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On the equivalence of isometric surfaces in E^4

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Two isometric surfaces in E^4 are globally non-equivalent even under suitable conditions on the positiveness of the curvatures. To ensure the equivalence, we have to add a further condition, this condition being automatically satisfied for surfaces in E^3.

Let $G \subset \mathbf{R}^2$ be a bounded domain and $M: G \to E^4$ a surface in the euclidean 4-space E^4; let us write $M = M(G)$, $\partial M = M(\partial G)$ being the boundary of M. On M, consider a field of orthonormal frames $\{m; v_1, v_2, v_3, v_4\}$; $m \in M$; $v_1, v_2 \in T_m(M)$; $v_3, v_4 \in N_m(M)$. Here, $T(M)$ is the tangent and $N(M)$ the normal bundle of M resp. Then we may write

$$\begin{aligned} dm &= \omega^1 v_1 + \omega^2 v_2, \\ dv_1 &= \qquad\quad \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4, \\ dv_2 &= -\omega_1^2 v_1 \qquad\;\; + \omega_2^3 v_3 + \omega_2^4 v_4, \\ dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 \qquad\;\; + \omega_3^4 v_4, \\ dv_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3 \end{aligned} \tag{1}$$

with the integrability conditions

$$d\omega^i = \omega^j \wedge \omega_j^i, \qquad d\omega_i^j = \omega_i^k \wedge \omega_k^j, \tag{2}$$

where

$$\omega^3 = \omega^4 = 0, \qquad \omega_i^j + \omega_j^i = 0. \tag{3}$$

From $(3_{1,2})$, we get the existence of functions $a_1, \ldots, b_3: G \to \mathbf{R}$ such that

$$\begin{aligned} \omega_1^3 &= a_1\omega^1 + a_2\omega^2, \qquad \omega_1^4 = b_1\omega^1 + b_2\omega^2, \\ \omega_2^3 &= a_2\omega^1 + a_3\omega^2, \qquad \omega_2^4 = b_2\omega^1 + b_3\omega^2. \end{aligned} \tag{4}$$

The further differentiation yields the existence of functions $\alpha_1, \ldots, \beta_4: G \to \mathbf{R}$ such that

$$\begin{aligned} & da_1 - 2a_2\omega_1^2 - b_1\omega_3^4 = \alpha_1\omega^1 + \alpha_2\omega^2, && db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 = \beta_1\omega^1 + \beta_2\omega^2, \\ & da_2 + (a_1 - a_3)\,\omega_1^2 - b_2\omega_3^4 && db_2 + (b_1 - b_3)\,\omega_1^2 + a_2\omega_3^4 \\ & = \alpha_2\omega^1 + \alpha_3\omega^2, && = \beta_2\omega^1 + \beta_3\omega^2, \\ & da_3 + 2a_2\omega_1^2 - b_3\omega_3^4 = \alpha_3\omega^1 + \alpha_4\omega^2, && db_3 + 2b_2\omega_1^2 + a_3\omega_3^4 = \beta_3\omega^1 + \beta_4\omega^2. \end{aligned} \tag{5}$$

The first fundamental form of M is given by

$$I = \langle dm, dm \rangle = (\omega^1)^2 + (\omega^2)^2. \tag{6}$$

For $m \in M$, introduce the mapping

$$II_m : N_m(M) \times T_m(M) \to \mathbf{R}; \qquad II_m(n_0, t) = -\langle tn, tn \rangle; \tag{7}$$

here, $t \in T_m(M)$ and $n : M \to N(M)$ is a section (defined in a neighborhood of m) such that $n(m) = n_0$. Obviously,

$$\begin{aligned} &II_m(xv_3 + yv_4, \omega^1 v_1 + \omega^2 v_2) \\ &= (xa_1 + yb_1)(\omega^1)^2 + 2(xa_2 + yb_2)\omega^1\omega^2 + (xa_3 + yb_3)(\omega^2)^2. \end{aligned} \tag{8}$$

For the unit normal vector $n = xv_3 + yv_4 \in N(M)$, let us introduce the Gauss and mean curvature with respect to n by

$$\begin{aligned} K_n &= (xa_1 + yb_1)(xa_3 + yb_3) - (xa_2 + yb_2)^2, \\ 2H_n &= x(a_1 + a_3) + y(b_1 + b_3). \end{aligned} \tag{9}$$

resp.
Be given another surface $M^* : G \to E^4$; let us suppose that M and M^* are isometric. Then we may choose the orthonormal frames on M^* in such a way that

$$\begin{aligned} dm^* &= \omega^1 v_1^* + \omega^2 v_2^*, \\ dv_1^* &= (\omega_1^2 + \tau_1^2) v_2^* + (\omega_1^3 + \tau_1^3) v_3^* + (\omega_1^4 + \tau_1^4) v_4^*, \\ dv_2^* &= -(\omega_1^2 + \tau_1^2) v_1^* + (\omega_2^3 + \tau_2^3) v_3^* + (\omega_2^4 + \tau_2^4) v_4^*, \\ dv_3^* &= -(\omega_1^3 + \tau_1^3) v_1^* - (\omega_2^3 + \tau_2^3) v_2^* + (\omega_3^4 + \tau_3^4) v_4^*, \\ dv_4^* &= -(\omega_1^4 + \tau_1^4) v_1^* - (\omega_2^4 + \tau_2^4) v_2^* - (\omega_3^4 + \tau_3^4) v_3^*. \end{aligned} \tag{10}$$

From the integrability conditions, we get

$$\omega^2 \wedge \tau_1^2 = 0, \qquad \omega^1 \wedge \tau_1^2 = 0, \tag{11}$$

$$\omega^1 \wedge \tau_1^3 + \omega^2 \wedge \tau_2^3 = 0, \qquad \omega^1 \wedge \tau_1^4 + \omega^2 \wedge \tau_2^4 = 0, \tag{12}$$

$$d\tau_1^2 = -\omega_1^3 \wedge \tau_2^3 - \tau_1^3 \wedge \omega_2^3 - \tau_1^3 \wedge \tau_2^3 - \omega_1^4 \wedge \tau_2^4 - \tau_1^4 \wedge \omega_2^4 - \tau_1^4 \wedge \tau_2^4, \tag{13}$$

$$\begin{aligned} d\tau_1^3 &= \omega_1^2 \wedge \tau_2^3 + \tau_1^2 \wedge \omega_2^3 + \tau_1^2 \wedge \tau_2^3 - \omega_1^4 \wedge \tau_3^4 - \tau_1^4 \wedge \omega_3^4 - \tau_1^4 \wedge \tau_3^4, \\ d\tau_2^3 &= -\omega_1^2 \wedge \tau_1^3 - \tau_1^2 \wedge \omega_1^3 - \tau_1^2 \wedge \tau_1^3 - \omega_2^4 \wedge \tau_3^4 - \tau_2^4 \wedge \omega_3^4 - \tau_2^4 \wedge \tau_3^4, \\ d\tau_1^4 &= \omega_1^2 \wedge \tau_2^4 + \tau_1^2 \wedge \omega_2^4 + \tau_1^2 \wedge \tau_2^4 + \omega_1^3 \wedge \tau_3^4 + \tau_1^3 \wedge \omega_3^4 + \tau_1^3 \wedge \tau_3^4, \\ d\tau_2^4 &= -\omega_1^2 \wedge \tau_1^4 - \tau_1^2 \wedge \omega_1^4 - \tau_1^2 \wedge \tau_1^4 + \omega_2^3 \wedge \tau_3^4 + \tau_2^3 \wedge \omega_3^4 + \tau_2^3 \wedge \tau_3^4, \\ d\tau_3^4 &= -\omega_1^3 \wedge \tau_1^4 - \tau_1^3 \wedge \omega_1^4 - \tau_1^3 \wedge \tau_1^4 - \omega_2^3 \wedge \tau_2^4 - \tau_2^3 \wedge \omega_2^4 - \tau_2^3 \wedge \tau_2^4. \end{aligned} \tag{14}$$

From (11),

$$\tau_1^2 = 0, \tag{15}$$

and (13) reduces to

$$\omega_1^3 \wedge \tau_2^3 + \tau_1^3 \wedge \omega_2^3 + \tau_1^3 \wedge \tau_2^3 + \omega_1^4 \wedge \tau_2^4 + \tau_1^4 \wedge \omega_2^4 + \tau_1^4 \wedge \tau_2^4 = 0. \tag{16}$$

From (12), we get the existence of functions $R_1, \ldots, S_3 : G \to \mathbf{R}$ such that

$$\begin{aligned} &\tau_1^3 = R_1\omega^1 + R_2\omega^2, \ \tau_2^3 = R_2\omega^1 + R_3\omega^3; \\ &\tau_1^4 = S_1\omega^1 + S_2\omega^2, \ \tau_2^4 = S_2\omega^1 + S_3\omega^2; \end{aligned} \tag{17}$$

a further exterior differentiation yields the existence of functions $R_1^*, \ldots, S_4^* : G \to \mathbf{R}$ such that

$$
\begin{aligned}
&dR_1 - 2R_2\omega_1^2 - b_1\tau_3^4 - S_1(\omega_3^4 + \tau_3^4) = R_1^*\omega^1 + R_2^*\omega^2,\\
&dR_2 + (R_1 - R_3)\,\omega_1^2 - b_2\tau_3^4 - S_2(\omega_3^4 + \tau_3^4) = R_2^*\omega^1 + R_3^*\omega^2,\\
&dR_3 + 2R_2\omega_1^2 - b_3\tau_3^4 - S_3(\omega_3^4 + \tau_3^4) = R_3^*\omega^1 + R_4^*\omega^2,\\
&dS_1 - 2S_2\omega_1^2 + a_1\tau_3^4 + R_1(\omega_3^4 + \tau_3^4) = S_1^*\omega^1 + S_2^*\omega^2,\\
&dS_2 + (S_1 - S_3)\,\omega_1^2 + a_2\tau_3^4 + R_2(\omega_3^4 + \tau_3^4) = S_2^*\omega^1 + S_3^*\omega^2,\\
&dS_3 + 2S_2\omega_1^2 + a_3\tau_3^4 + R_3(\omega_3^4 + \tau_3^4) = S_3^*\omega^1 + S_4^*\omega^2.
\end{aligned}
\tag{18}
$$

The equation (16) takes the form

$$
a_3R_1 - 2a_2R_2 + a_1R_3 + R_1R_3 - R_2^2 + b_3S_1 - 2b_2S_2 + b_1S_3 + S_1S_3 - S_2^2 = 0. \tag{19}
$$

Let $g \in G$, $m_0 = M(g)$, $m_0^* = M^*(g)$. Further, let $I_g : E^4 \to E^4$ be an isometry satisfying

$$
I_g \circ (dM^*)_g = (dM)_g. \tag{20}
$$

The most general (so-called tangent) isometry is given by

$$
\begin{aligned}
&I_g(m_0^*) = m_0, \qquad I_g(v_1^*) = v_1, \qquad I_g(v_2^*) = v_2,\\
&I_g(v_3^*) = \varepsilon(\cos\alpha \, . \, v_3 - \sin\alpha \, . \, v_4), \qquad I_g(v_4^*) = \sin\alpha \, . \, v_3 + \cos\alpha \, . \, v_4;\\
&\varepsilon = \pm 1.
\end{aligned}
\tag{21}
$$

Let $m = m(s)$ be a curve on M; let s be its arc and let $m_0 = m(s_0)$. Denote by $m^* = m^*(s)$ the corresponding curve on M^*. Then

$$
I_g\left(\frac{dm^*(s_0)}{ds}\right) = \frac{dm(s_0)}{ds}. \tag{22}
$$

Further,

$$
I_g\left(\frac{d^2m^*(s_0)}{ds^2}\right) = \frac{d^2m(s_0)}{ds^2} + \left(\{(\omega^1)^2 + (\omega^2)^2\}^{-1} \, . \, \mathcal{L}_{I_g}\right)\left(\frac{dm(s_0)}{ds}\right) \tag{23}
$$

with

$$
\begin{aligned}
\mathcal{L}_{I_g} = \{&(\omega^1\omega_1^3 + \omega^2\omega_2^3)(\varepsilon\cos\alpha - 1) + (\omega^1\omega_1^4 + \omega^2\omega_2^4)\sin\alpha\\
&+ (\omega^1\tau_1^3 + \omega^2\tau_2^3)\,\varepsilon\cos\alpha + (\omega^1\tau_1^4 + \omega^2\tau_2^4)\sin\alpha\}\, v_3\\
&+ \{-(\omega^1\omega_1^3 + \omega^2\omega_2^3)\,\varepsilon\sin\alpha + (\omega^1\omega_1^4 + \omega^2\omega_2^4)(\cos\alpha - 1)\\
&\quad - (\omega^1\tau_1^3 + \omega^2\tau_2^3)\,\varepsilon\sin\alpha + (\omega^1\tau_1^4 + \omega^2\tau_2^4)\cos\alpha\}\, v_4.
\end{aligned}
\tag{24}
$$

Thus, for each tangent isometry I_g, we get the quadratic mapping

$$
\mathcal{L}_{I_g} : T_{m_0}(M) \to N_{m_0}(M), \tag{25}
$$

its geometrical interpretation being given by (23).

Theorem. *Let $M, M^* : G \to E^4$ be two isometric surfaces. Let there exist a section $n : M \to N(M)$ such that, for each $g \in G$, there is a tangent isometry I_g satisfying*

$$
\mathcal{L}_{I_g}\left(T_{M(g)}(M)\right) \subset n(g) \quad \text{on } G \tag{26}
$$

and

$$\mathcal{L}_{I_g}\big(T_{M(g)}(M)\big) = 0 \qquad on\ \partial G. \tag{27}$$

Further, suppose

$$K_n > 0,\ H_n \neq 0 \qquad on\ M; \qquad H_n + H^*_{I_g^{-1}(n)} \neq 0 \qquad on\ M^*. \tag{28}$$

Then M and M are equivalent.*

Proof. The frames may be chosen in such a way that

$$n = v_3; \qquad I_g(v_3^*) = v_3, \qquad I_g(v_4^*) = v_4. \tag{29}$$

Then

$$\mathcal{L}_{I_g} = \{R_1(\omega^1)^2 + 2R_2\omega^1\omega^2 + R_3(\omega^2)^2\}\, v_3 + \{S_1(\omega^1)^2 + 2S_2\omega^1\omega^2 + S_3(\omega^2)^2\}\, v_4. \tag{30}$$

Our suppositions may be rewritten as follows:

$$S_1 = S_2 = S_3 = 0 \qquad \text{on } G, \tag{31}$$

$$R_1 = R_2 = R_3 = 0 \qquad \text{on } \partial G, \tag{32}$$

$$a_1a_3 - a_2^2 > 0, \qquad a_1 + a_3 \neq 0, \qquad 2a_1 + 2a_3 + R_1 + R_3 \neq 0 \qquad \text{on } G. \tag{33}$$

Write (19) in the form

$$(2a_3 + R_3)\, R_1 - 2(2a_2 + R_2)\, R_2 + (2a_1 + R_1)\, R_3 = 0, \tag{34}$$

from this,

$$\begin{gathered} R_1 = A_1(R_1 - R_3) + A_2R_2, \qquad R_3 = A_3(R_1 - R_3) + A_2R_2; \\ A_1 := A(2a_1 + R_1), \qquad A_2 := 2A(2a_2 + R_2), \qquad A_3 := -A(2a_3 + R_3), \\ A := (2a_1 + 2a_3 + R_1 + R_3)^{-1}. \end{gathered} \tag{35}$$

Let us write

$$\omega_3^4 = c_1\omega^1 + c_2\omega^2, \qquad \tau_3^4 = x\omega^1 + y\omega^2. \tag{36}$$

From (18_{4-6}),

$$\begin{aligned} (a_2 + R_2)\, x - (a_1 + R_1)\, y &= c_2R_1 - c_1R_2, \\ (a_3 + R_3)\, x - (a_2 + R_2)\, y &= c_2R_2 - c_1R_3, \end{aligned} \tag{37}$$

i. e.,

$$\begin{gathered} x = B_1(R_1 - R_3) + B_2R_2, \qquad y = B_3(R_1 - R_3) + B_4R_2; \\ B_1 := -B\,\{c_2(a_2 + R_2)\, A_1 + c_1(a_1 + R_1)\, A_3\}, \\ B_2 := B\,\{(a_2 + R_2)\,(c_1 - c_2A_2) + (a_1 + R_1)\,(c_2 - c_1A_2)\}, \\ B_3 := -B\,\{c_2(a_3 + R_3)\, A_1 + c_1(a_2 + R_2)\, A_3\}, \\ B_4 := B\,\{(a_2 + R_2)\,(c_2 - c_1A_2) + (a_3 + R_3)\,(c_1 - c_2A_2)\}, \\ B := (a_1a_3 - a_2^2)^{-1}. \end{gathered} \tag{38}$$

From (18), it follows

$$\begin{gathered} d(R_1 - R_3) - 4R_2\omega_1^2 - (b_1 - b_3)\,\tau_3^4 = (R_1^* - R_3^*)\,\omega^1 + (R_2^* - R_4^*)\,\omega^2, \\ dR_2 + (R_1 - R_3)\,\omega_1^2 - b_2\tau_3^4 = R_2^*\omega^1 + R_3^*\omega^2. \end{gathered} \tag{39}$$

On M, choose isothermic coordinates (u, v) such that

$$I = r^2(du^2 + dr^2), \qquad r = r(u, v) > 0 \tag{40}$$

and the vector fields v_1, v_2 are chosen in such a way that

$$\omega^1 = r du, \qquad \omega^2 = r dv. \tag{41}$$

From $d\omega^1 = -\omega^2 \wedge \omega_1^2$, $d\omega^2 = \omega^1 \wedge \omega_1^2$, we get

$$\omega_1^2 = r^{-1}\left(-\frac{\partial r}{\partial v} du + \frac{\partial r}{\partial u} dv\right). \tag{42}$$

From (39),

$$\begin{aligned}
\frac{\partial(R_1 - R_3)}{\partial u} &= (R_1^* - R_3^*)\, r + C_1(R_1 - R_3) + C_2 R_2, \\
\frac{\partial(R_1 - R_3)}{\partial v} &= (R_2^* - R_4^*)\, r + C_3(R_1 - R_3) + C_4 R_2, \\
\frac{\partial R_2}{\partial u} &= R_2^* r + C_5(R_1 - R_3) + C_6 R_2, \\
\frac{\partial R_2}{\partial v} &= R_3^* r + C_7(R_1 - R_3)\, C_8 R_2;
\end{aligned}$$

$$\begin{aligned}
&C_1 := B_1(b_1 - b_3)\, r, && C_2 := B_2(b_1 - b_3)\, r - 4r^{-1}\frac{\partial r}{\partial v}, \\
&C_3 := B_3(b_1 - b_3)\, r, && C_4 := B_4(b_1 - b_3)\, r + 4r^{-1}\frac{\partial r}{\partial u}, \\
&C_5 := B_1 b_2 r + r^{-1}\frac{\partial r}{\partial v}, && C_6 := B_2 b_2 r, \\
&C_7 := B_3 b_2 r - r^{-1}\frac{\partial r}{\partial u}, && C_8 := B_4 b_2 r.
\end{aligned} \tag{43}$$

The differentiation of (34) yields

$$\begin{aligned}
a_3 R_1^* - 2a_2 R_2^* + a_1 R_3^* = &- (b_3 c_1 + \alpha_3 + b_3 x + R_3^*)\, R_1 \\
&+ 2(b_2 c_1 + \alpha_2 + b_2 x + R_2^*)\, R_2 \\
&- (b_1 c_1 + \alpha_1 + b_1 x + R_1^*)\, R_3 \\
&- (a_3 b_1 - 2a_2 b_2 + a_1 b_3)\, x, \\
a_3 R_2^* - 2a_2 R_3^* + a_1 R_4^* = &- (b_3 c_2 + \alpha_4 + b_3 y + R_4^*)\, R_1 \\
&+ 2(b_2 c_2 + \alpha_3 + b_2 y + R_3^*)\, R_2 \\
&- (b_1 c_2 + \alpha_2 + b_1 y + R_2^*)\, R_3 \\
&- (a_3 b_1 - 2a_2 b_2 + a_1 b_3)\, y.
\end{aligned} \tag{44}$$

Taking regard of (35) and (38), (44) may be written in the form

$$\begin{aligned}
a_3 R_1^* - 2a_2 R_2^* + a_1 R_3^* &= D_1(R_1 - R_3) + D_2 R_2, \\
a_3 R_2^* - 2a_2 R_3^* + a_1 R_4^* &= D_3(R_1 - R_3) + D_4 R_2.
\end{aligned} \tag{45}$$

From (43) and (45),

$$a_3 \frac{\partial(R_1 - R_3)}{\partial u} - 2a_2 \frac{\partial R_2}{\partial u} + (a_1 + a_3) \frac{\partial R_2}{\partial v}$$
$$= \{rD_1 + a_3(C_1 + C_7) - 2a_2C_5 + a_1C_7\} (R_1 - R_3)$$
$$+ \{rD_2 + a_3(C_2 + C_8) - 2a_2C_6 + a_1C_8\} R_2,$$
$$- a_1 \frac{\partial(R_1 - R_3)}{\partial v} + (a_1 + a_3) \frac{\partial R_2}{\partial u} - 2a_2 \frac{\partial R_2}{\partial v}$$
$$= \{rD_3 + a_3C_5 - 2a_2C_7 + a_1(C_5 - C_3)\} (R_1 - R_3)$$
$$+ \{rD_4 + a_3C_6 - 2a_2C_8 + a_1(C_6 - C_4)\} R_2. \tag{46}$$

Let us recall a well known theorem from the theory of elliptic systems of partial differential equations: Let the functions $f, g: G \to \mathbf{R}$ satisfy the system

$$a_{11} \frac{\partial f}{\partial u} + a_{12} \frac{\partial f}{\partial v} + b_{11} \frac{\partial g}{\partial u} + b_{12} \frac{\partial g}{\partial v} = c_{11}f + c_{12}g,$$
$$a_{21} \frac{\partial f}{\partial u} + a_{22} \frac{\partial f}{\partial v} + b_{21} \frac{\partial g}{\partial u} + b_{22} \frac{\partial g}{\partial v} = c_{21}f + c_{22}g; \tag{47}$$

$a_{11} = a_{11}(u, v), \ldots, c_{22} = c_{22}(u, v)$; let the form

$$\Phi = (a_{12}b_{22} - a_{22}b_{12}) \mu^2 - (a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11}) \mu\nu$$
$$+ (a_{11}b_{21} - a_{21}b_{11}) \nu^2 \tag{48}$$

be definite on G and $f = g = 0$ on ∂G; then $f = g = 0$ on G. Now, (46) is of the form (47), the associated form (48) being

$$\Phi = (a_1 + a_3)(a_1\mu^2 + 2a_2\mu\nu + a_3\nu^2); \tag{49}$$

it is definite because of (33). From (32), $R_1 - R_3 = R_2 = 0$ in G. Thus, (31) implies $\tau_1^3 = \tau_2^3 = 0$, (35) implies $\tau_1^4 = \tau_2^4 = 0$, and, finally, (36_2) and (38) imply $\tau_3^4 = 0$. QED.

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