) (1971), 124—136.
[4] Kolář, I.: On some operations with connections, Math. Nachr. 69 (1975), 297—306.
[5] Kolář, I.: Generalized G-structures and G-structures of higher order, Boll. Un. Mat. Ital. (4) 12, Suppl. fasc. 3 (1975), 245—256.
[6] Libermann, P.: Parallélismes, J. Diff. Geometry 8 (1973), 511—539.
[7] Ouzilou, R.: Expression symplectique des problèmes variationnels, Symp. Math. 14, Geom. simplett. Fis. mat., Teor. geom. Integr. Var. minim., Convegni 1973, p. 85—98.

[8] Pradines, J.: Suites exactes vectorielles doubles et connexions, C. R. Acad. Sci. Paris, Sér. A, 278 (1974), 1587–1590.
[9] Vilms, J.: Connections on tangent bundles, J. Diff. Geometry 1 (1967), 235–243.
[10] Vilms, J.: Curvature of nonlinear connections, Proc. Amer. Math. Soc. 19 (1968), 1125 bis 1129.

Manuskripteingang: 17. 5. 1978

VERFASSER:

Ivan Kolář, Institute of Mathematics of ČSAV, branch
Brno, ČSSR