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On the theory of linearly compact rings I

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§ 1. Introduction and preliminaries

In ring theory the celebrated Wedderburn-Artin structure theorem is of central significance. It states that a ring with descending chain condition on left ideals is *semisimple* if and only if it is a finite direct sum of rings of linear transformations of finite dimensional vector spaces over division rings. It was LEPTIN [6] who succeeded in eliminating both finiteness conditions from this characterization; he proved that *linearly compact semisimple rings* are just complete direct sums of rings of linear transformations of vector spaces over division rings.
The purpose of this paper is to give various simple characterizations of linearly compact semisimple rings and to apply them to strictly linearly compact rings.
We consider only rings with a multiplicative unit element distinct from the zero element ("the zero ring is not a ring"). Homomorphisms are required to preserve the unit. All modules are unitary, and, unless explicitly stated otherwise, they are left modules. By a *topological ring* we shall mean a ring R which is at the same time a Hausdorff topological space such that the maps $R \times R \to R$ given by $(x, y) \to x - y$ and $(x, y) \to xy$ are continuous. By a *topological R-module* we shall mean an R-module M over a topological ring R which is at the same time a Hausdorff topological abelian group such that the composition map $R \times M \to M$ is continuous with respect to the product topology. We recall that a topological module is *linearly topologized* if the open submodules form a fundamental system of neighborhoods of zero. A linearly topologized module is *linearly compact* if every filter base of cosets of closed submodules has an adherent point. An R-module M is *strictly linearly compact* (i.e. a s.l.k. module in [6, 7]) if it is the inverse limit of discrete R-modules satisfying the descending chain condition on submodules, or equivalently if M is linearly compact and every continuous epimorphism from M onto any linearly topologized R-module is open. We assume familiarity with basic properties of linearly compact and strictly linearly compact modules, discussed in [6, 7] or [2, Exercises 14–22, pp. 236–241]. A ring R is called *linearly topologized, linearly compact*, and *strictly linearly compact*, respectively, if it is such as an R-module. By this definition, every continuous isomorphism between strictly linearly compact rings is always a homeomorphism, i.e. it is a topological mapping.
In what follows, the terms linearly topologized, linearly compact, and strictly linearly compact will be abbreviated as l.t., l.c., and s.l.c., respectively.
In a ring R of linear transformations of a vector space V over a division ring we can introduce a topology as follows. Let $L(u_1, \ldots, u_n)$ be the set of all linear transformations on V which map the elements $u_1, \ldots, u_n$ of V into the zero element. If $\{L(u_1, \ldots,$

$u_n)$, $u_1, \ldots, u_n \in V$, $n = 1, 2, \ldots\}$ is considered as an open base for a neighborhood system of the zero element, then this introduces a topology on R, which is called the *finite topology*. In the following, if a ring of linear transformations of a vector space over a division ring comes up, we always mean this endowed with the finite topology.

For non-defined notions or more details on the results in § 1 we refer to [2, 6, 7]. Throughout this paper, J will stand for the Jacobson radical of the ring R.

The *l.c. semisimple rings* are characterized by the following structure theorems, most of them due to LEPTIN.

(1) [6, Sätze 12 and 13]: A l.c. semisimple ring is a complete direct sum of rings which are rings of linear transformations of vector spaces over division rings, moreover, the l.c. semisimple rings are s.l.c.

(2) [7, (1.2)]: A l.c. ring R is semisimple if and only if every l.c. R-module is a complete direct sum of minimal submodules, hence every l.c. over a l.c. semisimple ring is s.l.c.

(3) [11, Satz 2]: A l.c. ring is semisimple if and only if all closed left ideals of it have right unit elements.

(4) [7, Satz 15]: A s.l.c. ring R is a complete direct sum of indecomposable left ideals $R_\mu = Re_\mu$ generated by orthogonal idempotents e_μ:

$$R = \sum Re_\mu, \quad e_\mu \cdot e_\nu = e_\mu \cdot \delta_{\mu\nu}.$$

(5) [6, Satz 1]: Let M be any l.c. module, K be any closed submodule of M. If $\{N_\mu\}$ is any filter base consisting of closed submodules N_μ of M, then

$$\bigcap_\mu (N_\mu + K) = \Big(\bigcap_\mu N_\mu\Big) + K.$$

(6) [1, Folgerung 4]: Let R be a s.l.c. ring. If $\bigcap_{n=1}^{\infty} \overline{J^n} = 0$, where $\overline{J^n}$ is the closure of J^n, then R is an inverse limit of noetherian modules.

Let R be a topological ring and A an arbitrary ideal in R. Consider the following ideals in R:

$$A_1 = \bar{A}, \qquad \underset{1}{A} = \bar{A},$$

$$A_{\mu+1} = \overline{A_\mu \cdot A}, \qquad \underset{\mu+1}{A} = \overline{\underset{\mu}{A} \cdot \underset{\mu}{A}},$$

$$A_\lambda = \bigcap_{\mu<\lambda} A_\mu, \qquad \underset{\lambda}{A} = \bigcap_{\mu<\lambda} \underset{\mu}{A}$$

if λ is a limit ordinal, where $\bar{B}$ denotes the closure of the set B.

If there exists an ordinal ξ such that $A_\xi = 0$ and $\underset{\xi}{A} = 0$, the ideal A is said to be *transfinitely r-nilpotent* and *transfinitely nilpotent*, respectively. Since $\underset{\xi}{A} \subseteq A_\xi$ holds for all ordinals ξ, every transfinitely r-nilpotent ideal is transfinitely nilpotent.

(7) [6, Satz 9]: The Jacobson radical of a s.l.c. ring is transfinitely r-nilpotent.

As an application of assertion (7) we can prove the following result.

Proposition 1.1. *Every closed left ideal L of a s.l.c. ring R with radical J has the form $Re + L \cap J$, with an appropriately chosen (not necessarily non-zero!) idempotent e.*

Proof. Let L be any closed left ideal in R. Then $\bar{L} = (L + J)/J$ is a direct summand of the semisimple l.c. ring $\bar{R} = R/J$ according to (3), and there exists a (not necessarily non-zero) idempotent $\bar{e}^*$ with $\bar{L} = \bar{R}\bar{e}^*$ and $e^* \in L$. We may suppose that, for all ordinals $\mu < \lambda$, there are cosets $e_\mu + \underset{\mu}{J}$ with $e_\mu \in L$ such that $\mathfrak{M}_\lambda = \left\{e_\mu + \underset{\mu}{J}, \mu < \lambda\right\}$ is a filter base of idempotent cosets with $e_\mu \in \bar{e}^*$. Since $e_\mu \in L$ holds, $L \cap \left(e_\mu + \underset{\mu}{J}\right) \neq \emptyset$ for every $\mu < \lambda$. If λ is a limit ordinal, then we can choose an element $e_\lambda \in \bigcap_{\mu<\lambda}\left(e_\mu + \underset{\mu}{J}\right) \cap L = \downarrow\mathfrak{M}_\lambda \cap L$ $\left(\text{here } \downarrow\mathfrak{M}_\lambda = \bigcap_{\mu<\lambda}\left(e_\mu + \underset{\mu}{J}\right)\right)$ and we have the idempotent coset $e_\lambda + \underset{\lambda}{J} \subseteqq \downarrow\mathfrak{M}_\lambda \subseteqq \bar{e}^*$, $e_\lambda \in L$. If $\lambda = \varkappa + 1$, so $e_\varkappa^2 = e_\varkappa + q$ with $q \in L \cap \underset{\varkappa}{J}$ and hence $q^2 \in L \cap \underset{\lambda}{J}$. By setting $e_\lambda = e_\varkappa - 2e_\varkappa q + q \in L$, $e_\lambda + \underset{\lambda}{J}$ is an idempotent coset and it is contained in $\bar{e}^*$, too. So we have constructed for every ordinal λ an idempotent coset $e_\lambda + \underset{\lambda}{J}$ with $e_\lambda \in L$ contained in $\bar{e}^*$. Since $\underset{\xi}{J} = 0$ holds for some ξ, there exists an idempotent $e = e_\xi + \underset{\xi}{J} = e_\xi \in L$ with $\bar{e} = \bar{e}^*$. This implies the equality $L = Re + L \cap J$.

If L is a closed left ideal of R, an idempotent e for which $L = Re + L \cap J$ will be called a *fundamental idempotent* of L.

Corollary 1.2. *If L is a closed left ideal in a s.l.c. ring R, then L is contained in the radical J if and only if L contains no non-zero idempotent.*

Proof. Since $J_\xi = 0$ holds for some ordinal ξ, it is clear that J contains no non-zero idempotent. On the other hand, the relation $L = Re + L \cap J$ implies that $L \subseteqq J$ if L contains no non-zero idempotent.

Corollary 1.3. *Let R be a s.l.c. ring, and let I be a closed two-sided ideal of R. If $I = Re + I \cap J$, then $ex - xe \in J$ for all $x \in R$ and $I = eR + I \cap J$.*

Proof. Form the ring R/J and let $^-$ be the canonical map $R \to R/J = \bar{R}$. Since $I + J = Re + J$, we have $\overline{I + J} = \bar{R}\bar{e}$. Now $I + J$ is a closed two-sided ideal of R, hence $\overline{I + J}$ is a closed two-sided ideal of $\bar{R}$. By (1) $\overline{I + J}$ is a complete direct sum of rings of linear transformations of vector spaces over division rings which are direct summands of $\bar{R}$, therefore $\overline{I + J}$ has a unit element e^*. Since $\overline{I + J} = \bar{R}\bar{e}$ and $\bar{e} \in \overline{I + J}$, there is an element $v \in \bar{R}$ with $v\bar{e} = e^*$. This implies $\bar{e} = e^*\bar{e} = (v\bar{e})\,\bar{e} = v\bar{e}^2 = v\bar{e} = e^*$. From this we have $ex - xe \in J$ for all $x \in R$. This in turn yields $Re \subseteqq eR + J$, and also $Re \subseteqq eR + I \cap J$, since I is two-sided. Hence $I = Re + I \cap J \subseteqq eR + I \cap J \subseteqq I$, which gives the desired conclusion.

Proposition 1.4. *If I is a two-sided ideal of a s.l.c. ring R, then $I \subseteqq J$ if and only if I contains no non-zero idempotent.*

Proof. If $I \subseteqq J$ then, as we have seen already, I contains no non-zero idempotent. On the other hand, if I is not contained in J, then there is a closed maximal left ideal L which does not contain I. Hence $I + L = R$. Let e be the fundamental idempotent of L, then $L = Re + J$, therefore $I + J + Re = R$. This implies $I + Re = R$. Multiplying on the right by $1 - e$ and recalling that I is two-sided, we obtain that $1 - e \in I$.

Proposition 1.5. *Let I be a two-sided ideal of R. Any idempotent contained in $I + J$ is in I.*

Proof. Suppose e is an idempotent in $I + J$. Then $R(1 - e) + I + J = R$, so that $R(1 - e) + I = R$, from which we conclude that $e \in Re \subseteqq I$.

§ 2. Characterisations of linearly compact semisimple rings

The Wedderburn-Artin structure theorem characterizes artinian semisimple rings as the finite direct sums of rings, each of which is the ring of linear transformations of a finite dimensional vector space over a division ring. There are two obvious directions in which this class of rings may be enlarged: one may drop the requirement that the sum has only finitely many summands, and one may drop the finite-dimensionality of the vector spaces. Doing both of them, the resulting class of rings is, according to LEPTIN [6], that of all l.c. semisimple rings, and this has been the subject of intensive investigations, principally by JACOBSON [4], and LEPTIN [6, 7].
In this section we give various simple characterizations of rings of this enlarged class. We begin with the following theorem.

Theorem 2.1. *Let R be a l.c. ring. The following conditions are equivalent:*

1. *R is semisimple,*
2. *every discrete R-module is semisimple,*
3. *every discrete R-module is projective,*
4. *all exact sequences $0 \to A \to B \to C \to 0$ of R-modules where at least one of B and C is a discrete R-module, split.*

Proof. 1. $\Rightarrow$ 2.: Let R be semisimple and M be a discrete R-module. Let M_1 be a submodule of M, we wish to show that M_1 is a direct summand of M. A trivial application of Zorn's lemma asserts the existence of a submodule M_2, maximal with respect to $M_1 \cap M_2 = 0$. What we must show is that $M_1 + M_2 = M$. If we set $N = M_1 + M_2$, the maximality of M_2 implies that N has the following property: if L is a non-zero submodule of M then $N \cap L \neq 0$. This is simply seen as follows: if $L \cap N = 0$ then $M_1 \cap (M_2 + L) = 0$, hence $L \subseteq M_2$, while $M_2 \cap L \subseteq N \cap L = 0$. Thus we are led to the verification of $N = M$ whenever N has the property just described.
Let x be any element of M. Since M is discrete, N is closed and therefore $A = \{a \in R \mid ax \in N\}$ is a closed left ideal in R. Since R is semisimple, the closed left ideal A has a right unit element e according to (3). Consider the submodule $L = R(1 - e)\,x$ of M. If $L = 0$, then $1 - e \in A$ so that $A = R$ and $x \in N$. On the other hand, if $b(1 - e)\,x \in L \cap N$, then $b(1 - e) \in A = Re$, which clearly shows that $b(1 - e) = 0$. Thus $L \cap N = 0$ and hence $L = 0$ and $x \in N$.

2. $\Rightarrow$ 1.: Let L be any open left ideal of R. Then R/L is a discrete R-module, hence R/L is a semisimple R-module by assumption. Let J be the radical of R, we wish to show that $J \subseteq L$, i.e. $J \cdot R/L = 0$. Since R/L is semisimple, it is a sum of simple submodules, and the verification that $J \cdot R/L = 0$ reduces to its verification in case R/L is simple. If x is any non-zero element of R/L, then the annihilator of x is an open maximal regular left ideal K. But $J \subseteq K$, hence $Jx = 0$, or $J \cdot R/L = 0$. Since R has a basis of neighborhoods of 0 consisting of such L, it follows that $J = 0$, i.e. R is semisimple.

2. $\Rightarrow$ 3.: By assumption, every discrete R-module M is semisimple, i.e. it is a discrete direct sum of its simple submodules M_μ. Let x be any non-zero element of M_μ, then $Rx = M_\mu$, because M_μ is simple. The annihilator of x is an open maximal regular left ideal K of R. Then K is a direct summand of R, therefore M_μ is a direct summand of R. This implies that M is a direct summand of a free R-module, i.e. M is a projective R-module.

3. $\Rightarrow$ 1.: Instead of verifying the implication 3. $\Rightarrow$ 1. we prove the following stronger statement:
A complete, l.t. ring R is a l.c. semisimple ring if every discrete R-module is projective.

In particular, let L be any open left ideal in R. The factor module R/L is a discrete R-module, therefore $R = L \oplus R/L$. This implies the existence of an element $e \in R/L$ with $Re = R/L$. Since the factor module Re/K is also discrete for every submodule K of R/L, Re/K is projective, therefore K is a direct summand of Re. Hence Re is semisimple. Now a finitely generated semisimple module is a direct sum of a finite number of simple modules, from which we easily deduce the descending chain condition on submodules of Re. This shows that R is a s.l.c. ring. We must prove that even R is semisimple. For this let M be any discrete R-module, and M_1 be a submodule of M. We wish to show that M_1 is a direct summand of M. Since M is discrete, the factor module M/M_1 is also discrete, therefore M/M_1 is projective, i.e. M_1 is a direct summand of M. Hence M is semisimple and by 2. $\Rightarrow$ 1. R is semisimple.
Since 4. is a well-known characterization of semisimple and projective modules, 4. is also equivalent with conditions 1., 2. and 3.

Remark. For any l.c. semisimple ring, its discrete left ideals are projective, injective, and semisimple, but the converse is false. For let P denote the ring of all p-adic integers. If $\{p^nP, n = 1, 2, \ldots\}$ is considered as an open base for a neighborhood system of 0 in P, then $\{p^nP, n = 1, 2, \ldots\}$ induces a l.c. (moreover, a s.l.c.) topology on P. As is easy to see, pP is the radical of P. On the other hand, every non-zero ideal of P has the form p^nP for some $n \geqq 0$. Hence 0 is the unique discrete ideal of P. 0 is trivially projective, injective, and semisimple, but P is not semisimple.

It is a classical result that an artinian ring is semisimple if and only if all unitary modules over it are injective. Now we can ask whether a l.c. ring is semisimple if and only if every discrete module over it is injective. The answer is negative. There are l.c. semisimple rings such that they have ideals which are non-injective discrete modules with the discrete topology. The reason is that a discrete direct sum of injective modules is not necessarily injective. Nevertheless this is valid for projective modules.

The following theorem determines the role of artinian semisimple rings in the class of l.c. rings and may be regarded as a stronger assertion of the above mentioned result.

Theorem 2.2. *A l.c. ring R is an artinian semisimple ring if and only if every discrete R-module is injective.*

Proof. The necessity is well known. Conversely, assume that every discrete R-module is injective. First we prove that R is semisimple. Since every submodule N of a discrete R-module M is also discrete, it is injective by assumption. This shows that N is a direct summand of M, i.e. M is semisimple. So R is semisimple according to Theorem 2.1. On account of (1) R is a complete direct sum $\sum_{\alpha \in \Gamma} R_\alpha$ of rings R_α of linear transformations of vector spaces V_α over division rings. We must exhibit that, on the one hand, each V_α is finite dimensional and, on the other hand, Γ is a finite set. If V_α is infinite dimensional, the left socle S of R_α is a proper, dense ideal of R_α. It is easy to see that S, endowed with the discrete topology, which is different from the relative topology induced by R_α on S, is a topological R-module. By assumption S is injective, therefore S is a direct summand of R_α. Since R_α is unitary, S has a right unit element. This shows that S is a closed ideal of R_α, which is inconsistent with S being a dense ideal of R_α. So each V_α is finite dimensional, i.e. each R_α is an artinian simple ring. If Γ is infinite, then the discrete direct sum R^* of the R_α's is a proper dense ideal of R, which is a topological R-module with the discrete topology, as is easy to verify. Hence R^* is injective, therefore R^* is a direct summand of R, which is a contradiction. This completes the proof.

Since a complete direct sum of modules is injective if and only if each summand is injective, the following theorem is true.

Theorem 2.3. *A l.c. ring is semisimple if and only if all l.c. modules over it are injective.*

Proof. Necessity: By (1) a l.c. semisimple ring R is a complete direct sum of rings R_α of linear transformations of vector spaces V_α over division rings. On the other hand, each ring R_α is an injective left R_α-module by [9, Corollary 1.5, pp. 246]. It is interesting to note that if V_α is infinite dimensional, then R_α is not an injective right R_α-module. By [9, Corollary 1.5, pp. 246] R is also an injective left R-module. By (2) every l.c. module is a complete direct sum of its minimal submodules. As we have seen, every simple R-module is isomorphic to a minimal left ideal, therefore to a direct summand of R. So every l.c. module is injective.
Sufficiency: Since every closed left ideal of R is a l.c. R-module, it is injective, therefore it is a direct summand of R. This implies that every closed left ideal of R has a right unit element, so R is semisimple on account of (3). This ends the proof of Theorem 2.3.

Corollary 2.4. *A l.c. ring R is an artinian semisimple ring if and only if all exact sequences $0 \to A \to B \to C \to 0$ of R-modules where A is a discrete R-module, split.*

Corollary 2.5. *A l.c. ring R is semisimple if and only if all exact sequences $0 \to A \to B \to C \to 0$ of R-modules where A is a l.c. module, split.*

GOLDMAN and SAH [3] have introduced the notion of topological injective modules. We denote by R an arbitrary topological ring. A left R-module Q will be called *injective* if it has the following property: if L is an open submodule of a topological R-module M and $f: L \to Q$ is a continuous R-homomorphism, then f extends to an R-homomorphism from M to Q, which is automatically continuous.

The following theorem can be regarded as a topological analogon of the well-known characterization of artinian semisimple rings stating that a ring is a semisimple artinian ring if and only if each module over this ring is injective.

Theorem 2.6. *Let R be a complete, l.t. ring. Then R is a l.c. semisimple ring if and only if every topological R-module Q is injective.*

Proof. Assume that R is a l.c. semisimple ring. Let Q be an arbitrary topological R-module. Let L be any open submodule of a topological R-module M and let f denote a continuous R-homomorphism from L into Q. Since L is open, M/L is a discrete R-module. Hence M/L is projective according to Theorem 2.1. By a well-known result about projective modules stating that all exact sequences $0 \to A' \to A \to P \to 0$ split for a projective module P, we obtain a decomposition $M = L \oplus M/L$. If we put $\bar{f}(x + y) = f(x)$ for all $x \in L$ and $y \in R/L$, then $\bar{f}$ is a continuous extension of f from M into Q, i.e. Q is an injective module.
For the sufficiency of the theorem we prove first that R is s.l.c., hence l.c. Let $\{L_\mu\}$ be a basis of neighborhoods of 0 consisting of left ideals in R. By the assumption L_μ is a topological injective R-module. Since L_μ is an open left ideal in R, L_μ is a direct summand of R by Theorem 3.2 [3], i.e. $R = L_\mu \oplus K_\mu$, $L_\mu \cap K_\mu = 0$. This implies that K_μ is a discrete R-module. Since every submodule of K_μ is again injective and open in K_μ, it is a direct summand of K_μ according to Theorem 3.2 [3]. Therefore K_μ is semisimple. As K_μ is a direct summand of R, it is generated by an idempotent, i.e. $K_\mu = Re_\mu$ holds. Now a finitely generated semisimple module is a direct sum of a finite number of simple submodules, from which we easily deduce the descending

chain condition on submodules of K_μ. This shows that R is l.c. Now let J denote the radical of R. Since R/L_μ is a direct summand of a finite number of simple submodules, L_μ is an intersection of a finite number of maximal regular left ideals in R. Thus by the definition of the Jacobson radical, J is contained in any open left ideal L_μ, which implies $J \subseteqq \bigcap L_\mu = 0$, therefore R is semisimple.

The concept of semisimple modules in a topological context was generalized by GOLDMAN and SAH [3] as follows. Let R be an arbitrary topological ring and N a topological R-module. We shall say that N is *topologically semisimple* if for every submodule M, closed or not, of N there is a submodule K such that $M \cap K = 0$ and $M + K$ is everywhere dense. We shall say that R is (left) *topologically semisimple* if every left R-module is topologically semisimple. It holds namely:

Theorem 2.7. *A complete, l.t. ring R is l.c. and semisimple if and only if R is topologically semisimple.*

Proof. Assume that R is l.c. and semisimple. Let N denote an arbitrary submodule, closed or not, of a topological R-module M. A trivial application of Zorn's lemma implies the existence of a submodule K maximal with respect to $N \cap K = 0$. If we set $E = N + K$, the maximality of K shows that E has a non-zero intersection with every non-zero submodule of M. We must show that E is everywhere dense. Let $\bar{E}$ denote the closure of E. For any element x of M denote by A the left ideal $\{a \in R \mid ax \in \bar{E}\}$. Then A is closed in R. Because R is semisimple, the left ideal A has a right unit element e. Consider the submodule $L = R(1 - e)x$ of M. If $L = 0$, then $1 - e \in A$, so that $A = R$ and $x \in \bar{E}$. On the other hand, if $b(1 - e)x \in L \cap \bar{E}$, then $b(1 - e) \in A = Re$, which clearly shows that $b(1 - e) = 0$. Thus $L \cap \bar{E} = 0$, hence $L = 0$ and $x \in \bar{E}$.

Conversely, if R is topologically semisimple, then every left R-module is injective according to Proposition 7.6 [3]. The previous theorem gives that R is l.c. and semisimple.

Proposition 2.8. *Let R be any l.c. ring and M be any topological R-module with an open submodule N. Then M/N is semisimple if and only if $JM \subseteqq N$.*

Proof. If $JM \subseteqq N$, then M/N as an R-module behaves just as it does as an R/J-module. For if $\alpha_1 \equiv \alpha_2 \bmod J$, $\alpha_1 + \eta = \alpha_2$, $\eta \in J$, then for any element x of M, $\alpha_2 x = \alpha_1 x + \eta x \equiv \alpha_1 x \bmod N$, that is, α_1 and α_2 produce the same operation on M/N. But the discrete R/J-module M/N is semisimple according to Theorem 2.1, since R/J is a l.c. semisimple ring.

For the second part of the proposition, suppose M/N is semisimple, then M/N is a discrete direct sum of simple submodules. Thus the verification of $JM \subseteqq N$ reduces to its verification in case M/N is simple, which is obvious.

Now let M be any discrete module over a s.l.c. ring R. We define for each ordinal μ the submodule ${}^\mu M$ of M. Let ${}^0 M$ be the trivial submodule of M, consisting of the zero element. If a submodule ${}^\mu M$ is defined, then let ${}^{\mu+1}M$ be the sum of all minimal submodules of M containing ${}^\mu M$. If ${}^\mu M$ is defined for all ordinals $\mu < \lambda$ where λ is a limit ordinal, then let ${}^\lambda M = \bigcup_{\mu<\lambda} {}^\mu M$. The smallest ordinal σ with ${}^\sigma M = M$ is called the *layer number* of M. We define the layer number of an arbitrary complete, l.t. module as the least upper bound of the layer numbers of its discrete factor modules.

Theorem 2.9. *Let R be a s.l.c. ring with radical J. If M is a discrete, faithful R-module, then we have for all ordinals μ:*

$$ {}^\mu M = \{x \mid J_\mu x = 0\} \qquad (J_0 = R). $$

Proof. Put $M_\mu = \{x \mid J_\mu x = 0\}$. The statement is valid for $\mu = 0$. Assume its validity for $\varkappa$. From $J \cdot {}^{\varkappa+1}M \subseteqq {}^{\varkappa}M$ it follows by ([7], (1.12)) that

$$J_{\varkappa+1} \cdot {}^{\varkappa+1}M = J_\varkappa \cdot J^{\varkappa+1}M \subseteqq J_\varkappa \cdot {}^{\varkappa}M = 0,$$

hence ${}^{\varkappa+1}M \subseteq M_{\varkappa+1}$. On the other hand, $M_{\varkappa+1}$ is also contained in ${}^{\varkappa+1}M$, because from $J(M_{\varkappa+1}/{}^{\varkappa}M) = 0$ it follows by Proposition 2.8 that $M_{\varkappa+1}$ modulo ${}^{\varkappa}M$ is a sum of its simple submodules. If now ${}^{\mu}M = M_\mu$ for all ordinals $\mu < \lambda$ where λ is a limit ordinal, then $J_\lambda{}^{\lambda}M = J_\lambda(\bigcup {}^{\mu}M) = 0$, therefore ${}^{\lambda}M \subseteqq M_\lambda$. Let x be any element of M_λ, then $J_\lambda x = \left(\bigcap\limits_{\mu<\lambda} J_\mu\right) x = 0$ holds. This implies that $\bigcap\limits_{\mu<\lambda} J_\mu$ is contained in the annihilator of x, which is an open left ideal A of R since M is discrete. From (5) we have

$$A = A + \left(\bigcap_{\mu<\lambda} J_\mu\right) = \bigcap_{\mu<\lambda} (A + J_\mu).$$

Since R is s.l.c., $J_\mu + A = A$ holds for some $\mu < \lambda$, i.e. $x \in M_\mu = {}^{\mu}M \subseteqq {}^{\lambda}M$. Therefore ${}^{\lambda}M = M_\lambda$ holds also here and the proof is complete.

Let R be a s.l.c. ring with radical J. Then J is transfinitely r-nilpotent. The smallest ordinal τ with $J_\tau = 0$ is said to be the *index* of J and R, respectively.

Corollary 2.10. *Let R be a s.l.c. ring and M be a faithful discrete R-module. Then the index of R is equal with the layer number of M.*

Theorem 2.11. *Let R be any s.l.c. ring and M be any complete, l.t. faithful R-module. Then the index of R is equal to the layer number of M.*

Proof. If M is a discrete R-module, the statement follows from Corollary 2.10. Let τ be the index of R, J be the radical of R and σ be the layer number of M. If A is an arbitrary closed, two-sided ideal of R, then it is easy to see that $(J + A/A)_\mu = J_\mu + A/A$ holds for all ordinals μ. For any open submodule L of M, M/L is a faithful R/C-module where $C = C_L$ is the annihilator of L. Theorem 2.9 implies that the layer number of M/L is at most equal to τ since $J + C/C$ is the radical of R/C according to [6, Satz 14] and $(J + C/C)_\tau = J_\tau + C/C = C/C = 0$ and therefore $\sigma \leqq \tau$ is valid. Conversely, it follows from Theorem 2.9 that $(J + C/C)_\sigma = 0$, hence $J_\sigma + C = C$, i.e. $J_\sigma \subseteqq C$. This shows that $J_\sigma \subseteqq \bigcap\limits_{L} C_L = 0$, $J_\sigma = 0$, $\tau \leqq \sigma$. So the proof of Theorem 2.11 is complete.

Remark. We can construct for any ordinal μ another subspace M^μ which can be regarded as the dual of ${}^{\mu}M$. Let $M^0 = M$ and assume that we have constructed M^μ, then let $M^{\mu+1}$ be such a submodule of M^μ that $M^\mu/M^{\mu+1}$ is maximal in the set of the semisimple R-factor modules of M^μ. If all M^μ are constructed for all ordinals $\mu < \lambda$ where λ is a limit ordinal, then let $M^\lambda = \bigcap\limits_{\mu<\lambda} M^\mu$. It follows from this definition that these submodules are unique. For, if $\mu = 0$, the statement is trivial. Assume that the statement is true for the ordinal μ, then if K and K' are two submodules of M^μ such that M^μ/K and M^μ/K' are maximal semisimple modules, then $M^\mu/K \cap K'$ is also semisimple, since by Proposition 2.8, $JM^\mu \subseteqq K'$ and $JM^\mu \subseteqq K$, hence $JM^\mu \subseteqq K \cap K'$. But $M^\mu/K \cap K' \supseteqq M^\mu/K$, and as M^μ/K is maximal, the equality must hold, and this implies $K' \subseteqq K$. Similarly, $K \subseteqq K'$, i.e. $K = K'$. If the M^μ are unique for all $\mu < \lambda$ where λ is a limit ordinal, then $M^\lambda = \bigcap\limits_{\mu<\lambda} M^\mu$ is obviously unique. For ${}^{\mu}M$ we have proved that ${}^{\mu}M = \{x \in M \mid J_\mu x = 0\}$ if M is a faithful module, but it is not true for its dual that $M^\mu = J_\mu M$ as in the case of artinian rings. For this aim consider the ring P of p-adic integers with p-adic topology, which is the full endomorphism ring of the quasicyclic group $C(p^\infty)$. Then $C(p^\infty)$ may be regarded as a faithful discrete

l.c. (moreover s.l.c.) P-module. As is well-known, $\{P_n = p^n P,\ n = 1, 2, \dots\}$ are all the non-zero ideals of P, $\bigcap_{n=1}^{\infty} P_n = 0$, P_1 is the radical of P.
Furthermore $P_n C(p^\infty) = C(p^\infty)$ for all integers $n = 1, 2, \dots$ This implies $M^1 = M^2 = \dots = M^\omega = \bigcap_{n=1}^{\infty} M^n = C(p^\infty)$. But $J_\omega C(p^\infty) = \left(\bigcap_{n=1}^{\infty} P_n\right) C(p^\infty) = 0$, i.e. $M^\omega \neq J_\omega \cdot M$.

As an application of Theorem 2.1 we prove the following theorem about the semi-simplicity of endomorphism rings of abelian groups.

Theorem 2.12. *The endomorphism ring of an abelian group G is a l.c. semisimple ring with the natural topology (i.e. the finite topology) if and only if G has the form*

$$G = \sum^{\oplus} \mathfrak{R}_0 \oplus \sum_{\mathfrak{p}_i}^{\oplus} \sum^{\oplus} C(\mathfrak{p}_i) \tag{1}$$

where $\mathfrak{R}_0$ is the additive group of rational numbers and $C(\mathfrak{p}_i)$ is a cyclic group of prime order $\mathfrak{p}_i$.

Proof. Assume that G has the form (1). Then the endomorphism ring of G is a complete direct sum of the endomorphism rings of $\sum^{\oplus} \mathfrak{R}_0$ and $\sum^{\oplus} C(\mathfrak{p}_i)$, respectively. The abelian groups $\sum^{\oplus} \mathfrak{R}_0$ and $\sum^{\oplus} C(\mathfrak{p}_i)$ are vector spaces over the prime fields K and $K_{\mathfrak{p}_i}$ of characteristics 0 and $\mathfrak{p}_i$, respectively. The endomorphism rings of $\sum^{\oplus} \mathfrak{R}_0$ and $\sum^{\oplus} C(\mathfrak{p}_i)$ consist of all linear transformations of these vector spaces, respectively. Hence they are l.c. and semisimple, and therefore the endomorphism ring of G is l.c. and semisimple.
Conversely, assume that the endomorphism ring R of G is l.c. and semisimple with the finite topology. Then G as an R-module is discrete. According to Theorem 2.1, G is a discrete direct sum of simple submodules G_i. So G_i is a faithful simple R/C_i-module where C_i is the annihilator of G_i. Let Γ_i denote the R/C_i-endomorphism ring of G_i. By Schur's lemma Γ_i is a division ring. Then G_i may be regarded as a vector space over the division ring Γ_i for every i. From this it follows that G has the form (1), since the additive group of a division ring is of the form $\sum^{\oplus} C(\mathfrak{p}_i)$ or $\sum^{\oplus} \mathfrak{R}_0$.

§ 3. Strictly linearly compact rings

Two natural problems arising in the theory of topological rings are to determine the consequences of each of the following conditions:

1. every ideal (or left ideal) is closed,
2. every non-zero ideal (or non-zero left ideal) is open.

Wagner [10] has solved these problems for compact rings (by definition, a topological ring is always a Hausdorff space). Here we answer them for s.l.c. rings such that $\bigcap_{n=1}^{\infty} J^n = 0$ where J is the radical. For our considerations we need some preparations first.

Proposition 3.1. *Let M be a l.c. module. Then the topological space M is a Baire space.*

Proof. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of closed sets whose union is a linearly compact module M. Assuming that no M_n contains a non-void open set, we shall derive a contradiction. Let $\{L_\mu,\ \mu \in \Gamma\}$ be a basis of neighborhoods of 0 consisting of open submodules L_μ of M. Since $M - M_1$ is non-void and open, it contains a coset $x_1 + L_{\mu_1}$ for some index $\mu_1 \in \Gamma$ where x_1 is an arbitrary point of $M - M_1$. Since M_2 has no interior point, the intersection $(x_1 + L_{\mu_1}) \cap (M - M_2)$ is not empty, i.e. there exist a point x_2 and an index μ_2 such that $x_1 + L_{\mu_1}$ contains $x_2 + L_{\mu_2}$ and the intersection $(x_2 + L_{\mu_2}) \cap M_2$ is vacuous. By repeating the same argument we obtain a sequence $\{x_n + L_{\mu_n}\}$ of cosets of open (and hence closed) submodules L_{μ_n} with the properties:

$$x_{n+1} + L_{\mu_{n+1}} \subseteqq x_n + L_{\mu_n}, \qquad (x_n + L_{\mu_n}) \cap M_n = \emptyset.$$

By the linear compactness of M there exists at least one point x for which x lies in $x_n + L_{\mu_n}$ for every n. Hence x is in none of the sets M_n, therefore also not in the union $\bigcup_{n=1}^{\infty} M_n = M$, contrary to $x \in M$.

Proposition 3.2. *Let R be a s.l.c. ring with radical J satisfying $\bigcap \overline{J^n} = 0$ where $\overline{J^n}$ is the closure of J^n. If $\bigcup_{n=1}^{\infty} L_n$ is closed whenever $(L_n)_{n=1}^{\infty}$ is an increasing sequence of closed (left) ideals L_n in R, then there exists a natural number k with $L_n = L_k$ for all $n \geqq k$, i.e. this sequence breaks off.*

Proof. Let $L = \bigcup_{n=1}^{\infty} L_n$. Since L is closed, L is l.c. as an R-module, whence L is a Baire space. As each L_n is closed in L, therefore L_{k^*} has an interior point for the relative topology of L for some $k^* \geqq 1$. Consequently, L_{k^*} is open for the relative topology of L, so L_n is open in L for all $n \geqq k^*$. There exists also an open left ideal U in R such that $L \cap U = L_{k^*}$, hence $L_{k^*} \subseteqq L_n \cap U \subseteqq L \cap U \subseteqq L_{k^*}$ for all $n \geqq k^*$. By the second isomorphism theorem we have

$$L_n + U/U \cong L_n/L_n \cap U = L_n/L_{k^*}$$

as R-modules for all $n \geqq k^*$. By (6) R/U satisfies the ascending chain condition, hence L_n/L_{k^*} gets stationary at some k and the proof is complete.

First we shall determine the topology of s.l.c. rings all of whose ideals are closed, provided the rings satisfy an additional condition on their radicals.

Theorem 3.3. *Let R be a s.l.c. ring with radical J satisfying $\bigcap_{n=1}^{\infty} \overline{J^n} = 0$ where $\overline{J^n}$ is the closure of J^n. Every ideal of R is closed if and only if R satisfies the ascending chain condition on ideals and every principal ideal of R is closed. In this case R/J is a semisimple artinian ring, hence J is open.*

Proof. The necessity of the condition is trivial by Proposition 3.2. Conversely, suppose I were a non-closed ideal of R. Then for any $a_1, \ldots, a_n \in I$, I would contain the sum $(a_1) + \cdots + (a_n)$ of the principal ideals $(a_1), \ldots, (a_n)$ properly, as this sum is closed. Hence there would exist a strictly increasing sequence of ideals contained in R, a contradiction.

Henceforth we assume that R possesses the equivalent properties mentioned. Every ideal in R/J is the range of an ideal in R under the canonical homomorphism from R onto R/J, hence it is closed. Since R/J is a semisimple l.c. ring, by (1) it is a complete direct sum of rings of linear transformations of vector spaces over division rings. Now on the one hand, R/J satisfies the ascending chain condition on ideals, hence there can be only a finite number of components in this direct sum, and on the other hand, all these vector spaces must be of finite dimension, because the socle (i.e. the sum of all minimal left ideals) of the ring P of all linear transformations on an infinite dimensional vector space over a division ring is a proper dense ideal in P, contrary to our assumption that R, hence R/J, contains closed ideals only. All this proves that R/J is a semisimple artinian ring, which implies that J is open.

Theorem 3.4. *Let R be a topological ring with radical J satisfying $\bigcap_{n=1}^{\infty} \overline{J^n} = 0$, where $\overline{J^n}$ is the closure of J^n. The following conditions are equivalent:*

1. *R is s.l.c. and every left ideal of R is closed,*
2. *R is s.l.c. and noetherian,*
3. *R is s.l.c. and satisfies the ascending chain condition on closed left ideals,*
4. *R is noetherian, the topology of R is the J-adic topology, R is complete for this topology, and R/J is a semisimple artinian ring.*

Proof. By Proposition 3.2 condition 1. implies condition 2. Condition 2. clearly implies condition 3.

3. $\Rightarrow$ 4.: By Proposition 3.2, R is a noetherian ring. By Theorem 3.3 R/J is a semisimple artinian ring, hence J is open in R, because every ideal in R is closed. The rings J^n/J^{n+1}, $n = 1, 2, \ldots$ may be regarded as left modules over the semisimple artinian ring R/J. In fact, $a_1, a_2 \in R$, $a_1 - a_2 \in J$ implies $a_1 u - a_2 u \in J^{n+1}$ for arbitrary $u \in J^n$, i.e. a_1 and a_2 have the same effect on the elements of J^n/J^{n+1}. So a subgroup G of J^n/J^{n+1} is an R-submodule of J^n/J^{n+1} if and only if G is an R/J-submodule of J^n/J^{n+1}. From this it follows that J^n/J^{n+1} satisfies the ascending chain condition on closed R/J-submodules. By (2) J^n/J^{n+1} is the complete direct sum of simple R/J-submodules with the product topology. Hence there can be only a finite number of components in this direct sum. This shows that the topology of J^n/J^{n+1} is discrete. Hence J^{n+1} is open for the relative topology of J^n. Since J is open in R, J^n is open in R for all $n \geqq 1$.
To show that the topology of R is the J-adic topology, it remains to prove that if L is an open left ideal, there exists an $n \geqq 1$ such that $J^n \subseteqq L$. As L is open and R is s.l.c., the R-module R/L satisfies the minimum condition on submodules, hence there exists a natural number n with $J^n + L = J^k + L$ for all $k \geqq n$. Now (5) implies $L = \bigcap_{i=1}^{\infty} J^i + L = \bigcap_{i=1}^{\infty} (J^i + L) = J^n + L$ and thus $J^n \subseteqq L$.

4. $\Rightarrow$ 1.: We prove first that R is s.l.c. as a left R-module. For this end it suffices to see that R/J^n as an R-module satisfies the minimum condition on submodules, since the topology of R is complete. For $n = 1$ the assertion is trivial, because R/J is a semisimple artinian ring. Assume that R/J^n satisfies the minimum condition on submodules. Since $(R/J^{n+1})/(J^n/J^{n+1}) \cong R/J^n$, we must prove only that J^n/J^{n+1} is an artinian R-module. J^n/J^{n+1} may be regarded as a left module over R/J. By a known result stating that each module over a semisimple artinian ring splits into the direct sum of minimal modules, we obtain a decomposition $J^n/J^{n+1} = L_1 + \cdots + L_k$ where the L_i, $i = 1, \ldots, k$, are simple R/J-modules. The finiteness of k is a consequence of the fact that the maximum condition holds in J^n/J^{n+1}. This implies that J^n/J^{n+1} satisfies the minimum condition on submodules, i.e. R is s.l.c.
It remains to show that every left ideal of R is closed. Let L be any left ideal in R. Since R is noetherian, $L = Ra_1 + \cdots + Ra_n$ with some elements $a_1, \ldots, a_n \in R$. The R-module L is the image under the continuous homomorphism $(x_1, \ldots, x_n) \to a_1 x_1 + \cdots + a_n x_n$ from the R-module R^n into R, and R^n is s.l.c.; therefore the R-module L is s.l.c., hence complete, and thus closed in R. This ends the proof of Theorem 3.4.

Theorem 3.5. *If R is a s.l.c. ring satisfying $\bigcap_{n=1}^{\infty} \overline{J^n} = 0$ where $\overline{J^n}$ is the closure of J^n, then every non-zero ideal of R is open if and only if every ideal of R is closed, the topology of R is the J-adic topology, and every non-zero prime ideal of R is a primitive ideal.*

Proof. Necessity: It is clear that every ideal of R is closed. By assumption, J^n is open for every n. To show that the topology of R is the J-adic one, it is sufficient to exhibit that if L is an open left ideal, there is an $n \geqq 1$ such that $J^n \subseteq L$. As L is open and R is s.l.c., the R-module R/L satisfies the minimum condition on submodules, hence there exists a natural number n with $J^n + L = J^k + L$ for all $k \geqq n$. Now (5) implies $L = \bigcap_{i=1}^{\infty} \overline{J^i} + L = \bigcap_{i=1}^{\infty} (J^i + L) = J^n + L$ and thus $J^n \subseteq L$.

Let P be a non-zero prime ideal. Then P is open, so $P \supseteq J^n$ for some $n \geqq 1$. But a prime ideal containing a product of ideals contains one of them, so $J \subseteq P$. It is then easy to verify that P/J is a proper prime ideal of R/J. Since R/J is a semisimple artinian ring by Theorem 3.3, every proper prime ideal of R/J is primitive.

Sufficiency: By Theorem 3.3 R satisfies the ascending chain condition on ideals. The argumentation of [12, p. 200] can be slightly modified to show that every non-zero ideal of R contains a product of non-zero prime ideals. Since every non-zero prime ideal is primitive, it contains J. Therefore every non-zero ideal of R contains J^n for some $n \geqq 1$ and hence is open, for the topology of R is the J-adic topology.

Corollary 3.6. *Let R be a topological ring with $\bigcap_{n=1}^{\infty} \overline{J^n} = 0$. The following are equivalent:*

1. *R is a s.l.c. ring, every non-zero ideal of R is open and every left ideal is closed,*
2. *R is a noetherian ring, the topology is the J-adic topology, R is complete for this topology, R/J is a semisimple artinian ring, and every non-zero proper prime ideal of R is a primitive ideal.*

Now let R be any s.l.c. ring. If 0 is an open ideal, then the topology of R is discrete, hence R is an artinian ring. So we assume that the topology of R is not discrete and $\bigcap_{n=1}^{\infty} \overline{J^n} = 0$ holds. Then every non-zero left ideal of R is open if and only if every non-zero ideal of R is open and every non-zero left ideal contains a non-zero ideal: indeed, the condition is clearly sufficient, and it is necessary since every left ideal of R is open, the topology of R is the J-adic topology by Theorem 3.5, so a non-zero left ideal contains the ideal J^n for some $n \geqq 1$, and $J^n \neq 0$ as R is not discrete. Actually, much more can be said about s.l.c. rings in which every non-zero left ideal is open.

Theorem 3.7. *Let R be a non-discrete s.l.c. ring with $\bigcap_{n=1}^{\infty} \overline{J^n} = 0$. Every non-zero left ideal of R is open if and only if R is a noetherian ring without zero divisors, every non-zero left ideal contains a non-zero ideal, the topology is the complete J-adic topology, R/J is a division ring, and J is the only non-zero proper prime ideal of R.*

Proof. The condition is sufficient by Theorem 3.5 and the preceding remark.

Necessity: We shall first show that R has no zero divisors. Suppose $xy = 0$ where $x \neq 0$ and $y \neq 0$. The left annihilator of y would therefore be a non-zero left ideal and hence would contain J^n for some $n \geqq 1$ as the topology of R is the J-adic topology by Theorem 3.5. The right annihilator of J^n would consequently be a non-zero ideal and hence would contain J^k for some $k \geqq 1$. As $J^n \cdot J^k = 0, J^{n+k} = 0$, so the topology of R would be discrete, which is a contradiction.

R is a noetherian ring by Proposition 3.2. By (1) R/J is the complete direct sum of rings of linear transformations of vector spaces over division rings. As Proposition 1.1 shows, every idempotent $\bar{e}$ of R/J is the image of an idempotent of R under the canonical homomorphism from R onto R/J. Since R has no zero divisors, R/J has only two idempotents, 0 and 1. Consequently, R/J is a division ring. Therefore J is the only primitive ideal, hence by Theorem 3.5 it is also the only non-zero prime ideal.

Let f be a homomorphism from a topological R-module M into a topological R-module N. The *graph* of f is the set of all points in the product space $M \times N$ of the form $(x, f(x))$ with $x \in M$. The following proposition is the analogon of Banach's closed graph theorem for strictly linearly compact modules.

Proposition 3.8. *A homomorphism $f\colon M \to N$ where M and N are s.l.c. R-modules, is continuous if and only if its graph is closed.*

Proof. If f is continuous, then the homomorphism $x \to (x, f(x))$ is continuous and therefore the graph of f is closed, since M and N are s.l.c.

Conversely, if the graph G of f is closed, then G as an R-module is s.l.c. The map p: $G \to M$, $p((x, f(x))) = x$ of G onto M is one-to-one, linear, and continuous. Since G is s.l.c., this implies that the inverse p^{-1} is continuous. Thus $f = j \circ p^{-1}$ is continuous, where j is the projection $j\colon G \to N$ defined by $j((x, f(x))) = f(x)$.

§ 4. Projective modules

In what follows let R denote a s.l.c. ring with nilpotent radical J. We give some results on projective modules over such a ring, giving analogues of results of Morita — Kawada — Tachikawa [8].

Theorem 4.1. *Let P be a projective R-module with P/JP being a topological R-module with the discrete topology. Then P has the form $\sum_{\mu \in \Gamma}^{\oplus} Re_\mu$ where the e_μ are primitive idempotents in R.*

Proof. Since P/JP is a discrete R-module, P/JP may be regarded as a discrete R/J-module. By Theorem 2.1 we have

$$P/JP = \sum^{\oplus} Re_\mu / J(\sum^{\oplus} Re_\mu).$$

Let us consider the module $\sum^{\oplus} Re_\mu$. Since the Re_μ are direct summands of a free R-module, therefore $\sum^{\oplus} Re_\mu$ is projective. As P is projective, there exists a homomorphism $f\colon P \to \sum^{\oplus} Re_\mu$ with $j_2 \circ f = j_1$ where $j_1\colon P \to P/JP$ and j_2: $\sum^{\oplus} Re_\mu \to \sum^{\oplus} Re_\mu / J(\sum^{\oplus} Re_\mu)$ are canonical homomorphisms:

$$\begin{array}{ccc} P & \xrightarrow{j_1} & P/JP \\ \downarrow f & & \| \\ \sum^{\oplus} Re_\mu & \xrightarrow[j_2]{} & \sum^{\oplus} Re_\mu/J(\sum^{\oplus} Re_\mu). \end{array}$$

This shows that $\sum^{\oplus} Re_\mu = \operatorname{im} f + J \cdot \sum^{\oplus} Re_\mu$. This implies further that

$$\sum^{\oplus} Re_\mu = \operatorname{im} f + J^n \cdot \sum Re_\mu, \qquad n = 1, 2, \ldots,$$

since

$$\sum^{\oplus} Re_\mu = \operatorname{im} f + J(\operatorname{im} f + J \cdot \sum^{\oplus} Re_\mu) = \operatorname{im} f + J^2 \cdot \sum^{\oplus} Re_\mu.$$

Therefore f is an epimorphism, since J is nilpotent. Moreover we have $P = \ker f \oplus P'$ and $f'\colon P' \to \sum^{\oplus} Re_\mu$ is an isomorphism where $f' = f/P'$. If we put $\varphi(x) = f'^{-1}(x)$ for every $x \in \sum^{\oplus} Re_\mu$, then we have a homomorphism $\varphi: \sum^{\oplus} Re_\mu \to P$ with $j_1 \circ \varphi = j_2$. This shows that $J \cdot \sum Re_\mu \neq \operatorname{im} \varphi = P$, hence φ is an epimorphism. Thus we have $\ker f = 0$ and f is an isomorphism, which completes the proof.

Theorem 4.2. *Let P be a projective R-module with P/JP being a discrete topological R-module and L be an arbitrary R-module. For a submodule L_0 of L we denote by Φ_0 the set of homomorphisms φ of P into L such that $\varphi(P) \subseteqq L_0$. If every simple R-module is isomorphic to a quotient module of P, then we have*

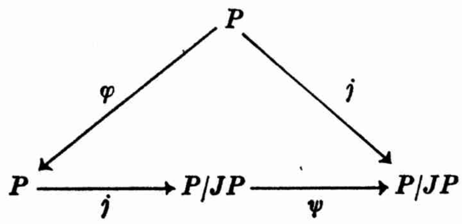
$$L_0 = \sum \varphi(P)$$

where φ ranges over the set Φ_0.

Proof. Let x_0 be any element of L_0 which is not contained in $\sum \varphi(P)$. Then we have $Rx_0 + \sum \varphi(P) = Jx_0 + \sum \varphi(P)$; otherwise we would have $Rx_0 + \sum \varphi(P) = J^i x_0 + \sum \varphi(P)$, $i = 1, 2, \ldots$ which is a contradiction, since J is nilpotent. Hence $Rx_0 + \sum \varphi(P)/Jx_0 + \sum \varphi(P)$ is by Theorem 2.1 a semisimple module in which the residue class $\{x_0\}$ containing x_0 is not zero. Since this module is finitely generated, it is a direct sum of finitely many simple submodules. By the assumption on P there exists an epimorphism ξ of P onto $Rx_0 + \sum \varphi(P)/Jx_0 + \sum \varphi(P)$. Hence there exists a homomorphism η of P into $Rx_0 + \sum \varphi(P)$ such that in the module $Rx_0 + \sum \varphi(P)/Jx_0 + \sum \varphi(P)$ the residue class $\{\eta(x_0)\}$ containing $\eta(x_0)$ is the residue class $\{x_0\}$ containing x_0. This implies that $x_0 - \eta(x_0) \in Jx_0 + \sum \varphi(P)$, i.e. $x_0 \in Jx_0 + \sum \varphi(P)$, contrary to the assumption that the residue class of x_0 is not zero. This proves that $L_0 \subseteqq \sum \varphi(P)$. Since by definition we have $\sum \varphi(P) \subseteqq L_0$, the desired relation is proved hereby.

Theorem 4.3. *Let P be a projective R-module and let Φ and Ψ be the endomorphism rings of P and P/JP, respectively. If we denote by Φ_J the set of all endomorphisms φ of P such that $\varphi(P) \subseteqq JP$, then Φ_J is a two-sided nilpotent ideal of Φ and the factor ring Φ/Φ_J is isomorphic to Ψ; more precisely, if $J^n = 0$, then $\Phi_J^n = 0$.*

Proof. Since P is projective, any endomorphism Ψ can be extended to an endomorphism φ of P with $\Psi \circ j \circ \varphi = j$ where j is the canonical homomorphism from P onto P/JP:



On the other hand, $\varphi(JP) \subseteqq JP$ for any $\varphi \in \Phi$. Therefore we have a ring isomorphism $\Phi/\Phi_J \simeq \Psi$ by the correspondence $\Phi \ni \varphi \to \tilde{\varphi} \in \Psi$ where $\tilde{\varphi}(x + JP) = \varphi(x) + JP$. Since $\varphi(P) \subseteqq JP$, $\varphi(J^i P) \subseteqq J^i \varphi(P) \subseteqq J^{i+1}P$ holds. This shows $\varphi(J^{n-1}P) = 0$ for every $\varphi \in \Phi_J$, because $J^n = 0$. Let φ_j, $j = 1, 2, \ldots, n$ be any elements of Φ_J. Then $\varphi_n(P), \ldots, \varphi_1(P) \subseteq J^n P = 0$. Thus we have $\Phi_J^n = 0$. This completes the proof of Theorem 4.3.

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