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Pivot theorems in n-space

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1. Introduction

If points are marked on the edges of a simplex in n-space, one on each, and a sphere is drawn through each vertex and the points marked on those edges which meet in it, then these spheres all meet in a point. This, known as Haskell's pivot theorem [1], is the first in a series of theorems to be called as *pivot theorems* in n-space. If we are given a set of points in n-space with more points than is necessary to form the vertices of a simplex in n-space, still we can mark points on the joins of these points to each other so as to have a sphere going through each point of the set and the points marked on the joins of it to the other points, and it will be found that these spheres all meet in a point in like manner. And this results in a chain of pivot theorems, arising as we add one point after another to a set of points which form the vertices of a simplex in n-space.

2. The first pivot theorem

The following will be a simple proof of the first pivot theorem in E_n. Let $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the vertices of a simplex in E_n. Let $\boldsymbol{p}_{ij}$ $(= \boldsymbol{p}_{ji})$ be the point marked on the edge $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle$; as a point on $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle$, $\boldsymbol{p}_{ij} = g^j_i \boldsymbol{e}_i + g^i_j \boldsymbol{e}_j$, where $g^j_i + g^i_j = 1$. Let S_i be the sphere going through $\boldsymbol{e}_i$ and the points $\boldsymbol{p}_{ij}$ marked on the edges which meet in $\boldsymbol{e}_i$; $\boldsymbol{p}_i$ be its centre. Let $\boldsymbol{p} = \sum_{i=0}^{n} x_i \boldsymbol{e}_i$, where $\sum_{i=0}^{n} x_i = 1$, be the centre and r, the radius of the common orthogonal sphere of the spheres $S_0, S_1, \ldots, S_n$.

To prove that the spheres S_i all meet in a point, it is enough to show that $r = 0$, that is, the common orthogonal sphere is a point sphere.

Since the sphere S_i with centre $\boldsymbol{p}_i$ and radius $|\boldsymbol{p}_i - \boldsymbol{e}_i|$ is cut orthogonally by the common orthogonal sphere we have: $r^2 = (\boldsymbol{p} - \boldsymbol{p}_i)^2 - (\boldsymbol{p}_i - \boldsymbol{e}_i)^2$. That is,

$$r^2 = -\boldsymbol{e}_i^2 + 2\boldsymbol{p}_i \cdot \boldsymbol{e}_i - 2\boldsymbol{p}_i \cdot \boldsymbol{p} + \boldsymbol{p}^2. \tag{1}$$

Also since $\boldsymbol{p}_{ij}$ $(j = 0, 1, \ldots, n)$, are points on S_i, we have

$$(\boldsymbol{p}_{ij} - \boldsymbol{p}_i)^2 = (\boldsymbol{p}_i - \boldsymbol{e}_i)^2, \quad \text{or,} \quad (\boldsymbol{p}_{ij} - \boldsymbol{e}_i) \cdot (\boldsymbol{p}_{ij} + \boldsymbol{e}_i - 2\boldsymbol{p}_i) = 0,$$

i.e., $g^i_j(\boldsymbol{e}_j - \boldsymbol{e}_i) \cdot \big(\boldsymbol{e}_j + \boldsymbol{e}_i + g^j_i(\boldsymbol{e}_i - \boldsymbol{e}_j) - 2\boldsymbol{p}_i\big) = 0$, which gives

$$2\boldsymbol{p}_i \cdot \boldsymbol{e}_i = g^j_i(\boldsymbol{e}_j - \boldsymbol{e}_i)^2 + \boldsymbol{e}_i^2 - \boldsymbol{e}_j^2 + 2\boldsymbol{p}_i \cdot \boldsymbol{e}_j.$$

On substitution for $2p_i \,.\, e_i$ in (1), we get

$$r^2 = g^j_i(e_j - e_i)^2 - e_j^2 + 2p_i \,.\, e_j - 2p_i \,.\, p + p^2.$$

This being true for $j = 0, 1, \ldots, n$, we have

$$r^2 = \sum_{j=0}^{n} x_j[g^j_i(e_j - e_i)^2 - e_j^2 + 2p_i \,.\, e_j - 2p_i \,.\, p + p^2]$$

$$= \sum_{j=0}^{n} x_j g^j_i(e_j - e_i)^2 - \sum_{j=0}^{n} x_j e_j^2 + p^2,$$

since $\sum x_j = 1$ and $\sum x_j e_j = p$. This is true for each $i = 0, 1, \ldots, n$ and so we get

$$r^2 = \sum_{i=0}^{n} x_i \left[\sum_{j=0}^{n} g^j_i x_j(e_j - e_i)^2 - \sum_{j=0}^{n} x_j e_j^2 + p^2\right]$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n} x_i g^j_i x_j(e_j - e_i)^2 - \sum_{j=0}^{n} x_j e_j^2 + p^2.$$

But this last expression is easily seen to vanish, since

$$\sum_{i=0}^{n} \sum_{j=0}^{n} x_i g^j_i x_j(e_j - e_i)^2 = \sum_{i,j} x_i(g^j_i + g^i_j)\, x_j(e_j - e_i)^2$$

$$= \sum_{j=0}^{n} x_j e_j^2 - \left(\sum_{i=0}^{n} x_i e_i\right)^2 = \sum_{j=0}^{n} x_j e_j^2 - p^2.$$

It follows that $r = 0$.

3. Pivot theorem of $n + 2$ points

Theorem. *Let a set of $n + 2$ points, $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{n+1}$, be given in E_n, of which no set of $n + 1$ lie in the same prime. Let a sphere S_0 going through $\mathbf{e}_0$ meet the joins of $\mathbf{e}_0$ to the other points in points $\mathbf{p}_{01}, \mathbf{p}_{02}, \ldots, \mathbf{p}_{0,n+1}$; a sphere S_1, going through $\mathbf{e}_1$ and $\mathbf{p}_{01}$, meet the joins of $\mathbf{e}_1$ to $\mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_{n+1}$ in points $\mathbf{p}_{12}, \mathbf{p}_{13}, \ldots, \mathbf{p}_{1,n+1}$; a sphere S_2 going through $\mathbf{e}_2, \mathbf{p}_{02}, \mathbf{p}_{12}$ meet the joins of $\mathbf{e}_2$, to $\mathbf{e}_3, \mathbf{e}_4, \ldots, \mathbf{e}_{n+1}$ in points $\mathbf{p}_{23}, \mathbf{p}_{24}, \ldots, \mathbf{p}_{2,n+1}$; and so on. Finally let the sphere S_n going through $\mathbf{e}_n, \mathbf{p}_{0n}, \mathbf{p}_{1n}, \ldots, \mathbf{p}_{n-1,n}$ meet $\langle \mathbf{e}_n, \mathbf{e}_{n+1} \rangle$ in $\mathbf{p}_{n,n+1}$. Then*

(i) *there is a sphere S_{n+1} going through $\mathbf{e}_{n+1}$ and the points $\mathbf{p}_{0,n+1}, \mathbf{p}_{1,n+1}, \ldots, \mathbf{p}_{n,n+1}$; and*

(ii) *the spheres $S_0, S_1, \ldots, S_{n+1}$ all meet in a point.*

In proving the theorem it will be useful to know the condition in order that, given a set of $n + 2$ points, $a_0, a_1, \ldots, a_{n+1}$, in E_n, there is a sphere going through them all. The vector a_0 can be expressed as

$$a_0 = \sum_{i=1}^{n+1} g_i a_i, \quad \text{where} \quad \sum_{i=1}^{n+1} g_i = 1. \tag{2}$$

If the sphere, through the $n + 1$ points $a_1, a_2, \ldots, a_{n+1}$, has r, c for radius and centre

$$(a_i - c)^2 = r^2, \quad \text{i.e.,} \quad a_i^2 - 2a_i \,.\, c + c^2 - r^2 = 0 \quad (i = 1, 2, \ldots, n + 1)$$

which means:

$$\sum_{i=1}^{n+1} g_i(\boldsymbol{a}_i^2 - 2\boldsymbol{a}_i \cdot \boldsymbol{c} + \boldsymbol{c}^2 - r^2) = 0,$$

that is,

$$\sum_{i=1}^{n+1} g_i\boldsymbol{a}_i^2 - 2\boldsymbol{a}_0 \cdot \boldsymbol{c} + \boldsymbol{c}^2 - r^2 = 0,$$

since $\sum_{i=1}^{n+1} g_i = 1$ and $\sum_{i=1}^{n+1} g_i\boldsymbol{a}_i = \boldsymbol{a}_0$. This sphere goes through $\boldsymbol{a}_0$ also, that is, $\boldsymbol{a}_0^2 - 2\boldsymbol{a}_0 \cdot \boldsymbol{c} + \boldsymbol{c}^2 - r^2 = 0$, if and only if $\sum_{i=1}^{n+1} g_i\boldsymbol{a}_i^2 = \boldsymbol{a}_0^2$. Thus the necessary and sufficient condition in order that the points $\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n+1}$, have a sphere going through them all, is

$$\boldsymbol{a}_0^2 - \sum_{i=1}^{n+1} g_i\boldsymbol{a}_i^2 = 0. \tag{3}$$

This condition may be expressed in a more convenient form. Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ be a set of linearly independent vectors. Then solving for $g_1, g_2, \ldots, g_{n+1}$ from the equations

$$\boldsymbol{a}_0 \cdot \boldsymbol{u}_k = \sum_{i=1}^{n+1} g_i\boldsymbol{a}_i \cdot \boldsymbol{u}_k \quad (k = 1, 2, \ldots, n),$$

$$1 = \sum_{i=1}^{n+1} g_i,$$

and substituting in (3), the condition reduces to the vanishing of the determinant

$$\begin{vmatrix} \boldsymbol{a}_0^2 & \boldsymbol{a}_1^2 & \ldots & \boldsymbol{a}_{n+1}^2 \\ \boldsymbol{a}_0 \cdot \boldsymbol{u}_1 & \boldsymbol{a}_1 \cdot \boldsymbol{u}_1 & \ldots & \boldsymbol{a}_{n+1} \cdot \boldsymbol{u}_1 \\ \cdot & \cdot & \cdot & \cdot \\ \boldsymbol{a}_0 \cdot \boldsymbol{u}_n & \boldsymbol{a}_1 \cdot \boldsymbol{u}_n & \ldots & \boldsymbol{a}_{n+1} \cdot \boldsymbol{u}_n \\ 1 & 1 & \ldots & 1 \end{vmatrix} = \det(d_0, d_1, \ldots, d_{n+1}),$$

where $d_k = (\boldsymbol{a}_k^2, \boldsymbol{a}_k \cdot \boldsymbol{u}_1, \ldots, \boldsymbol{a}_k \cdot \boldsymbol{u}_n, 1)$, written as a column.

4. Proof of the theorem

Being a point on $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle$, $\boldsymbol{p}_{ij}$ $(= \boldsymbol{p}_{ji})$ may be expressed as $\boldsymbol{p}_{ij} = g_i^j\boldsymbol{e}_i + g_j^i\boldsymbol{e}_j$, where $g_i^j + g_j^i = 1$ $(i, j = 0, 1, \ldots, n+1;\ i \neq j)$. Setting $g_k^k = 1/2$ for $k = 0, 1, \ldots, n+1$, we may occasionally represent $\boldsymbol{e}_i$ as $\boldsymbol{p}_{ii} = g_i^i\boldsymbol{e}_i + g_i^i\boldsymbol{e}_i$.
The condition for a sphere to be going through $\boldsymbol{e}_i$ $(= \boldsymbol{p}_{ii})$ and the points $\boldsymbol{p}_{i0}, \boldsymbol{p}_{i1}, \ldots, \boldsymbol{p}_{i,n+1}$ on the joins of $\boldsymbol{e}_i$ to the other points, is that

$$\det(d_0^i, d_1^i, \ldots, d_{n+1}^i) = 0,$$

where

$$d_j^i = (\boldsymbol{p}_{ij}^2, \boldsymbol{p}_{ij} \cdot \boldsymbol{u}_1, \ldots, \boldsymbol{p}_{ij} \cdot \boldsymbol{u}_n, 1).$$

This determinant is easily seen to be, but for a non-zero factor, equal to $\det(\bar{d}_0^i, \bar{d}_1^i, \ldots, \bar{d}_{n+1}^i)$, where

$$\bar{d}_j^i = ([\boldsymbol{e}_j^2 - g_i^j(\boldsymbol{e}_i - \boldsymbol{e}_j)^2], \boldsymbol{e}_j \cdot \boldsymbol{u}_1, \ldots, \boldsymbol{e}_j \cdot \boldsymbol{u}_n, 1);$$

for, while $\bar{d}^i_i = d^i_i$, we have

$$p^2_{ij} = g^i_j[e^2_j - g^j_i(e_i - e_j)^2] + g^j_i e^2_i$$

so that $d^i_j = g^i_j \bar{d}^i_j + g^j_i d^i_i$.

And with the vector e_{n+1} expressed as

$$e_{n+1} = \sum_{j=0}^{n} y_j e_j, \quad \sum_{j=0}^{n} y_j = 1,$$

and the values of $y_0, y_1, \ldots, y_n$ obtained from the equations

$$\sum_{j=0}^{n} y_j e_j \cdot u_k = e_{n+1} \cdot u_k \quad (k = 1, \ldots, n)$$

and $\sum_{j=0}^{n} y_j = 1$, we find that det $(\bar{d}^i_0, \bar{d}^i_1, \ldots, \bar{d}^i_{n+1})$ is equal to the expression

$$e^2_{n+1} - g^{n+1}_i(e_i - e_{n+1})^2 - \sum_{j=0}^{n} y_j[e^2_j - g^j_i(e_i - e_j)^2]$$

multiplied by the non-vanishing determinant

$$\begin{vmatrix} e_0 \cdot u_1 & \ldots & e_n \cdot u_1 \\ \cdots & \cdots & \cdots \\ e_0 \cdot u_n & \ldots & e_n \cdot u_n \\ 1 & \ldots & 1 \end{vmatrix}.$$

Hence

$$\begin{aligned} \det (d^i_0, d^i_1, \ldots, d^i_{n+1}) = 0 \Leftrightarrow \det (\bar{d}^i_0, \bar{d}^i_1, \ldots, \bar{d}^i_{n+1}) = 0 \\ \Leftrightarrow e^2_{n+1} - g^{n+1}_i(e_i - e_{n+1})^2 \\ - \sum_{j=0}^{n} y_j e^2_j + \sum_{j=0}^{n} g^j_i y_j (e_i - e_j)^2 = 0. \end{aligned}$$

This last relation, using the identity

$$e^2_{n+1} - \sum_{j=0}^{n} y_j e^2_j = (e_i - e_{n+1})^2 - \sum_{j=0}^{n} y_j (e_i - e_j)^2,$$

takes the form

$$g^i_{n+1}(e_i - e_{n+1})^2 - \sum_{j=0}^{n} g^i_j y_j (e_i - e_j)^2 = 0; \tag{4}$$

which thus expresses the necessary — also sufficient — condition for a sphere to be going through the $n + 2$ points $p_{i0}, p_{i1}, \ldots, p_{i,n+1}$.

Now, it is given that a sphere passes through the $n + 2$ points $p_{i0}, p_{i1}, \ldots, p_{ii}$ $(= e_i), \ldots, p_{i,n+1}$, for $i = 0, 1, \ldots, n$. Thus (4) is true for each $i = 0, 1, \ldots, n$; wherefrom at once we obtain

$$\sum_{i=0}^{n} y_i \left[g^i_{n+1}(e_i - e_{n+1})^2 - \sum_{j=0}^{n} g^i_j y_j (e_i - e_j)^2 \right] = 0,$$

i.e.,

$$\sum_{i=0}^{n} y_i g_{n+1}^i (e_{n+1} - e_i)^2 - \sum_{i=0}^{n} \sum_{j=0}^{n} y_i g_j^i y_j (e_i - e_j)^2 = 0.$$

Since

$$\sum_{i=0}^{n} \sum_{j=0}^{n} y_i g_j^i y_j (e_i - e_j)^2 = \sum_{i,j}^{n} y_i (e_i - e_j)^2 \, y_j = \sum_{j=0}^{n} y_j e_j^2 - \left(\sum_{j=0}^{n} y_j e_j\right)^2,$$

this means

$$e_{n+1}^2 - \sum_{j=0}^{n} y_j [e_j^2 - g_{n+1}^j (e_{n+1} - e_j)^2] = 0. \tag{5}$$

And as above,

$$\det (d_0^{n+1}, d_1^{n+1}, \ldots, d_{n+1}^{n+1}) = 0$$

$$\Leftrightarrow \det (\bar{d}_0^{n+1}, \bar{d}_1^{n+1}, \ldots, \bar{d}_{n+1}^{n+1}) = 0$$

$$\Leftrightarrow e_{n+1}^2 - \sum_{j=0}^{n} y_j [e_j^2 - g_{n+1}^j (e_{n+1} - e_j)^2] = 0.$$

It follows that (5) implies $\det (d_0^{n+1}, d_1^{n+1}, \ldots, d_{n+1}^{n+1}) = 0$, that is, there exists a sphere S_{n+1} going through the points $p_{0,n+1}, p_{1,n+1}, \ldots, p_{n,n+1}, e_{n+1}$.

5. The proof completed

It remains to prove that the spheres $S_0, S_1, \ldots, S_{n+1}$ all pass through a common point.

By the first pivot theorem, the theorem of $n + 1$ points, the spheres $S_0, S_1, \ldots, S_n$ all meet in a point. Let $p = \sum_{i=0}^{n} x_i e_i$, $\sum_{i=0}^{n} x_i = 1$, be the point in which they all meet. Then, for each $i = 0, 1, \ldots, n$, there being a sphere S_i going through the points $p, p_{i0}, \ldots, p_{ii} (= e_i), \ldots, p_{in}$, we have $\det (d, d_0^i, \ldots, d_n^i) = 0$, where $d = (p^2, p \,.\, u_1, \ldots, p \,.\, u_n, 1)$. With d_j^i, $\bar{d}_j^i$ denoting the same as before, it means $\det (d, \bar{d}_0^i, \ldots, \bar{d}_n^i) = 0$; which, with $x_0, x_1, \ldots, x_n$ given by the equations

$$\sum_{i=0}^{n} x_i e_i \,.\, u_k = p \,.\, u_k \quad (k = 1, 2, \ldots, n), \quad \text{and} \quad \sum_{i=0}^{n} x_i = 1$$

reduces to

$$p^2 - \sum_{j=0}^{n} x_j [e_j^2 - g_j^i (e_i - e_j)^2] = 0. \tag{6}$$

And this being true for each $i = 0, 1, \ldots, n$, we obtain

$$\sum_{i=0}^{n} y_i \left[p^2 - \sum_{j=0}^{n} x_j e_j^2 + \sum_{j=0}^{n} x_j g_j^i (e_i - e_j)^2\right] = 0,$$

that is,

$$p^2 - \sum_{j=0}^{n} x_j e_j^2 + \sum_{j=0}^{n} x_j \left[\sum_{i=0}^{n} g_j^i y_i (e_i - e_j)^2\right] = 0.$$

Since by (4)

$$\sum_{i=0}^{n} g_i^j y_i (e_i - e_j)^2 = g_{n+1}^j (e_j - e_{n+1})^2, \quad \text{for each } j = 0, 1, \ldots, n,$$

this means

$$p^2 - \sum_{j=0}^{n} x_j e_j^2 + \sum_{j=0}^{n} x_j g_{n+1}^j (e_j - e_{n+1})^2 = 0. \tag{7}$$

This gives: det $(d, \bar{d}_0^{n+1}, \bar{d}_1^{n+1}, \ldots, \bar{d}_n^{n+1}) = 0$, or equivalently,

$$\det (d, \bar{d}_{n+1}^{n+1}, \bar{d}_1^{n+1}, \ldots, \bar{d}_n^{n+1}) = 0$$

(since det $(\bar{d}_0^{n+1}, \bar{d}_1^{n+1}, \ldots, \bar{d}_{n+1}^{n+1}) = 0$ as by (5)).
Now to have S_{n+1} going through p, it is enough that p and the points $e_{n+1}, p_{1,n+1}, \ldots, p_{n,n+1}$ on S_{n+1} have a sphere going through them all. And a sphere goes through them all iff

$$\det (d, d_{n+1}^{n+1}, d_1^{n+1}, \ldots, d_n^{n+1}) = 0$$

$$\Leftrightarrow \det (d, \bar{d}_{n+1}^{n+1}, \bar{d}_1^{n+1}, \ldots, \bar{d}_n^{n+1}) = 0.$$

It follows that S_{n+1} goes through p as each S_i $(i = 0, 1, \ldots, n)$ does.

6. Pivot theorem of $k + 1$ points, $k > n + 1$

Theorem of $n + 3$ points. *Let a set of $n + 3$ points, $e_0, e_1, \ldots, e_{n+2}$, be given, no $n + 1$ of which lie in the same prime. Let spheres $S_0, S_1, \ldots, S_{n+1}$ be drawn, in order, through $e_0, e_1, \ldots, e_{n+1}$; S_0 going through e_0 and meeting the joins of e_0 to $e_1, e_2, \ldots, e_{n+2}$ in points $p_{01}, p_{02}, \ldots, p_{0,n+2}$; S_1 going through e_1 and p_{01}, and meeting the joins of e_1 to $e_2, e_3, \ldots, e_{n+2}$ in $p_{12}, p_{13}, \ldots, p_{1,n+2}$, and so on; finally S_{n+1} going through e_{n+1} and the points $p_{0,n+1}, p_{1,n+1}, \ldots, p_{n,n+1}$ — a sphere goes through them all as by the theorem of $n + 2$ points — and meeting $\langle e_{n+1}, e_{n+2}\rangle$ in point $p_{n+1,n+2}$. Then*

(i) *there exists a sphere S_{n+2} going through e_{n+2} and the points $p_{0,n+2}, p_{1,n+2}, \ldots, p_{n+1,n+2}$; and*

(ii) *the spheres $S_0, S_1, \ldots, S_{n+2}$ have a common point through which they all go.*

For the proof, we consider the set of points obtained on dropping from the given set one point other than e_{n+2}. On dropping e_0, for the set of $n + 2$ points that are left, we have, as for the theorem of $n + 2$ points, spheres drawn through them; S_1 going through e_1 and meeting the joins of e_1 to $e_2, e_3, \ldots, e_{n+2}$ in points $p_{12}, p_{13}, \ldots, p_{1,n+2}$; S_2 going through e_2 and p_{12} and meeting $\langle e_2, e_i\rangle$ in p_{2i} $(i = 3, 4, \ldots, n + 2)$; ...; S_{n+1} going through e_{n+1} and $p_{j,n+1}$ $(j = 1, 2, \ldots, n)$ and meeting the join of e_{n+1} to e_{n+2} in $p_{n+1,n+2}$. Therefore by the theorem of $n + 2$ points, there is a sphere S'_{n+2} going through e_{n+2} and the points $p_{1,n+2}, p_{2,n+2}, \ldots, p_{n+1,n+2}$ on the joins of e_{n+2} to $e_1, e_2, \ldots, e_{n+1}$; and further the spheres $S_1, S_2, \ldots, S_{n+1}, S'_{n+2}$ have a common point in which they all meet. Similarly, on dropping e_{n+1}, for the points left, at $e_0, e_1, \ldots, e_n$, there are spheres $S_0, S_1, \ldots, S_n$ drawn through them in like manner; and by the theorem of $n + 2$ points again, there is a sphere S''_{n+2} going through e_{n+2} and the points $p_{0,n+2}, \ldots, p_{n,n+2}$, and the spheres $S_0, S_1, \ldots, S_n, S''_{n+2}$ all meet in a point.
The points $e_{n+2}, p_{1,n+2}, p_{2,n+2}, \ldots, p_{n,n+2}$ determine a unique sphere going through them all. If S_{n+2} is this sphere, then as a sphere through all these $n + 1$ points S'_{n+2} coincides with it. So also S''_{n+2} coincides with it. Hence $p_{0,n+2}$ lying on S''_{n+2}, and

$p_{n+1,n+2}$ lying on S'_{n+2}, along with the above points, are of S_{n+2}. Thus S_{n+2} is a sphere going equally through all the points $p_{0,n+2}, p_{1,n+2}, \ldots, p_{n+1,n+2}$.
So also the point in which the spheres $S_1, S_2, \ldots, S_{n+1}, S_{n+2}$ meet and the point in which $S_0, S_1, \ldots, S_n, S_{n+2}$ meet must be identical, identical with the point in which $S_1, S_2, \ldots, S_n, S_{n+2}$ meet.

Further extensions. The way to further extensions is now clear. Having proved the theorem for sets with up to k points, $k > n + 1$, we can prove the same for a set with $k + 1$ points in a similar manner, that is, by dropping in turn one point from a pair picked up from the set of $k + 1$ points and applying the theorem already proved to the set of k points that will be left, as it was done above for the proof of the theorem of $n + 3$ points.
Thus adding one point after another to the set of points for which the theorem has been proved, we get a chain of theorems corresponding to the sequence of integers greater than n. In this we will have also the theorem for the general case of a set with an indefinite number of points.

7. The general case

As for the general case, considering the given set of points as the vertices and their joins to each other as the edges of a polyhedron in n-space, the theorem may be seen to come to the following propositions:

(i) *Given a polyhedron in n-space, points may be marked directly on its edges — with as many as $n(n + 1)/2$ chosen arbitrary — so as to have a sphere go through each vertex and the points marked on the edges which meet in it.*

(ii) *If points are marked on the edges of a polyhedron in n-space, one on each edge, so that a sphere can be drawn through each vertex and the points marked on those edges which meet in it, then all the spheres that may be so drawn, one at each vertex, will have a common point in which they all meet.*

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