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Remark on finitly projected modular lattices of breadth two

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To Prof. O.-H. Keller on his 75th birthday

1. Introduction

A lattice L in $\boldsymbol{K}$ is called *finitly* $\boldsymbol{K}$*-projected* if for any surjective $f\colon k \twoheadrightarrow L$ in $\boldsymbol{K}$ there is a finite sublattice of K whose image under f is L. These lattices are important by the investigation of subvarieties of K, namely every finitly $\boldsymbol{K}$-projected subdirectly irreducible lattice L is splitting in $\boldsymbol{K}$, i.e. there is a largest subvariety of K not containing L (see A. DAY [1]). So it is important to characterize finitly projected lattices and splitting lattices in the variety $\boldsymbol{M}$ (i.e. the variety of all modular lattices) and (in particular) in the modular lattices of breadth two. Our goal here is to give some characterizations for a lattice of breadth two to be $\boldsymbol{M}$-projected and $\boldsymbol{M}$-splitting.

2. Preliminaries

First we introduce some concepts.
We call an ordered five-tuple (σ, x, y, z, μ) of elements from a modular lattice a *diamond* if these elements form a copy of M_3 with σ and μ as the bottom and the top elements, respectively. If $a \mid b$ and $c \mid d$ are quotients in a lattice we write $a \mid b \nearrow c \mid d$ and we say that $a \mid b$ *transposes up* to $c \mid d$ if $a \wedge d = b$ and $a \vee d = c$. In this case we also say that $c \mid d$ *transposes down* to $a \mid b$, written $c \mid d \searrow a \mid b$. We also say that $a \mid b$ and $c \mid d$ are *transposes*. The quotients $a \mid b$, $c \mid d$ are said to be *projective* (in symbol $a \mid b \approx c \mid d$) if there exists a sequence of quotients $a \mid b = a_0 \mid b_0, a_1 \mid b_1, \ldots, a_n \mid b_n = c \mid d$ such that $a_k \mid b_k$ and $a_{k+1} \mid b_{k+1}$ are transposes for every $k = 0, 1, \ldots, n-1$.
A sublattice K of L is called an *isometric sublattice* if a prime quotient in K is a prime quotient in L.

Definition 1.1 (A. MITSCHKE, E. T. SCHMIDT, R. WILLE [3]).

(I) The diamond $D_1 = (\sigma_1, x_1, y_1, z_1, \mu_1)$ is said to be *translate up* to the diamond $D_2 = (\sigma_2, x_2, y_2, z_2, \mu_2)$ if one of the quotients $\mu_1 \mid x_1$, $\mu_1 \mid y_1$, $\mu_1 \mid z_1$ transposes up to one of the quotients $x_2 \mid \sigma_2$, $y_2 \mid \sigma_2$, $z_2 \mid \sigma_2$ and we write $D_1 \nearrow D_2$.
In this situation we also say that D_2 *translate down* to D_1, written $D_2 \searrow D_1$.

(II) A sequence $D_0, D_1, \ldots, D_{n-1}$ is called a *diamond circle* if the followings are satisfied:
  (i) for every $i = 1, 2, \ldots, n-1$ D_i translate up or translate down to D_{i+1},
  (ii) $D_0 \nearrow D_1$ and $D_{n-1} \searrow D_0$ such that $\mu_0 \mid x_0$ transposes up to one of the quotients $x_1 \mid \sigma_1$, $y_1 \mid \sigma_1$, $z_1 \mid \sigma_1$ and $\mu_0 \mid z_0$ translate up to one of the quotients $x_{n-1} \mid \sigma_{n-1}$, $y_{n-1} \mid \sigma_{n-1}$, $z_{n-1} \mid \sigma_{n-1}$.

Definition 1.2.

(I) We say that $a \mid b$ *transposes up* to the diamond $D = (\sigma, x, y, z, \mu)$ if $a \mid b$ transposes up to one of the quotients $x \mid \sigma, y \mid \sigma, z \mid \sigma$, written $a \mid b \nearrow D$ or $D \searrow a \mid b$. And we write $a \mid b \searrow D$ dually, if $a \mid b$ transposes down to one of the quotients $\mu \mid x, \mu \mid y, \mu \mid z$.

(II) Let T_i be a diamond or a quotient for $i \in I$. $T_1 \times T_2$ denotes that no $T_1 \nearrow T_2$ and no $T_1 \searrow T_2$ are satisfied. A sequence $T_1, T_2, \ldots, T_m$ is called "clear" if for $i = 1, 2, \ldots, m-1$

a) $T_1 \nearrow T_{i+1}$ or $T_i \searrow T_{i+1}$,

b) for T_i, T_{i+1}, T_{i+2} where T_{i+1} is a diamond $T_i \times T_{i+2}$ must be satisfied.

(III) A sequence $D_1, D_2, \ldots, D_m$ is called a *diamond halfcircle* if there exist $a \mid b$, $c \mid d$ and $a \mid b \times c \mid d$ such that

a) either

$a \mid b \nearrow D_1$ and $D_m \searrow c \mid d$

or $a \mid b \searrow D_1$ and $D_m \nearrow c \mid d$,

b) $a \mid b, D_1, D_2, \ldots, D_m, c \mid d$ form a clear sequence.

Remark 1. From sequence $T_1, T_2, \ldots, T_n$ of satisfying to that either $T_i \nearrow T_{i+1}$ or $T_i \searrow T_{i+1}$ we can choose a subsequence $T_{i_1} = T_1, T_{i_2}, \ldots, T_{i_m} = T_n$ to be clear.

2. From diamond circle $D_1, \ldots, D_n$ we can choose a diamond circle to be clear. A lattice L has a diamond circle if and only if L has a clear diamond circle.

Example:

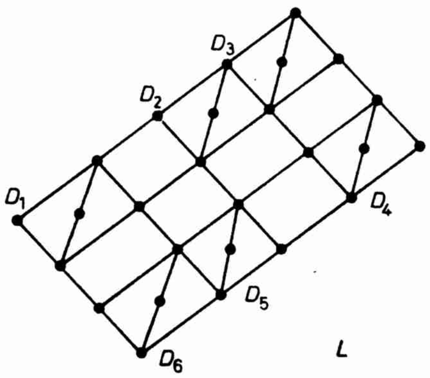


Fig. 1

The sequence D_1, D_2, D_3, D_4, D_6 in the lattice L is a diamond circle, but it isn't clear. The diamond circle D_1, D_3, D_4, D_6 is clear. However the sequence D_1, D_2, D_3, D_4, D_5 isn't a diamond circle.

Beside we introduce a useful so-called Hall-Dilworth construction. The generalization of this construction is found in [2].

Definition 1.3 (Hall-Dilworth construction). Let L_1 and L_2 be two modular lattices with isomorphic sublattice $C \cong C'$ where C is a filter of L_1 and C' is an ideal of L_2. Then $L = L_1 \cup L_2$ can be made into modular lattice by defining $x \leq y$ if and only if one of the following conditions is satisfied: $x \leq y$ in L_1 or $x \leq y$ in L_2 or $x \leq c$ in L_1 and $c' \leq y$ in L_2 where c, c' are corresponding elements under the isomorphism $C \cong C'$. We say that L is the *lattice obtained by gluing together* L_1 and L_2 identifying the corresponding elements under the isomorphism $C \cong C'$, and we write

$L = L_1 + L_2$ (C, Hall-Dilworth) or $L = L_1 + L_2$ ($L_1 \cap L_2$, Hall-Dilworth), or $L = L_1 + L_2$ (Hall-Dilworth).

Now we can enumerate results.

Theorem 1.1 (A. MITSCHKE, E. T. SCHMIDT, R. WILLE [3]).

a) *Let (σ, x, y, z, μ) be an isometric diamond of finitly M-projected lattice L such that $a \wedge b \leqq y$ $(a, b \nleqq y)$ implies that $a \wedge b \leqq \sigma$. Then $L' = L \setminus y^*$ is a sublattice of L, where $y^* = \{t \mid t \vee \sigma = y\}$, and the quotients $x \mid \sigma$ and $z \mid \sigma$ are not projective in the sublattice L'.*

b) *Let L be a finite modular lattice of breadth two. If L is finitly M-projected, then L doesn't contain a diamond circle and a sublattice isomorphic to M_4.*

Definition 1.4. We say that lattice L has *γ-property*, if L doesn't contain a diamond circle and a sublattice isomorphic to M_4.

Theorem 1.2 (E. T. SCHMIDT [4]).

a) *Let (σ, x, y, z, μ) be an isometric diamond of a splitting modular lattice L. If y is double-irreducible then the quotients $x \mid \sigma$ and $z \mid \sigma$ are not projected in the sublattice $L_y = L \setminus \{y\}$.*

b) *A finite subdirectly irreducible planar modular lattice L is splitting modular if and only if L has γ-property.*

For the Hall-Dilworth construction E. T. SCHMIDT have given two interesting necessary conditions for a lattice to be M-projected.

Theorem 1.3 (E. T. SCHMIDT [5]). *Let $M = L_1 + L_2$ ($C = L_1 \cap L_2$, Hall-Dilworth) be a finite modular lattice where C is a chain. If C has such two different prime quotients $a \mid b$ and $c \mid d$ which are projective in L_1 and L_2, then M isn't finitly M-projected.*

Theorem 1.4 (E. T. SCHMIDT [5]). *Let $M = L_1 + L_2$ ($C = L_1 \cap L_2$, Hall-Dilworth) be a finite modular lattice where C is a Boolean lattice. If C has such two prime quotients $a \mid b$ and $c \mid d$ which are projective in L_1 and L_2 but no in C, then M isn't finitly M-projected.*

Remark. 1. The condition "two different prim quotients $a \mid b$ and $c \mid d$" in Theorem 1.3 exactly means that $a \mid b$ and $c \mid d$ are not projective in C.

2. The lattice C in above theorems is either a chain, or a Boolean lattice.

Generally in the class of all modular lattices for the Hall-Dilworth construction we only know Schmidt's Theorems 1.3 and 1.4. There is a question what is the fact in "smaller" class of modular lattices. To answer this question we give related theorems for modular lattices of breadth two.

3. Results

First of all we prove such theorems for modular lattices of breadth two which are interesting in itself too.

Theorem 3.1. *Let L be a finite modular lattice of breadth two, and let $a_1 \mid b_1$, $p \mid q$, $a_2 \mid b_2$ be prime quotients of L such that $a_1 \mid b_1 \nearrow p \mid q \searrow a_2 \mid b_2$ (or $a_1 \mid b_1 \searrow p \mid q \nearrow a_2 \mid b_2$, respectively) and $a_1 \mid b_1 \not\sim a_2 \mid b_2$. Then there exists a diamond $D = (\sigma, x, y, z, \mu)$ for which $a_1 \mid b_1 \nearrow D \searrow a_2 \mid b_2$ such that $a_1 \mid b_1 \nearrow x \mid \sigma$, $z \mid \sigma \searrow a_2 \mid \upsilon$ (or $a_1 \mid b_1 \searrow D \nearrow a_2 \mid b_2$ such that $a_1 \mid b_1 \searrow \mu \mid x$, $\mu \mid z \nearrow a_2 \mid b_2$, respectively).*

Proof. We assume that $a_1 \mid b_1 \nearrow p \mid q \searrow a_2 \mid b_2$ and $a_1 \mid b_1 \times a_2 \mid b_2$. Now let us consider elements x' and z' such that $a_1 \leqq x' \prec P$, $a_2 \leqq z' \prec P$. If $x' \equiv z'$ then $a_1 \mid b_1 \nearrow x' \mid z' \wedge q \searrow a_2 \mid b_2$. In this case if $p' :=: x' = z'$ and $q' :=: x' \wedge q$ then we can consider again elements x'' and z'' such that $a_1 \leqq x'' \prec P'$, $a_2 \leqq z'' \prec P''$ and so on. Since $a_1 \mid b_1 \times a_2 \mid b_2$, so after all by step n we can get elements $\mu :=: x^{(n)}$ and $y :=: x^{(n)} \wedge q$ such that $a_1 \mid b_1 \nearrow \mu \mid y \searrow a_2 \mid b_2$, and if $a_1 \leqq x \prec \mu$ and $a_2 \leqq z \prec \mu$ then $x \neq z$. So x, y, z are different elements. Since L is of breadth two, so if $\sigma := : x \wedge z = x \wedge y = y \wedge z$, then σ, x, y, z, μ form a diamond such that $a_1 \mid b_1 \nearrow x \mid \sigma$, $z \mid \sigma \searrow a_2 \mid b_2$.
And likewise dually.

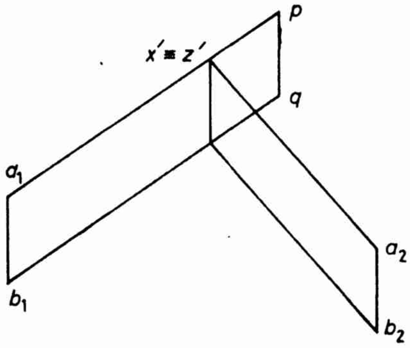


Fig. 2

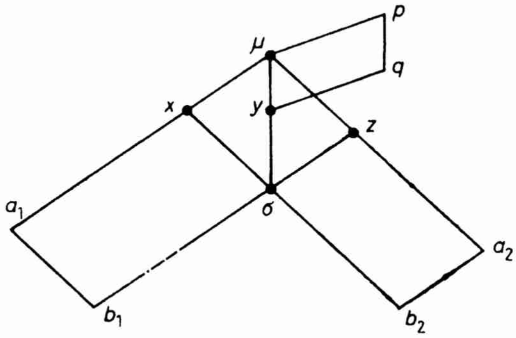


Fig. 3

Theorem 3.2. *Let L be a finite modular lattice of breadth two and $D_0, \ldots, D_n$ $(\subseteqq L)$ form a diamond sequence. A sequence $D_0, \ldots, D_n$ is a diamond circle if and only if there are quotients $a \mid b$, $c \mid d$ and $a \mid b \times c \mid d$ such that $a \mid b$, $D_{i_1}, \ldots, D_{i_m}$, $c \mid d$ and $c \mid d$, $D_{j_{m+1}}, \ldots, D_{j_k}$, $a \mid b$ are two diamond halfcircles* (see Definition 1.2) *satisfying $a \mid b \nearrow D_{i_1}$, $D_{i_m} \searrow c \mid d$ and $c \mid d \searrow D_{j_{m+1}}$, $D_{j_k} \nearrow a \mid b$.*

Proof. 1. Let a sequence $D_1, \ldots, D_n$ be a (clear) diamond circle. Put $a \mid b :=: \mu_0 \mid x_0$ and $c \mid d :=: \mu_0 \mid z_0$, then we get two required diamond halfcircles for which $a \mid b \nearrow D_1$, $\ldots, D_{n-1} \searrow c \mid d$ and $c \mid d \searrow D_0 \nearrow a \mid b$, and obviously $a \mid b = \mu_0 \mid x_0 \times \mu_0 \mid z_0 = c \mid d$.

2. Conversely, let $a \mid b \nearrow D_{i_1}, \ldots, D_{i_m} \searrow c \mid d$ and $c \mid d \searrow D_{j_{m+1}}, \ldots, D_{j_n} \nearrow a \mid b$ be two diamond halfcircles where $a \mid b \times c \mid d$. Since $c \mid d \searrow D_{j_{m+1}}, \ldots, D_{j_n} \nearrow a \mid b$ so by Theorem 3.1 and $c \mid d \times a \mid b$ there exists such D_0 for which $c \mid d \searrow \ldots, D_0,$ $\ldots \nearrow a \mid b$, and D_0 satisfies the conditions $(\mathrm{II})_{ij}$ of Definition 1.1. So we can get a diamond circle by fastening the sequence $a \mid b \nearrow D_{i_1}, \ldots, D_{i_m} \searrow c \mid d$ to the sequence $c \mid d \searrow \ldots, D_0, \ldots \nearrow a \mid b$. This completes our theorem.

Now we can prove further theorems.

Theorem 3.3. *Let $M = L_1 + L_2$ $(C = L_1 \cap L_2$, Hall-Dilworth$)$ be a finite modular lattice of breadth two. If C has such two prime quotients $a \mid b$ and $c \mid d$ which are projective in L_1 and in L_2 but no in C, then M isn't finitly M-projected.*

Remark. A lattice C here is an arbitrary lattice.

Proof of Theorem 3.3. Since $a \mid b$ and $c \mid d$ are not projective in C, so $a \mid b \times c \mid d$. But $a \mid b$ and $c \mid d$ are projective in L_1 and in L_2, so by Theorem 3.1 there exist diamond halfcircles separately in L_1 and in L_2. And because of the peculiarity of the Hall-Dilworth construction (it is that if $p \mid q \subseteqq L_1$, $r \mid s \subseteqq L_2$ then $p \mid q \searrow r \mid s$

isn't never satisfied) we can use Theorem 3.2 from which we get a diamond circle, and so by Theorem 1.1 this means that M isn't finitly M-projected. With this we have proved our Theorem.

We prove a sufficient theorem.

Theorem 3.4. *Let $M = L_1 + L_2$ ($C = L_1 \cap L_2$, Hall-Dilworth) be a lattice where L_1 and L_2 are finite modular lattices of breadth two which have γ-property. If C has no two prime quotients $a \mid b$ and $c \mid d$ which are projective in L_1 and in L_2 but no in C, then M has also γ-property.*

Proof. It is obvious that M is a finite modular lattice of breadth two by corollaries of Theorem 2 and Theorem 3 in [2]. In addition to that if L_1 and L_2 don't contain sublattice isomorphic to M_4 then $M = L_1 + L_2$ (Hall-Dilworth) doesn't contain it. So if M has no γ-property, then M contains a diamond circle $D_0, \ldots, D_n$ which is neither in L_1 nor in L_2. So for instance if $D_0 \subset L$, then partly there exists a least i such that $D_i \subset L$, $D_i \nsubseteq C$ but $D_{i+1} \subset L_2$ and $D_i \nearrow D_{i+1}$ such that $\mu_i \mid z_i \nearrow x_{i+1} \mid \sigma_{i+1}$, partly there exists a largest j such that $D_j \subset L_2$, $D_{j+1} \subset L_1$ but $D_{j+1} \nsubseteq C$ and $D_j \searrow D_{j+1}$ such that $x_j \mid \sigma_j \searrow \mu_{j+1} \mid z_{j+1}$. If

$$a :=: \mu_i \vee o_2 \in C, \; b :=: z_i \vee o_2 \in C,$$

$$c :=: \mu_{j+1} \vee o_2 \in C, \; d :=: z_{j+1} \vee o_2 \in C$$

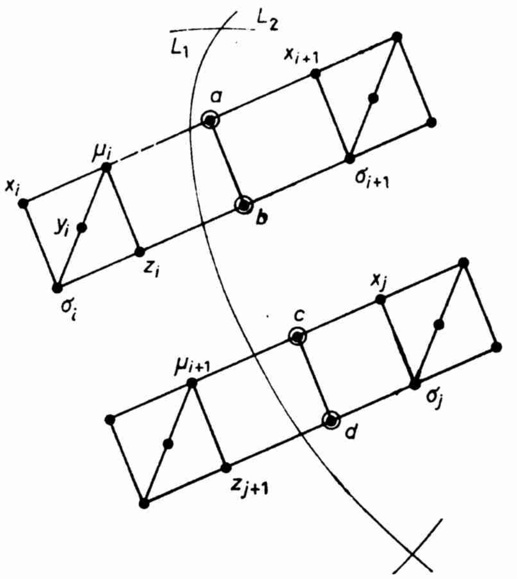
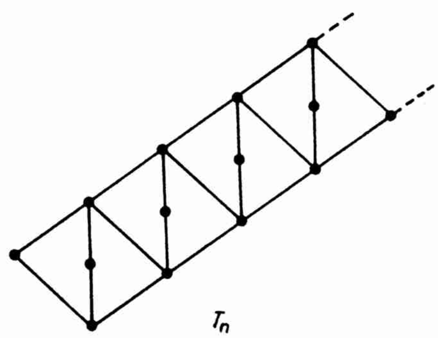


Fig. 4

where o_2 is the least element of L_2, then $\mu_i \mid z_i \nearrow a \mid b \nearrow x_{i+1} \mid \sigma_{i+1}$ and $\mu_{j+1} \mid z_{j+1} \nearrow c \mid d \nearrow x_j \mid \sigma_j$ where $a \mid b$ and $c \mid d$ are projective in L_1 and in L_2. Moreover $a \mid b \not\searrow c \mid d$ (namely if $a \mid b \searrow c \mid d$ then μ_i, c, b would form a sublattice isomorphic to 2^3) and $a \mid b$, $c \mid d$ are not projective in C (or else by Theorem 3.2 a diamond circle would exist) which is contradiction to the condition in this theorem. This completes our theorem.

Theorem 3.5. *Let $M = L_1 + L_2$ ($C = L_1 \cap L_2$, Hall-Dilworth) be a finite planar modular lattice where L_1 and L_2 are subdirectly irreducible lattices. M is splitting if* *and only if L_1 and L_2 are splitting and C has not two prime quotients $a \mid b$ and $c \mid d$ which are projective in L_1 and in L_2 but not in C. (It is well-known that a planar lattice is of breadth two.)*

Proof. If M is splitting, then by Theorem 1.2 M has γ-property, so L_1 and L_2 also has γ-property and so L_1 and L_2 are splitting. Moreover the proof of Theorem 3.3 guarantees the another condition. Conversely, if L_1 and L_2 are splitting then L_1 and L_2 have γ-property and so by Theorem 3.4 M also has γ-property. Besides M is a subdirectly irreducible lattice (namely if L_1 and L_2 are subdirectly irreducible, then $M = L_1 + L_2$ (Hall-Dilworth) is also subdirectly irreducible, because a finite modular lattice L is subdirectly irreducible iff all prime quotients of L are projective to one another) so by Theorem 1.2 M is a splitting lattice. This completes our theorem.



T_n Fig. 5

In application of Theorem 3.5, for instance, for T_n we can formulate such that T_n is constructed by gluing the lattices M_3 in step $n - 1$, and since M_3 is obviously splitting so T_n is also splitting.
Finally we have the following indepentently interesting theorem:

Theorem 3.6. *Let $M = L_1 + L_2$ ($C = L_1 \cap L_2$, Hall-Dilworth) be a lattice. M is a finite modular lattice of breadth two which has γ-property if and only if L_1 and L_2 are finite modular lattices of breadth two which have γ-property, and C has no two prime quotients $a \mid b$ and $c \mid d$ which are projective in L_1 and in L_2 but no in C.*

Proof. It is obvious by Theorems 3.3 and 3.4 and by corollaries of Theorems 2 and 3 in [2].

REFERENCES

[1] DAY, A.: Splitting algebras and weak notion of projectivity, Algebra Universalis **5** (1975), 153–162.
[2] CHUONG, H. M.: Gluing lattice-construction, Studia Sci. Math. Hungarica (to appear).
[3] MITSCHKE, A., E. T. SCHMIDT and R. WILLE: One finitly projected modular lattices of breadth two (in preparation).
[4] SCHMIDT, E. T.: On splitting modular lattices, submitting to the Proceedings of the Coll. on Universal Algebra held in Estergom, Hungary.
[5] SCHMIDT, E. T.: Remarks on finitly projected modular lattices, preprint.

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