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Finite order liftings in principal fibre bundles

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The concept of lifting, a covariant functor from a category of n-dimensional manifolds to a category of certain bundles over n-dimensional manifolds, is known as an important underlying notion for the theory of geometric objects. NIJENHUIS has used it in his definition of "natural bundles" and described its fundamental properties. In particular, he pointed out that the r-th order natural bundles coincide with fibre bundles associated to the bundles of holonomic r-frames [6]. Following NIJENHUIS' approach, SALVIOLI [7] has applied the notion of lifting to the definition of the Lie derivative of a field of geometric objects. The concept of lifting has also been used in the theory of generally invariant Lagrangian structures [4, 5].
The purpose of this note is to relate the finite order liftings in principal fibre bundles to the natural liftings defined by the bundles of holonomic r-frames. In studying these liftings we use the continuity condition due to CALABI [6] allowing us to prove that each principal fibre bundle obtained by a lifting of order r can be reduced to the bundle of r-frames. We have shown that this reduction procedure provides us a natural transformation of functors in the category of principal fibre bundles considered.

1. Liftings

All manifolds and maps considered in this work belong to the category C^∞. Our manifolds are supposed to be real, finite dimensional, Hausdorff manifolds with countably many components. If X is a manifold we denote by $\mathscr{D}_X$ the category defined by all local diffeomorphisms of X. The category $\mathscr{D}_n$, where $n \geqq 1$ is an integer, is defined by all n-dimensional manifolds and their diffeomorphisms.
If $\mathscr{A}$ is any category we write Ob $\mathscr{A}$ for the class of objects, and Mor $\mathscr{A}$ for the class of morphisms of $\mathscr{A}$.
Let P be a set. We denote by id_P the identity map of P. If Q is a subset of P and f a map defined on P then $f|_Q$ denotes the restriction of f to Q. The canonical injection of Q into P, $\mathrm{id}_P|_Q$, will be denoted by ι_Q. If P is a manifold and Q an open subset of P, then $\iota_Q \in \mathrm{Mor}\, \mathscr{D}_P$.
Let G be a Lie group, R^n the real, n-dimensional euclidean space, and consider the trivial principal G-bundle $(R^n \times G, \pi_{R^n}, R^n, G)$. The category defined by all local automorphisms of $(R^n \times G, \pi_{R^n}, R^n, G)$ will be denoted by $\mathscr{PB}_{R^n \times G}$. The category

whose objects are principal G-bundles over n-dimensional manifolds, and whose morphisms are G-isomorphisms of G-bundles, will be denoted by $\mathscr{PB}_n(G)$.

Definition 1. A covariant functor $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ assigning to each $X \in \mathrm{Ob}\, \mathscr{D}_n$ a principal fibre bundle $\tau X = (\tau_0 X, \pi_X, X, G) \in \mathrm{Ob}\, \mathscr{PB}_n(G)$ and to each $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$ a morphism $\tau\alpha = (\tau_0\alpha, \alpha, \mathrm{id}_G) \in \mathrm{Mor}\, \mathscr{PB}_n(G)$ is called a *lifting* to the group G if the following conditions hold:

I. $\tau R^n = (R^n \times G, \pi_{R^n}, R^n, G)$.

II. For every $X \in \mathrm{Ob}\, \mathscr{D}_n$ and every open subset U of X,

$$\tau_0 U = \pi_X^{-1}(U), \quad \pi_U = \pi_X|_{\tau_0 U}, \quad \tau_0 \iota_U = \iota_{\tau_0 U}.$$

Let (Y, π, X, G) be a principal G-bundle, let $(y, g) \to y \cdot g$ denote the right action of G on Y. We shall say that a system of pairs $((U_\varkappa, \varphi_\varkappa), \Phi_\varkappa)$, $\varkappa \in K$, is a *fibre atlas* on (Y, π, X, G) if

I. the system $(U_\varkappa, \varphi_\varkappa)$, $\varkappa \in K$, is an atlas on X,

II. for each $\varkappa \in K$, $\Phi_\varkappa$ is a diffeomorphism of $\pi^{-1}(U_\varkappa)$ onto $\varphi_\varkappa(U_\varkappa) \times G$ such that for every $y \in \pi^{-1}(U_\varkappa)$, $g \in G$,

$$\Phi_\varkappa(y) = \big(\varphi_\varkappa \pi(y), \bar{\Phi}_\varkappa(y)\big), \quad \bar{\Phi}_\varkappa(y \cdot g) = \bar{\Phi}_\varkappa(y) \cdot g.$$

Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting to the group G. The following two propositions are easy to prove.

Proposition 1. *Let $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$, $\alpha : X_1 \to X_2$, and let U be an open subset of X_1. Then*

$$\tau_0(\alpha|_U) = \tau_0\alpha|_{\tau_0 U}.$$

Proposition 2. *Let $X \in \mathrm{Ob}\, \mathscr{D}_n$, and let $(U_\varkappa, \varphi_\varkappa)$, $\varkappa \in K$, be an atlas on X. Then the pairs $((U_\varkappa, \varphi_\varkappa), \tau_0\varphi_\varkappa)$, $\varkappa \in K$, form a fibre atlas on $\tau X = (\tau_0 X, \pi_X, X, G)$. Moreover, if (V_ι, ψ_ι), $\iota \in I$, is an atlas on X, equivalent to the atlas $(U_\varkappa, \varphi_\varkappa)$, $\varkappa \in K$, then the fibre atlases $((U_\varkappa, \varphi_\varkappa), \tau_0\varphi_\varkappa)$, $\varkappa \in K$, and $((V_\iota, \psi_\iota), \tau_0\psi_\iota)$, $\iota \in I$, are equivalent.*

Denote by τ_{R^n} the restriction of τ to the subcategory $\mathscr{D}_{R^n}$ of $\mathscr{D}_n$. By definition, τ_{R^n} is a covariant functor from $\mathscr{D}_{R^n}$ to $\mathscr{PB}_{R^n \times G}$. Writing, for $U \in \mathrm{Ob}\, \mathscr{D}_n$ and $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$,

$$\tau_{R^n} U = (\tau_0 U, \pi_U, U, G), \quad \tau_{R^n}\alpha = (\tau_0\alpha, \alpha, \mathrm{id}_G)$$

we see that the functor τ_{R^n} obeys the following conditions:

I. $\tau_{R^n} R^n = (R^n \times G, \pi_{R^n}, R^n, G)$.

II. *For every $U \in \mathrm{Ob}\, \mathscr{D}_{R^n}$,*

$$\tau_0 U = \pi_{R^n}^{-1}(U), \quad \pi_U = \pi_{R^n}|_{\tau_0 U}, \quad \tau_0 \iota_U = \iota_{\tau_0 U}.$$

The next assertion gives us a method of constructing the liftings. One can prove it either using some general arguments [1, 3] or directly with the aid of transition functions [2].

Proposition 3. *Every covariant functor $\tau_{R^n} : \mathscr{D}_{R^n} \to \mathscr{PB}_{R^n \times G}$, $U \to (\tau_0 U, \pi_U, U, G)$, $\alpha \to (\tau_0\alpha, \alpha, \mathrm{id}_G)$, such that*

I. $\tau_{R^n} R^n = (R^n \times G, \pi_{R^n}, R^n, G)$.

II. *for every $U \in \mathrm{Ob}\, \mathscr{D}_{R^n}$,*

$$\tau_0 U = \pi_{R^n}^{-1}(U), \quad \pi_U = \pi_{R^n}|_{\tau_0 U}, \quad \tau_0 \iota_U = \iota_{\tau_0 U},$$

can be extended to a lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$.

2. Homogeneous liftings

Let G be a Lie group, e its identity element, $\tau : \mathscr{D}_n \to \mathscr{P}\mathscr{B}_n(G)$ a lifting to the group G. If $\alpha \in \operatorname{Mor} \mathscr{D}_{R^n}$, $\alpha : U \to R^n$, then $\tau_0\alpha$ is of the form

$$\tau_0\alpha(x, g) = \big(\alpha(x), \bar{\tau}\alpha(\alpha(x)) \cdot g\big), \tag{1}$$

where $(x, g) \in U \times G$, and $\bar{\tau}\alpha : \alpha(U) \to G$ is a map. If $\alpha_1, \alpha_2 \in \operatorname{Mor} \mathscr{D}_n$ are such that $\alpha_1\alpha_2$ is defined then it is easily seen that

$$\bar{\tau}\alpha_1\alpha_2\big(\alpha_1\alpha_2(x)\big) = \bar{\tau}\alpha_1\big(\alpha_1\alpha_2(x)\big) \cdot \bar{\tau}\alpha_2\big(\alpha_2(x)\big) \tag{2}$$

for every x from the domain of α_2. Obviously,

$$\bar{\tau}\ \mathrm{id}_{R^n}(x) = e. \tag{3}$$

Let $x_0 \in R^n$. Denote by $\mathscr{A}_{x_0}$ the class of all $\alpha \in \operatorname{Mor} \mathscr{D}_{R^n}$ defined at x_0 and such that $\alpha(x_0) = x_0$, and by G_{x_0} the set of all $g \in G$ which can be expressed as $\bar{\tau}\alpha(x_0)$, for some $\alpha \in \mathscr{A}_{x_0}$. The set G_{x_0} is a subgroup of G since for every $g_1, g_2 \in G_{x_0}$, $g_1 = \bar{\tau}\alpha_1(x_0)$, $g_2 = \bar{\tau}\alpha_2(x_0)$, the relation $g_1 \cdot g_2 = \bar{\tau}\alpha_1\alpha_2(x_0)$ holds independently of the representatives α_1, α_2 of g_1, g_2 respectively.
Let now $x_1, x_2 \in R^n$. Each $\beta \in \operatorname{Mor} \mathscr{D}_{R^n}$ such that $\beta(x_1) = x_2$ defines an isomorphism of the groups G_{x_1}, G_{x_2}. This is constructed as follows. Let $\alpha \in \mathscr{A}_{x_1}$. Then the composed map $\beta\alpha\beta^{-1}$ belongs to $\mathscr{A}_{x_2}$. Since $\bar{\tau}\beta\alpha\beta^{-1}(x_2) = \bar{\tau}\beta(x_2) \cdot \bar{\tau}\alpha(x_1) \cdot \bar{\tau}\beta^{-1}(x_1)$, $\bar{\tau}\beta^{-1}(x_1) = \big(\bar{\tau}\beta(x_2)\big)^{-1}$, we get

$$\bar{\tau}\beta\alpha\beta^{-1}(x_2) = \bar{\tau}\beta(x_2) \cdot \bar{\tau}\alpha(x_1) \cdot \big(\bar{\tau}\beta(x_2)\big)^{-1}. \tag{4}$$

The required isomorphism is given by

$$G_{x_1} \ni g \to \bar{\tau}\beta(x_2) \cdot g \cdot \big(\bar{\tau}\beta(x_2)\big)^{-1} \in G_{x_2}. \tag{5}$$

Proposition 4. *The following conditions are equivalent:*

I. *To every $x_1, x_2 \in R^n$ there exists $\beta \in \operatorname{Mor} \mathscr{D}_{R^n}$ sending x_1 to x_2 such that*

$$\tau_0\beta(x_1, e) = (x_2, e)$$

or, which is the same, such that $\bar{\tau}\beta(x_2) = e$.

II. *There exists a subgroup G_0 of G such that $G_x = G_0$ for every $x \in R^n$, and for every $\alpha \in \operatorname{Mor} \mathscr{D}_{R^n}$, $\bar{\tau}\alpha$ maps the domain of α to G_0.*

Proof. Choose x_1, x_2 and β satisfying the first condition. Then, by (1), $\bar{\tau}\beta(x_2) = e$, and the map (5) becomes the identity map. This proves that $G_{x_1} = G_{x_2}$. Let $\alpha \in \operatorname{Mor} \mathscr{D}_{R^n}$, $\alpha(x_1) = x_2$. Then $\beta^{-1}\alpha \in \mathscr{A}_{x_1}$ which shows that $\bar{\tau}\beta^{-1}\alpha(x_1) \in G_{x_1}$. But, according to (2) and (3),

$$\bar{\tau}\beta^{-1}\alpha(x_1) = \bar{\tau}\beta^{-1}(x_1) \cdot \bar{\tau}\alpha(x_2) = \big(\bar{\tau}\beta(x_2)\big)^{-1} \cdot \bar{\tau}\alpha(x_2) = \bar{\tau}\alpha(x_2)$$

which shows that the second condition must hold. Conversely, let us assume that II is satisfied. Choose $x_1, x_2 \in R^n$ and $\gamma \in \operatorname{Mor} \mathscr{D}_{R^n}$ sending x_1 to x_2. We have, by (2), $\bar{\tau}\beta(x_2) = e$ which proves that the map β satisfies the first condition. This proves Proposition 4.

Definition 2. A lifting $\tau : \mathscr{D}_n \to \mathscr{P}\mathscr{B}_n(G)$ satisfying the equivalent conditions of Proposition 4, is called a *homogeneous lifting*.

3. Finite order liftings

Many liftings which have been used in geometry and the calculus of variations in tensor bundles are, in a sense, of finite order. We define these liftings similarly as NIJENHUIS [6], our underlying spaces being, however, the principal fibre bundles.
The r-jet of a map f at a point x will be denoted by $j_x^r f$. The composition of jets will be denoted by $*$.
Let G be a Lie group, $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ a lifting to the group G, assigning to $X \in \mathrm{Ob}\, \mathscr{D}_n$ a principal fibre bundle $(\tau_0 X, \pi_X, X, G) \in \mathrm{Ob}\, \mathscr{PB}_n(G)$, and to each $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$ a morphism $(\tau_0\alpha, \alpha, \mathrm{id}_G) \in \mathrm{Mor}\, \mathscr{PB}_n(G)$. For a point $x \in X$, denote

$$\tau_0\alpha|_x = \tau_0\alpha|_{\pi_X^{-1}(x)}.$$

Definition 3. τ is said to be *of order* r, where $r \geq 0$ is an integer, if for every $X \in \mathrm{Ob}\, \mathscr{D}_n$, every $x \in X$, and every $\alpha_1, \alpha_2 \in \mathrm{Mor}\, \mathscr{D}_n$ defined at x, the condition

$$\tau_0\alpha_1|_x = \tau_0\alpha_2|_x$$

holds if and only if $j_x^r\alpha_1 = j_x^r\alpha_2$.

Let L_n^r be the Lie group of all r-jets of local diffeomorphisms of R^n with source and target $0 \in R^n$, the group operation being the composition of jets. Let $X \in \mathrm{Ob}\, \mathscr{D}_n$, and denote by $\mathscr{F}^r X = (\mathscr{F}_0^r X, \pi_{X,r}, X, L_n^r)$ the principal fibre bundle of holonomic r-frames on X. If $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$, $\alpha : X_1 \to X_2$, then there is defined an isomorphism $\mathscr{F}^r\alpha = (\mathscr{F}_0^r\alpha, \alpha, \mathrm{id}_{L_n^r})$ of $\mathscr{F}^r X_1$ to $\mathscr{F}^r X_2$ by

$$\mathscr{F}_0^r\alpha(j_0^r f) = j_0^r \alpha f = j_0^r\alpha * j_0^r f. \tag{6}$$

The correspondence $X \to \mathscr{F}^r X$, $\alpha \to \mathscr{F}^r\alpha$ is a lifting to the group L_n^r which is obviously homogeneous, and of order r.

Definition 4. Let $X_1, X_2 \in \mathrm{Ob}\, \mathscr{D}_n$, $x_1 \in X_1$, $x_2 \in X_2$. A map $\sigma : \pi_{X_1}^{-1}(x_1) \to \pi_{X_2}^{-1}(x_2)$ is *τ-admissible* if there exists $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$ such that

$$\sigma = \tau_0\alpha|_{x_1}.$$

Let t_x be the translation of R^n sending $x \in R^n$ to the origin $0 \in R^n$, let I be the identity map of R^n. Obviously,

$$t_{x_1+x_2} = t_{x_1} t_{x_2} \tag{7}$$

and for each $x_0 \in R^n$, $j_{x_0}^r t_x = (x_0, x - x_0, I, 0, \ldots, 0)$.
Let $x_1, x_2 \in R^n$. Denote by $\mathscr{F}^r(x_1, x_2)$ the set of all $\mathscr{F}^r$-admissible maps from $\pi_{R^n,r}^{-1}(x_1)$ to $\pi_{R^n,r}^{-1}(x_2)$. If $\sigma \in \mathscr{F}^r(x_1, x_2)$, $\sigma = \mathscr{F}_0^r\alpha|_{\pi_{R^n,r}^{-1}(x_1)}$, then we are given, by (6), the r-jet of α at x_1, $j_{x_1}^r\alpha$, and a τ-admissible map $\eta : \pi_{R^n}^{-1}(x_1) \to \pi_{R^n}^{-1}(x_2)$ by

$$\eta = \tau_0\alpha|_{x_1}.$$

If $\tau(x_1, x_2)$ denotes the set of all τ-admissible maps of $\pi_{R^n}^{-1}(x_1)$ to $\pi_{R^n}^{-1}(x_2)$ then we have constructed a map of $\mathscr{F}^r(x_1, x_2)$ to $\tau(x_1, x_2)$ which is obviously a bijection. The relation

$$\mathscr{F}_0^r\alpha(x_1, \bar{e}) = (x_2, j_0^r f),$$

where $\bar{e} \in L_n^r$ denotes the identity, defines a one-to-one correspondence between $\mathscr{F}^r(x_1, x_2)$ and L_n^r, and similarly the relation

$$\tau_0\alpha(x_1, e) = \big(x_2, \bar{\tau}\alpha(x_2)\big),$$

where $e \in G$ is the identity, defines an injection of $\tau(x_1, x_2)$ to G. Combining these two maps we obtain an injection

$$L_n^r \ni j_0^r f \to \nu_{x_1,x_2}(j_0^r f) = \bar{\tau}(t_{-x_2} f t_{x_1})\,(x_2) \in G.$$

Since

$$\mathscr{F}_0^r t_{x_1-x_2}(x_1, \bar{e}) = (x_2, e) = \mathscr{F}_0^r \,\mathrm{id}_{R^n}(x_2, e)$$

we have

$$\nu_{x_1,x_2}(\bar{e}) = \nu_{x_2,x_2}(\bar{e}) = e$$

i.e., by (3) and (7),

$$\bar{\tau}(t_{x_1} t_{-x_2})\,(x_2) = \bar{\tau} t_{x_1-x_2}(x_2) = e. \tag{8}$$

This relation together with (4) shows that the map ν_{x_1,x_2} is independent of the choice of x_1, x_2. We denote

$$\nu = \nu_{x_1,x_2}. \tag{9}$$

ν is an injective homomorphism of groups.

Proposition 5. *Each lifting of order r is homogeneous.*

Proof. Our assertion follows from (8).

4. Differentiability condition

Let $\tau : \mathscr{D}_n \to \mathscr{P}\mathscr{B}_n(G)$ be a lifting of order r. Using the same notation as before we define:

Definition 5. τ is said to satisfy the *differentiability condition* if for every $X \in \mathrm{Ob}\,\mathscr{D}_n$ and every differentiable map $J \times U \ni (t, x) \to \alpha(t, x) \in X$, where J is an open interval and U is an open subset of X, such that for each $t \in J$ the map α_t, defined by $\alpha_t(x) = \alpha(t, x)$, belongs to $\mathrm{Mor}\,\mathscr{D}_X$, the map

$$J \times \tau_0 U \ni (t, y) \to \tau_0 \alpha_t(y) \in \tau_0 X$$

is differentiable.

Theorem 1. *The following two conditions are equivalent:*

I. *τ satisfies the differentiability condition.*

II. *The map (9) is continuous.*

Proof. Assume that τ satisfies the differentiability condition, and choose on open interval J containing $0 \in R^1$ and a point $S \in L_n^r$, $S = j_0^r f$. Let $J \ni t \to \psi(t) \in L_n^r$ be any differentiable curve such that $\psi(0) = S$. There exist a neighbourhood U of $0 \in R^n$ and a differentiable map $J \times U \ni (t, x) \to \alpha(t, x) \in R^n$ such that for every $t \in J$, the map α_t belongs to $\mathrm{Mor}\,\mathscr{D}_{R^n}$, $\alpha_t(0) = 0$, and $\psi(t) = j_0^r \alpha_t$. The map α may easily be constructed by means of polynomials. Now, by (9), $\nu\psi(t) = \bar{\tau}\alpha_t(0)$, and our assumption ensures that the curve $t \to \bar{\tau}\alpha_t(0)$ is differentiable. Thus we see that for any differentiable curve $t \to \psi(t)$ passing through S, the curve $t \to \nu\psi(t)$ in G is differentiable. This means that ν is differentiable at S, hence continuous.

Conversely, let us assume that ν is continuous. Then ν is differentiable ([8], Chap. 5, Sec. 3). Consider a differentiable map $J \times U \ni (t, x) \to \alpha(t, x) = \alpha_t(x) \in R^n$ such that $\alpha_t \in \operatorname{Mor} \mathscr{D}_{R^n}$. Then by (1)

$$\tau_0\alpha_t(x, g) = \big(\alpha_t(x), \bar{\tau}\alpha_t(\alpha_t(x)) \cdot g\big), \tag{10}$$

where for each $t \in J$ the map $\bar{\tau}\alpha_t : \alpha_t(U) \to G$ is differentiable. Let $x \in U$. Since the map $J \ni t \to j_0^r(t_{\alpha_t(x)}\alpha_t t_{-x}) \in L_n^r$ is differentiable the composed map

$$J \ni t \to \nu\big(j_0^r(t_{\alpha_t(x)}\alpha_t t_{-x})\big) = \bar{\tau}\alpha_t\big(\alpha_t(x)\big) \in G$$

must be differentiable. Formula (10) now shows that the map $\big(t, (x, g)\big) \to \tau_0\alpha_t(x, g)$ is differentiable. Consequently, τ satisfies the differentiability condition. This completes the proof.

If τ satisfies the differentiability condition then the pair (ν, L_n^r) is a Lie subgroup of G. This leads to the following result.

Theorem 2. *Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting of order r satisfying the differentiability condition. Then there exists a natural transformation $N : \mathscr{F}^r \to \tau$ of functors such that for each $X \in \operatorname{Ob} \mathscr{D}_n$, $N_X : \mathscr{F}^r X \to \tau X$ is a reduction of the principal fibre bundle τX to the principal fibre bundle $\mathscr{F}^r X$.*

Proof. Let $X \in \operatorname{Ob} \mathscr{D}_n$, let (U_ι, φ_ι), $\iota \in I$, be an atlas on X. Denote by $\big((U_\iota, \varphi_\iota), \tau_0\varphi_\iota\big)$, $\iota \in I$, and $\big((U_\iota, \varphi_\iota), \mathscr{F}^r\varphi_\iota\big)$, $\iota \in I$, the corresponding fibre atlases on τX and $\mathscr{F}^r X$, respectively. For each $\iota \in I$ there is defined a map $N_{X,\iota} : \pi_{X,r}^{-1}(U_\iota) \to \pi_X^{-1}(U_\iota)$ by

$$N_{X,\iota} = (\tau_0\varphi_\iota)^{-1} \circ (\mathrm{id}_{\varphi_\iota(U_\iota)} \times \nu) \circ \mathscr{F}_0^r\varphi_\iota. \tag{11}$$

It is directly proved that on every $\pi_{X,r}^{-1}(U_\iota \cap U_\varkappa)$

$$N_{X,\iota} = N_{X,\varkappa}. \tag{12}$$

Denote by $N_X^{(0)}$ the map of $\mathscr{F}_0^r X$ to $\tau_0 X$ uniquely determined by the condition

$$N_X^{(0)}|_{\pi_{X,r}^{-1}(U_\iota)} = N_{X,\iota}$$

for every $\iota \in I$. It is easily seen that $N_X^{(0)}$ satisfies locally, and consequently globally, the condition

$$N_X^{(0)}(j_0^r f * j_0^r\varphi) = N_X^{(0)}(j_0^r f) \cdot \nu(j_0^r\varphi) \tag{13}$$

for every $j_0^r f \in \pi_{X,r}^{-1}(U_\iota)$ and every $j_0^r\varphi \in L_n^r$. Hence the triple $N_X = (N_X^{(0)}, \mathrm{id}_X, \nu)$ is a homomorphism of principal fibre bundles. Formula (11) shows that it is an isomorphism, which means that N_X is a reduction of τX to $\mathscr{F}^r X$.
To prove that the family N_X, $X \in \operatorname{Ob} \mathscr{D}_n$, defines a natural transformation of functors, we are to show that for every $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\alpha : X_1 \to X_2$,

$$\tau\alpha \circ N_{X_1} = N_{X_2} \circ \mathscr{F}^r\alpha. \tag{14}$$

Let (U, φ) be a chart on X_1, (V, ψ) a chart on X_2 such that $\alpha(U) \subset V$. Obviously,

$$\tau_0\alpha \circ N_{X_1}|_{\pi_{X_1}^{-1}(U)} = N_{X_2} \circ \mathscr{F}_0^r\alpha|_{\pi_{X_1,r}^{-1}(U)}$$

if and only if

$$\tau_0\psi \circ \tau_0\alpha \circ N_{X_1}|_{\pi_{X_1,r}^{-1}(U)} \circ (\tau_0\varphi)^{-1} = \tau_0\psi \circ N_{X_2} \circ \mathscr{F}_0^r\alpha|_{\pi_{X_1,r}^{-1}(U)} \circ (\tau_0\varphi)^{-1}.$$

The last condition is equivalent to saying that

$$\tau_0 \psi \alpha \varphi^{-1} \circ (\mathrm{id}_{\varphi(U)} \times \nu) \circ \mathscr{F}^r_U \varphi \alpha \psi^{-1} = \mathrm{id}_{\psi(V)} \times \nu .$$

It follows from (12) that this condition is satisfied which proves (14). This completes the proof of Theorem 2.

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