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Formulas and ultraproducts in categories[1])

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1. Introduction

This work belongs to the field which is sometimes called "representation of formulas and their validity as purely category theoretical things". This field has recently been receiving an increasing amount of interest among algebraists and model theoreticians. See[2]) e.g. HERRLICH-RINGEL [10], BANASCHEWSKI-HERRLICH [3]. In this paper we extend the range of investigation from "implicational" classes to "universally axiomatisable" classes. A systematisation of kinds of axiomatisability is obtained.

In more detail: The well-known characterisations of the universal classes, quasivarieties, varieties and UDE-classes of (total) algebras are generalised to "almost every" category and to "almost every" possible concept of homomorphic image (H) and subalgebra (S). Special cases of the same category theoretical result are the following known theorems:

1. A class of universal algebras is hereditary and ultra-closed iff it is axiomatisable by universal formulas.

2. A class of universal algebras is closed w.r.t. homomorphic images, subalgebras and ultraproducts iff it is axiomatisable by "universal disjunctions of equations" (UDE's).

The category theoretical result is then applied to partial algebras and other kinds of structures.

In section 2 the category theoretical concepts used in the paper are summed up. Section 3 contains the main theorem. Examples and applications are given in section 5.

2. Basic definitions

Throughout the paper H, $\mathscr{H}$, S stand for classes of arrows (morphisms) of a category. They are closed w.r.t. isomorphisms.

Notation. Let M be a class of arrows, then $\xrightarrow{f\ M}$ means $f \in M$. (I.e. $\xrightarrow{S}$ stands for an S-arrow or $\xrightarrow{H}$ stands for an H-arrow.)

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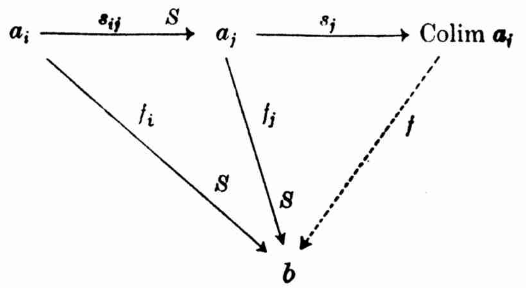
[1]) The results of this paper are taken from the dissertation of I. NÉMETI submitted in August 1976. The arrangement and some of the proofs are different here.

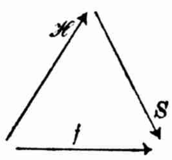
[2]) Some further works in this line are: SHAFAAT [21], HATCHER [9], JOHN [13], MATTHIESSEN [14], DIERS [4].

Definition. S is *inductive* if the directed union of S-subobjects is again an S-subobject.

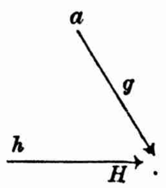
That is, S is inductive iff for any direct system $\left(a_i \xrightarrow{s_{ij}} a_j: i, j \in I\right)$ and cocone $\left(a_i \xrightarrow{f_i} b: i \in I\right)$ such that $s_{ij}, f_i \in S$ for every $i, j \in I$, also the induced map $\text{Colim } a_i \xrightarrow{f} b$ is in S. In diagram:



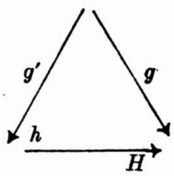
implies $f \in S$.
Recall that $(\mathscr{H}, S)$ is called a *factorisation system* iff every arrow has a unique $\mathscr{H}S$-decomposition and $\mathscr{H}$ and S are closed w.r.t. compositions. (I.e. for every $f \in$ Mor there is a unique



and $\mathscr{H} \cdot \mathscr{H} \subseteq \mathscr{H}$, $S \cdot S \subseteq S$.)
An object a is called *H-projective* (cf. MITCHELL [15], HERRLICH-STRECKER [11]) if to any



there exists an



(commuting of course).
The class of H-projective objects is denoted by $Pj(H)$.
The category *has enough H-projectives* (cf. MITCHELL [15]) if to any object a there exists an $a \xleftarrow{H} \cdot \in Pj(H)$.
Now we recall the definition of a strongly algebroidal category (cf. BANASCHEWSKI-HERRLICH [3], HERRLICH-STRECKER [11], 22E).

An object a is *s.small*[1]) if hom $(a, -)$ preserves direct limits. This means the following: An object a is s.small iff for any direct system $(h_{ij}: i, j \in I)$ with colimit $\left(\xrightarrow{h_i} b: i \in I\right)$, conditions (i) and (ii) below are satisfied.

(i) To any $a \xrightarrow{f} b$ there are $i \in I$ and g such that $gh_i = f$.

(ii) To any pair $a \underset{q}{\overset{p}{\rightrightarrows}}$ such that $ph_i = qh_i$ there exists $j \in I$ such that $ph_{ij} = qh_{ij}$.

A category is *s.algebroidal* if every object is a direct limit of some s.small objects. The adjective "algebroidal" was introduced in BANASCHEWSKI-HERRLICH [3] to refer to the analogy with algebraic lattices. Note that the concept of an s.small object is a generalisation of the concept of a compact element in a lattice.

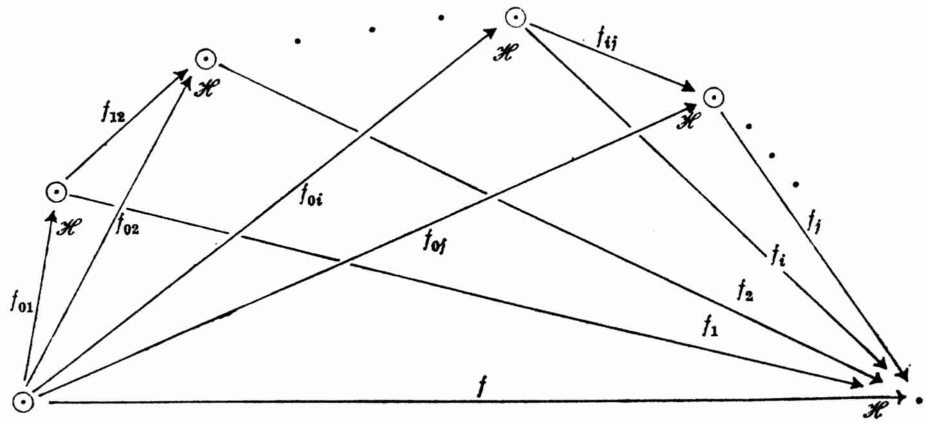
Notation. S.small objects will be denoted by $\odot$. (Thus $\odot \xrightarrow{f}$ or $\overset{a}{\odot}$ means that dom (f) is s.small or that a is s.small.)

Example. The category of relational structures (or models) of a fixed similarity type is s.algebroidal and the s.small objects are the "finite reducts" (of course every model is the directed union of its finite subreducts).

The following property means that the "$\mathscr{H}$-congruences" behave "inductively".

Definition. A category is *$\mathscr{H}$-arrow-algebroidal* if every $\odot \xrightarrow{f\ \mathscr{H}}$ is the direct limit of $\odot \xrightarrow{\mathscr{H}} \odot$-arrows.

More precisely, the condition holds iff for every $\odot \xrightarrow{f\ \mathscr{H}}$ there exists a direct system $\left(\odot \xrightarrow{f_{ij}} \odot: i, j \in I\right)$ with colimiting cocone[2]) $(f_i: i \in I)$ such that $f_{0j} \in \mathscr{H}$ for every $j \in I$, and $f = f_0$. See the diagram.



Examples. The category of relational structures is Epi-arrow-algebroidal and also Strongepi-arrow-algebroidal; the category of algebras is a variety and is Regular-arrow-algebroidal (Regular = Strongepi in this category), the category of partial algebras is Regular-arrow-algebroidal and also Surjections-arrow-algebroidal, every

[1]) s.small stands for strongly small.

[2]) Recall, that the colimiting cocone is the arrow-part of the direct limit. That is, the direct limit of $(f_{ij}: i, j \in I)$ is the pair $\left((f_i: i \in I), b\right)$ where $b = \text{cod}\ (f_i)$ for every $i \in I$. If it causes no misunderstanding, this b is said to be the direct limit.

regular and algebroidal category is Regular-arrow-algebroidal, every s.algebroidal category with coequalisers and direct limits is also of such a kind (cf. Corollary 3), etc.

Now, we recall the definition of *reduced product* and ultraproduct (of objects of a category) (cf. FAKIR-HADDAD [5], NÉMETI [17]).
A reduced product is nothing but a direct limit of (direct) products directed by their natural projections.
In more detail: Let $(a_i : i \in I)$ be a family of objects. $\Pi(a_i : i \in I) = \prod_{i \in I} a_i$ stands for the (direct) product of these objects. Let F be a filter[1]) on the powerset of I. For $Y \in F$ we define $\Pi_Y a \overset{d}{=} \Pi(a_i : i \in Y)$, and for $Y \supseteq X \in F$ define $\Pi_Y a \xrightarrow{\pi^Y_X} \Pi_X a$ to be the induced projection. Now, define $\left(\Pi_Y a \xrightarrow{\pi_Y} \Pi a/F : Y \in F\right)$ as the direct limit of the direct system $\left(\Pi_Y a \xrightarrow{\pi^Y_X} \Pi_X a : Y \supseteq X \in F\right)$. The object $\Pi a/F$ is called the F-*reduced product* of the system $(a_i : i \in I)$ or simply *a reduced product* of it. If F is an ultrafilter, $\Pi a/F$ is called an *ultraproduct*.
Throughout the paper products and direct limits (of sets of objects) are supposed to exist. Also $\mathscr{A}$ always denotes a class of objects (of the category under consideration). $P\mathscr{A}$, $P^r\mathscr{A}$, $Up\mathscr{A}$, $L\mathscr{A}$, $S\mathscr{A}$ and $H\mathscr{A}$ denote the class of direct products, reduced products, ultraproducts, direct limits, S-subobjects and H-homomorphic images of objects of $\mathscr{A}$, respectively. E.g.:

$$P^r\mathscr{A} \overset{d}{=} \{\Pi a/F : \{a_i : i \in \cup F\} \subseteq \mathscr{A}\},$$

$$S\mathscr{A} \overset{d}{=} \{\text{dom}(f) : \text{cod}(f) \in \mathscr{A} \quad \text{and} \quad f \in S\}$$
$$= \{a : a \xrightarrow{S} \cdot \in \mathscr{A}\},$$

$$H\mathscr{A} \overset{d}{=} \{\text{cod}(f) : \text{dom}(f) \in \mathscr{A} \quad \text{and} \quad f \in H\}$$
$$= \{a : a \xleftarrow{H} \cdot \in \mathscr{A}\}.$$

Above we defined six operations on the class of objects Ob of our category. P, P^r, Up, L, S and H are operations which to any class of objects $\mathscr{A}$ correlate another class of objects $P\mathscr{A}$, $H\mathscr{A}$ etc.
Note, that these operations are monotonic and extensive, e.g. $\mathscr{A} \subseteq P\mathscr{A}$. It is a frequently asked question (especially in universal algebra) which combinations of them are *closure operations*, i.e. idempotent.
In ANDRÉKA-MÁRKI-NÉMETI [1] we proved that P^r and Up are closure operations in every s.algebroidal category and also in some others. MATTHIESSEN has shown that if $S =$ Mono and the category is strongly finitary and s. algebroidal, then Mono P^r is a closure operation. In the category of relational structures (or universal algebras) of a fixed similarity type if $H =$ "Regular-epimorphisms", $S =$ Mono, then HSP, HS, SP, HSUp, SPUp, SUp and some others are known to be closure operations (cf. GRÄTZER [8]).

In the following we introduce a category theoretical generalisation of universal *axiomatisability*. A *universal formula* will be represented by a *cone*. A *finitary universal formula* will be represented by a "*small cone*".

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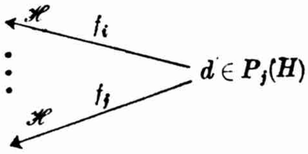
[1]) Talking about filters is not important: any family of subsets of I directed downward by inclusion will do.

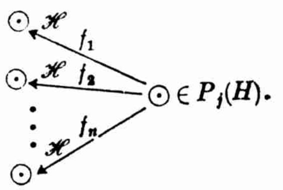
Definition. Let $(\mathscr{H}, S)$ be a factorisation system.

1. An HS-*cone* has the shape:



that is, an HS-cone is a system of arrows $(d, f_i)_{i\in I}$, such that dom $(f_i) = d$ is H-projective and $f_i \in \mathscr{H}$ for all $i \in I$.

2. An HS-cone $(d, f_i)_{i\in I}$ is *small* if I is finite and $\odot \xrightarrow{f_i\, \mathscr{H}} \odot$ for every $i \in I$:



The class of HS-cones will be denoted by K^{HS}, and that of the small HS-cones will be denoted by K_0^{HS}. If there is no danger of confusion, then we shall omit the superscript "HS". Note, that K^{HS} contains the "emety cones" as well (the latter are of the form $(d, f_i)_{i\in\emptyset}$).

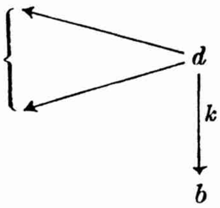
3. A *valuation* of the free variables of the cone $(d, f_i)_{i\in I}$ is a morphism $d \xrightarrow{k} b$. The cone $(d, f_i)_{i\in I}$ is *true* in the object b under the valuation k iff k factors through the cone, i.e. $k = f_i g$ for some $i \in I$ and g. See the diagram below.

The notation $b \models (d, f_i)_{i\in I}[k]$ stands for the statement, that "the cone $(d, f_i)_{i\in I}$ is true in the object b under the valuation $d \xrightarrow{k} b$".

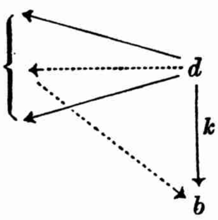
4. The cone $(d, f_i)_{i\in I}$ is *valid* in the object b iff it is true in b under every valuation $d \xrightarrow{k} b$.

Validity is denoted by $b \models (d, f_i)_{i\in I}$.

Note, that a cone is valid in an object iff the object is *injective* w.r.t. that cone (cf. Herrlich-Strecker [11]). That is, to any



here exists an



A cone φ is valid in a class $\mathscr{A}$ iff it is valid in every element of it.

The validity relation $\models$ is a binary relation between the objects and the cones: $\models \subseteq (\text{Ob} \times K)$. This binary relation induces a Galois-correspondence between objects and cones, familiar from model theory and universal algebra.
$K^{HS}\mathscr{A}$ (or $K_0^{HS}\mathscr{A}$) stands for the class of HS-cones (or small HS-cones, respectively) valid in $\mathscr{A}$.

$$K^{HS}\mathscr{A} \stackrel{d}{=} \{\varphi \in K^{HS} : \mathscr{A} \models \varphi\}.$$

Similarly for $K_0^{HS}\mathscr{A}$.

Let $T \subseteq K$ be an arbitrary class of cones. Then the "models" of the "theory" T are those objects in which every cone $\varphi \in T$ is valid. The class of these is denoted by Inj T. By more category theoretical terms, Inj T is the class of the T-injective objects.

$$\text{Inj } T \stackrel{d}{=} \{b \in \text{Ob}: b \models \varphi \text{ for every } \varphi \in T\}.$$

Inj $K^{HS}\mathscr{A}$ is the HS-*axiomatic* or HS-*injective* hull of $\mathscr{A}$.
Clearly Inj K^{HS} and Inj K_0^{HS} are *closure-operations* on classes of objects. We are going to characterise these closure operations.

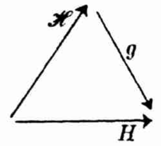
Examples. Let the category be that of all models of a first order language. Let H = Isomorphisms $\stackrel{d}{=}$ Is and S = Strong-monoes (submodels in the strong sense). Now, K_0^{IsS} coincides with the set of universal formulas (of the first order language). This is easy to check since the positive diagram of any s.small model is a finite conjunction of atomic formulas. By this we also have Inj $K_0^{IsS}\mathscr{A} = \text{SUp}\mathscr{A}$, in this case.

Remark. Our notions K^{HS} and K_0^{HS} generalise the concepts of implications and finitary implications first introduced by BANASCHEWSKI-HERRLICH [3].

3. The main result

Theorem 1. *Let the classes H and S of morphisms of a category $\mathscr{C}$ satisfy the following conditions:*

1. a. *There are enough H-projectives, i.e. $HPj(\text{H}) = \text{Ob}\,\mathscr{C}$.*
   b. *Every H-projective object is a direct limit of s.small H-projectives.*
2. *There exists an $\mathscr{H} \subseteq$ Epi such that:*
   a. *$(\mathscr{H}, S)$ is a factorisation system.*
   b. *The category is $\mathscr{H}$-arrow-algebroidal (and $\mathscr{H}$-cowell-powered).*
   c. *S is inductive.*
3. *There exists an $\mathscr{H}$ such that $(\mathscr{H}, S)$ is a factorisation system and commutativity of*



always implies $g \in H$.

Now,

A. *Exactly those classes of objects of $\mathscr{C}$ are closed w.r.t.* HSUp *which are axiomatisable by small* HS*-cones, i.e.* $\mathrm{HSUp}\mathscr{A} = \mathscr{A}$ *iff* $\mathrm{Inj}\, K_0^{\mathrm{HS}}\mathscr{A} = \mathscr{A}$, *for every* $\mathscr{A} \subseteq \mathrm{Ob}\,\mathscr{C}$. *Moreover,* $\mathrm{Inj}\, K_0^{\mathrm{HS}}\mathscr{A} = \mathrm{HSUpUp}\mathscr{A}$.

B. *If in addition $\mathscr{C}$ is s.algebroidal, then* $\mathrm{Inj}\, K_0^{\mathrm{HS}}\mathscr{A} = \mathrm{HSUp}\mathscr{A}$, *for every* $\mathscr{A} \subseteq \mathrm{Ob}\,\mathscr{C}$.

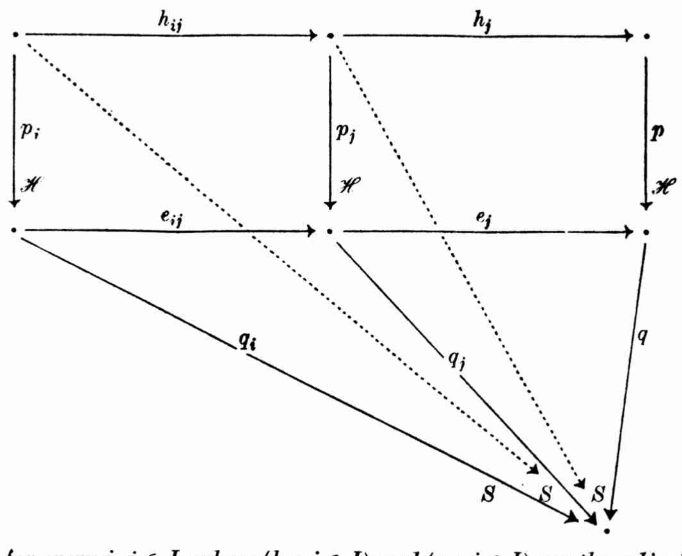
Remark on the conditions of the theorem. Conditions 1.a, 2.a state that our concepts of "subobject" (S) and "homomorphic image" (H) are reasonable, while condition 3 states that they are in a certain (fairly loose) connection. The weakness of condition 3 is important in applications to partial algebras, relational structures and other unusual situations. (See section 5.) These conditions are sufficient for $\mathrm{Inj}\, K^{\mathrm{HS}}\mathscr{A} = \mathrm{HS}\mathscr{A}$. See Corollary 4.
The remaining conditions are needed to treat ultraproducts and "smallness". These conditions are in connection with the subalgebra and the congruence-lattice-properties of the total algebras. 2.b is related to the fact that the lattice of all congruences of an algebra is generated by its compact elements. 2.c generates the fact that the subalgebras of an algebra form an inductive closed-set system.

Before proving the theorem, we prove a lemma.

Lemma 2. *Let $(\mathscr{H}, S)$ be a factorisation system,* $\mathscr{H} \subseteq \mathrm{Epi}$. *Then conditions* (i), (ii) *below are equivalent.*

(i) *S is inductive.*

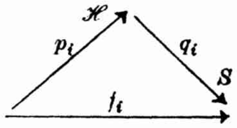
(ii) *The direct limit of $\mathscr{H}S$-decompositions is an $\mathscr{H}S$-decomposition, i.e.:*
For any direct system $(h_{ij}: i, j \in I)$ and cocone $(f_i: i \in I)$ commuting over it there is a direct system $(e_{ij}: i, j \in I)$ such that everything commutes in the diagram:



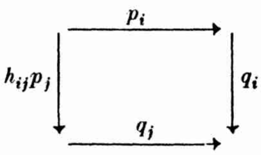
for every $i, j \in I$, where $(h_i: i \in I)$ and $(e_i: i \in I)$ are the colimits of $(h_{ij}: i, j \in I)$ and $(e_{ij}: i, j \in I)$, respectively. p_iq_i is the $\mathscr{H}S$-decomposition of f_i.

Proof. (i) $\Rightarrow$ (ii). Suppose, S is inductive and $(h_{ij}: i, j \in I)$, $(f_i: i \in I)$ are as required in the hypotheses of (ii).

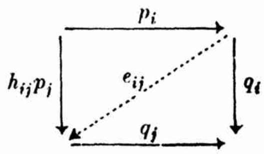
By factorisation system,



exists for every i and the commutative square



has a diagonal fill-in making



commute. Thus e_{ij} exists whenever h_{ij} exists. Since p_i is epi, the system $(e_{ij}: i, j \in I)$ commutes and is a direct system. Let $(h_i: i \in I)$ and $(e_i: i \in I)$ be the corresponding colimits, and let p and q be the maps induced by $(p_i e_i: i \in I)$ and $(q_i: i \in I)$. Since $\mathscr{H}$ is closed w.r.t. colimits for any factorisation system $(\mathscr{H}, S)$, we have $p \in \mathscr{H}$. By factorisation system, $e_{ij} \in S$ and therefore by inductivity of S also $q \in S$. Now, it is easy to check that everything commutes in the diagram, as claimed in the statement of (ii).

(ii) $\Rightarrow$ (i) is obvious by uniqueness of S-decompositions.

Proof of Theorem 1. Let $K_{0\infty}$ denote the class of not necessarily finite cones consisting of $\odot \xleftarrow{\mathscr{H}\ f_i} \odot \in Pj(H)$ arrows. I.e. $K_{0\infty} \subseteq K^{\mathrm{HS}}$ and $(a, f_i)_{i\in I} \in K_{0\infty}$ iff $\odot \xrightarrow{f_i\ \mathscr{H}} \odot$ for every $i \in I$, and $a \in Pj(H)$. (But I is not necessarily finite.)

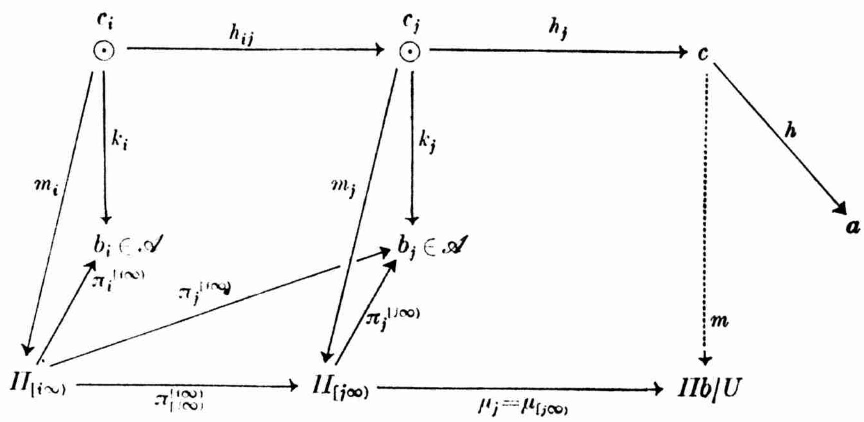
1. First we prove that $\mathrm{Inj}\ K_{0\infty} \subseteq \mathrm{HSUp}\mathscr{A}$.

Let $a \in \mathrm{Inj}\ K_{0\infty}\mathscr{A}$ be arbitrary. By condition 1.a there is an $a \xleftarrow{H\ h} c \in Pj(H)$. By condition 1.b there is a direct system of s.small H-projectives $\left(\overset{c_i}{\odot} \xrightarrow{h_{ij}} \overset{c_j}{\odot}: i, j \in I\right)$ with direct limit $\left(\overset{c_i}{\odot} \xrightarrow{h_i} c: i \in I\right)$.

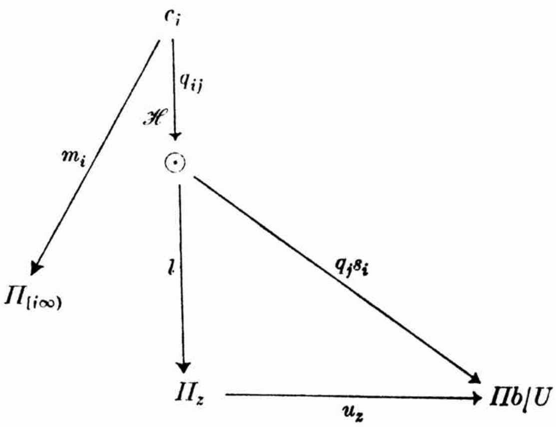
Now, to every $i \in I$ we define a cone φ_i. This cone φ_i is the class of those arrows $c_i \xrightarrow{f\ \mathscr{H}} \odot$ for which $a \not\models f[h_i h]$, i.e. to which there exists no g satisfying $fg = h_i h$. By $\mathscr{H}$-cowell-poweredness we can say, that φ_i is a set. Since $c_i \in Pj(H)$, also $\varphi_i \in K_{0\infty}$. But by definition $a \not\models \varphi_i$, and therefore $\mathscr{A} \not\models \varphi_i$. The latter means the existence of an $c_i \xrightarrow{k_i} b_i \in \mathscr{A}$ to every $i \in I$, which does not factor through φ_i, i.e. $b_i \not\models \varphi_i[k_i]$.

Let U be an ultrafilter over I such that $[i, \infty) \overset{\mathrm{d}}{=} \{j \in I: h_{ij} \text{ exists}\} \in U$ for every $i \in I$. We show, that a is an H-image of an S-subobject of the ultraproduct $\Pi b/U$. To each $i \in I$ there exists an arrow $c_i \xrightarrow{m_i} \Pi_{[i\infty)} b$ satisfying $m_i \pi^{[i\infty)}_{\{j\}} = h_{ij} k_j$ for every $j \in [i, \infty)$.

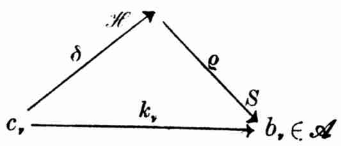
The following diagram represents these definitions:



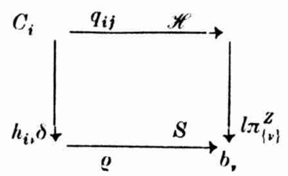
Let $(u_X: X \in U)$ denote the colimiting cocone of $\Pi b/U$. Denote $u_i \overset{d}{=} u_{[i\infty)}$. Now, $(m_i u_i: i \in I)$ is a commuting cocone for the direct system $(h_{ij}: i, j \in I)$, since $h_{ij} m_j = m_i \pi^{[i\infty)}_{[j\infty)}$ by basic properties of products. Thus m exists such that $h_i m = m_i u_i$ for every $i \in I$.
Let (g_i, s_i) be the $\mathscr{H}S$-decomposition of the arrow $m_i u_i$, i.e. $g_i s_i = m_i u_i$ and $g_i \in \mathscr{H}$, $s_i \in S$.
Now, we claim that to every $i \in I$ there exists f_i such that $g_i f_i = h_i h$.
By $\mathscr{H}$-arrow-algebroidalness, g_i is the direct limit of some $\left(c_i \xrightarrow{q_{ij}\ \mathscr{H}} \odot : j \in J\right)$. It holds $g_i = q_{ij} q_j$ if $(q_j: j \in J)$ is the colimiting cocone. By s.smallness of cod q_{ij}, there s a $Z \in U$ such that



commutes. We can suppose, that $Z \subseteq [i\infty)$. Moreover, $m_i \cdot \pi_Z^{[i\infty)} = q_{ij} l$ can also be supposed, since $m_i \cdot \pi_Z^{[i\infty)} u_Z = h_i m = q_{ij} l u_Z$ and c_i is s.small.
Let $\nu \in Z$. Let



commute. By $\mathscr{H}$-arrow-algebroidalness δ is the direct limit of some $\left(c_\nu \xrightarrow{\delta_n \mathscr{H}} \odot : n \in M\right)$. Since $b_\nu \models \delta_n[k_\nu]$, also $a \models \delta_n[h_\nu h]$ by the definition of k_ν and b_ν. This means $\delta_n\lambda_n = h_\nu h$ for all $n \in M$. Since $\mathscr{H} \subseteq$ Epi, the cocone $(\lambda_n : n \in M)$ commutes over the direct limit and induces a $\delta\lambda = h_\nu h$. Since $(\mathscr{H}, S)$ is a factorisation system, the commutative square



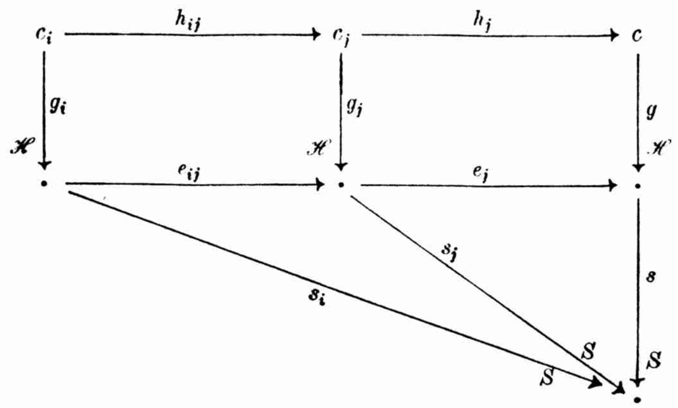
has a diagonal fill-in: $q_{ij}p = (h_{i\nu}\delta)$. Thus, $q_{ij}(p\lambda) = h_{i\nu}h_\nu h = h_i h$.

Denote $\varrho_j \overset{\mathrm{d}}{=} p\lambda$. Since q_{ij} is epi, the cocone $(\varrho_j : j \in J)$ commutes over the colimit, and induces an arrow $g_i\varrho = h_i h$. Denote $f_i = \varrho$.

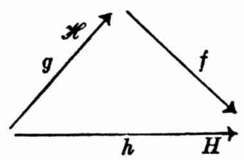
The claim is proved: there exists a cocone $(f_i : i \in I)$ such that $g_i f_i = h_i h$ for every $i \in I$.

Let (g, s) be the $\mathscr{H}S$-decomposition of m.

By lemma 2 there exists a direct system $(e_{ij} : i, j \in I)$ with colimit $(e_i : i \in I)$ such that everything commutes in the following diagram:



Since g_i is epi, the cocone $(f_i : i \in I)$ commutes over the direct system $(e_{ij} : i, j \in I)$ and induces an arrow f such that $f_i = e_i f$. Now, $h_i h = g_i f_i = g_i e_i f = h_i g f$. Since $(h_i : i \in I)$ is a colimit, this implies $h = gf$. But



implies $f \in H$ by condition 1.c.

We have $a \xleftarrow{H\ f} \cdot \xrightarrow{s\ S} \Pi b/u \in Up\,\mathscr{A}$. This completes the proof of 1.

2. Now we prove Inj $K_0\,\mathscr{A} \subseteq$ Inj $K_{0\infty}\,Up\mathscr{A}$.

It is sufficient to prove, that to any $(d, f_i)_{i\in I} \in K_{0\infty}Up\mathscr{A}$ there exists a finite $I_0 \subseteq I$ for which $(d, f_i)_{i\in I_0} \in K_0\mathscr{A}$.

Let $(d, f_i)_{i\in I} \in K_{0\infty}$ be such that to every finite $I_0 \subseteq I$ there exists a $d \xrightarrow{k_{I_0}} a_{I_0} \in \mathscr{A}$

for which $a_{I_0} \models (d, f_i)_{i \in I_0} [k_{I_0}]$. Let J stand for the class of finite subsets of I and let U be an ultrafilter over this J such that $\{I_0 \in J : i \in I_0\} \in D$ for every $i \in I$.
Now, we claim that $\Pi a/U \not\models (d, f_i)_{i \in I}$ and thus the latter is not in $K_{0\infty} Up\mathscr{A}$. Let k stand for the arrow $d \xrightarrow{k} \Pi(a_{I_0} : I_0 \in J)$ induced by the cone $(k_{I_0} : I_0 \in J)$. That is, $k\pi^J_{\{I_0\}} = k_{I_0}$. Let $(u_X : X \in U)$ denote the cocone of the ultraproduct. We prove $\Pi a/U \not\models (d, f_i)_{i \in I}[ku_J]$ by contradiction. Suppose that $f_i g = ku_J$ for some $i \in I$ and g. By smallness of d there exist Z and g' for which $g = g'u_Z$. By s.smallness we can suppose $k\pi^J_Z = f_i g'$, since $k\pi^J_Z u_Z = f_i g = f_i g' u_Z$. Let $I_0 \in Z \cap \{I_0 \in J : i \in I_0\}$ be arbitrary. Now, $f_i(g'\pi^Z_{\{I_0\}}) = k_{I_0}$ contradicting our hypothesis $a_{I_0} \not\models (d, f_i)_{i \in I_0}[k_{I_0}]$. This completes the proof of 2.

3. Now, we prove $\mathrm{HSUpUp}\mathscr{A} \subseteq \mathrm{Inj}\, K_0\mathscr{A}$.

(i) The proof of $Up\mathscr{A} \subseteq \mathrm{Inj}\, K_0\mathscr{A}$ is straightforward, by using the fact that if U is an ultrafilter, $\bigcup_{i<n} X_i \in U$ and $n \in \omega$, then also $X_i \in U$ for some $i < n$.

(ii) $H\mathscr{A} \subseteq \mathrm{Inj}\, K_0\mathscr{A}$, because the domains of HS-cones are H-projective.

(iii) $S\mathscr{A} \subseteq \mathrm{Inj}\, K_0\mathscr{A}$ because the members of HS-cones are elements of $\mathscr{H}$ and $(\mathscr{H}, S)$ is a factorisation system.

4. By 1, 2, and 3 we have proved that $\mathrm{Inj}\, K_0\mathscr{A} = \mathrm{HSUpUp}\mathscr{A}$. In ANDRÉKA-MÁRKI-NÉMETI [1] we proved that $\mathrm{UpUp}\mathscr{A} = \mathrm{Up}\mathscr{A}$ in every s.algebroidal category.
This completes the proof of Theorem 1.

Problem. Are the conditions of A sufficient for the statement of B in the above theorem?

Note that, by the above proof, the theorem also holds if we replace the assumption of s.algebroidalness by the weaker assumption used in ANDRÉKA-MÁRKI-NÉMETI [1].

Corollary 3. *Let the category be regular and s.algebroidal. Let* Is *denote the class of isomorphisms, and let* $S = \mathrm{Mono}$. *Now,*

$$\mathrm{Inj}\, K_0^{\mathrm{IsS}}\mathscr{A} = \mathrm{SUp}\mathscr{A},$$

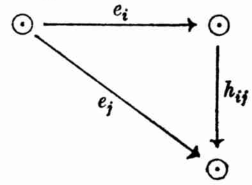
for every class $\mathscr{A}$ *of objects of the category.*

Proof. Let R stand for the class of regular epimorphisms. It is sufficient to prove that ad 2.b the category is R-arrow-algebroidal, and ad 2.c Mono is inductive. It is quite straightforward to prove that Mono is inductive in any algebroidal category. Thus 2.b remains to be shown. Actually, more is true: every s.algebroidal category with coequalisers and direct limits is R-arrow-algebroidal.

1. It is straightforward to show that for any pair of arrows

$$\odot \overset{p}{\underset{q}{\rightrightarrows}} \odot, \text{ also their coequaliser: } \odot \xrightarrow{\text{coeq }(pq)} \odot.$$

2. Let $\odot \xrightarrow{f}$ be regular. Then, by definition, $f = \mathrm{coeq}\,(p, q)$ for some $a \overset{p}{\underset{q}{\rightrightarrows}} \odot \xrightarrow{f}$. By s.algebroidalness, there is a direct system $\left(\odot \xrightarrow{k_{ij}} \odot : i, j \in I\right)$ with direct limit $\left(\odot \xrightarrow{k_i} a : i \in I\right)$. Define $e_i \overset{\mathrm{d}}{=} \mathrm{coeq}(k_i p, k_i q)$. If $i \leq j$ (i.e. k_{ij} exists), then there exists a commuting



because $(k_i p)\, e_j = (k_i q)\, e_j$. Since e_i is epi, $(h_{ij}: i, j \in I)$ is again a direct system. Let $(h_i: i \in I)$ denote its direct limit.
By the definition of coequalisers, there exists a cocone $(r_i: i \in I)$ such that $e_i r_i = f$. Since e_i is epi, this cocone commutes over the direct system, and thus induces an arrow r such that $(e_i h_i)\, r = f$.
Using the epi-property of the cocone $(k_i: i \in I)$, it is easy to show that $p(e_i h_i) = q(e_i h_i)$. Since f is the coequaliser, $fl = e_i h_i$ for some l. Clearly l and r are isomorphisms, by epi-properties of f and $(h_i: i \in I)$, and thus f is the colimit of the system

$$\left(\odot \xrightarrow{e_i R} \odot : i \in I\right). \qquad \square$$

Let K_1^{HS} and K_{01}^{HS} denote the classes of those elements $(d, f_i)_{i \in I}$ of K^{HS} and K_0^{HS}, respectively, for which $I = 1$. I.e.

$$K_1^{\mathrm{HS}} \stackrel{\mathrm{d}}{=} \left\{h: \xleftarrow{\mathscr{H}\; h} \cdot \in Pj(H)\right\}.$$

Is denotes the isomorphisms.

Corollary 4. *Let the category be s.algebroidal, and let S satisfy condition 2 of theorem 1. Then*

(i) $\mathrm{Inj}\; K_0^{\mathrm{IsS}}\mathscr{A} = \mathrm{SUp}\mathscr{A}$, $\quad \mathrm{Inj}\; K_{01}^{\mathrm{IsS}}\mathscr{A} = \mathrm{SPUp}\mathscr{A}$,
$\mathrm{Inj}\; K^{\mathrm{IsS}}\mathscr{A} = \mathrm{S}\mathscr{A}$, $\quad \mathrm{Inj}\; K_1^{\mathrm{IsS}}\mathscr{A} = \mathrm{SP}\mathscr{A}$.

(ii) *Let further H satisfy condition 1 and 3 of theorem 1. Then*
$\mathrm{Inj}\; K_0^{\mathrm{HS}}\mathscr{A} = \mathrm{HS\;Up}\mathscr{A}$, $\quad \mathrm{Inj}\; K_{01}^{\mathrm{HS}}\mathscr{A} = \mathrm{HSPUp}\mathscr{A}$,
$\mathrm{Inj}\; K^{\mathrm{HS}}\mathscr{A} = \mathrm{HS}\mathscr{A}$, $\quad \mathrm{Inj}\; K_1^{\mathrm{HS}}\mathscr{A} = \mathrm{HSP}\mathscr{A}$.

Statements (i) 3, (i) 4, (ii) 3, (ii) 4 *hold also, if we relax all the conditions related to algebroidalness* (i.e. conditions 1.b, 2.b, 2.c).

Proof. (i) clearly follows from (ii) by choosing $H = \mathrm{Is}$, and checking that condition 1 is always satisfied by Is.
To prove (ii) 3 we should repeat the proof of theorem 1 without the "small objects" considerations.
It remains to prove (ii) 2 and (ii) 4.
Statement (*). An arbitrary cone $(d, f_i)_{i \in I}$ is valid in $P\mathscr{A}$ iff there exists an $i \in I$ for which f_i is valid in $\mathscr{A}$.
The proof of this statement is straightforward: one direction is a simplified version of step 2 in the proof of theorem 1.
The above statement (*) yields (ii) 2 from (ii) 1 and also (ii) 4 from (ii) 3.
Let us see e.g. (ii) 2. By theorem 1 (ii) 1 holds, i.e. $\mathrm{Inj}\; K_0^{\mathrm{HS}}\mathscr{A} = \mathrm{HSUp}\mathscr{A}$. By statement (*), $\mathrm{Inj}\; K_{01}^{\mathrm{HS}}\mathscr{A} \subseteq \mathrm{Inj}\; K_0^{\mathrm{HS}}P\mathscr{A}$. Since products preserve K_1, we have

$$\mathrm{HSPUp}\mathscr{A} \subseteq \mathrm{Inj}\; K_{01}^{\mathrm{HS}}\mathscr{A} \subseteq \mathrm{Inj}\; K_0^{\mathrm{HS}}\mathrm{P}\mathscr{A} = \mathrm{HSUpP}\mathscr{A}.$$

$\mathrm{UpP}\mathscr{A} \subseteq \mathrm{SPUp}\mathscr{A}$ will be proved at theorem 6, and thus we have $\mathrm{HSUpP}\mathscr{A} \subseteq \mathrm{HSPUp}\mathscr{A}$ completing the proof.

Remark. Corollary 4 consists of 2^3 statements of the form $\mathrm{Inj}\; K^Q\mathscr{A} = Q\mathscr{A}$. The presence of H in Q causes the cones in K^Q to have H-projective domains. The presence of P in Q causes the cones in K^Q to consist of exactly one member. The presence of Up in Q causes the cones in K^Q to be small. K^Q is completely determined by the above three statements as a subclass of K^{IsS}, i.e. the effects of H, P and Up are independent of each other. Thus Corollary 4 can be reduced to 3 statements from 2^3 statements.

4. Closure operations (or, on the semigroup generated by H, S, P, P^r, Up, L)

Corollary 5. *Under the conditions of Theorem* 1:

1. HSUp, HSPUp, HS, HSP, SUp, SPUp, SP, *S are closure operations.*

2. *These are the only closure operations in a certain sense: Let the conditions of Theorem* 1 *together with $H \supseteq$* Is *hold, and let Q be a combination of some of the operations H, S, P, P^r, Up such that S occures in Q. Then Q is a closure operation iff it coincides with one of the operations listed in* 1.

Some of the conditions can be eliminated from the above Corollary. To prove (i) and (ii) of the following theorem, the adjective "strong" can be dropped everywhere, i.e. "algebroidalness" and "small object" are enough.

Theorem 6. *Let the category be algebroidal, let $(\mathscr{H}, S)$ be a factorisation system, $\mathscr{H} \subseteq$* Epi, *and let the category be $\mathscr{H}$-cowell-powered (at least w.r.t. the small H-projective objects). Then* (i)—(iv) *below hold.*

(i) LSP *is a closure operation.*

(ii) *If H satifies condition* 1 *and* 3 *of theorem* 1 *("strongly" can be dropped), then* HLSP *is a closure operation.*

(iii) *If the conditions of theorem* 1 *are satisfied, then* HLSP = HSLP = HSPU*p* = HSP^r *and therefore* LSP = SLP = SPUp = SP^r.

(iv) *In any s.algebroidal category if S =* Mono, *then* SLP = SPUp = SP^r *is a closure operation.*

Proof. (i) is a special case of (ii), namely $H =$ Is and thus needs no proof.

ad (ii).

Statement (**). Let the conditions of (ii) be satisfied, and let a be a small H-projective object. Let Q be an arbitrary sequence of the operations H, L, S and P (e.g. Q might be HLSPHLSP).

Now, there is an arrow $a \xrightarrow{u\ \mathscr{H}} \cdot \in \mathrm{SP}\mathscr{A}$ "universal over" $Q\mathscr{A}$, i.e. every $a \xrightarrow{f} \cdot \in Q\mathscr{A}$ factors through u uniquely.

The proof is straightforward (cf. Pásztor [20], Gergely-Németi-Pásztor [7], or Andréka-Németi [2], where u is called "free $\mathscr{H}$-extension towards $\mathscr{A}$").

Let $a \in Q\mathscr{A}$. (Here $\odot$ denotes a small object instead of s.small.) By condition 1 of theorem 1, there are an $a \xleftarrow{H\ h} c$ and a direct system $\left(\overset{c_i}{\odot} \xrightarrow{h_{ij}} \overset{c_j}{\odot} : i, j \in I\right)$ with colimit $\left(c_i \xrightarrow{h_i} c : i \in I\right)$ such that $c_i \in Pj(H)$ for every $i \in I$.

Let $c_i \xrightarrow{u_i\ \mathscr{H}} \cdot \in \mathrm{SP}\mathscr{A}$ be the universal arrow of (**) for every i. Clearly, there is a direct system $(k_{ij}: i, j \in I)$ with colimit $(k_i: i \in I)$ such that

$$\begin{array}{ccccccc}
c_i & \xrightarrow{h_{ij}} & c_j & \xrightarrow{h_j} & c & \xrightarrow{h\ \ H} & a \\
\mathscr{H}\downarrow u_i & & \mathscr{H}\downarrow u_j & & \mathscr{H}\downarrow u & & \\
b_i & \xrightarrow{k_{ij}} & b_j & \xrightarrow{k_j} & b & &
\end{array}$$

commutes, where u is the induced arrow.

Since u_i is universal over $Q\mathscr{A}$ and $a \in Q\mathscr{A}$, there is a cocone $(q_i: i \in I)$ commuting over $(k_{ij}: i, j \in I)$ (since u_i is epi) such that $h_i h = u_i q_i$. For the induced arrow this implies $h_i uq = u_i k_i q = h_i h$. By "universalness" of $(h_i: i \in I)$, we have $uq = h$. Since the direct limit of $\mathscr{H}$-arrows is again an $\mathscr{H}$-arrow, $u \in \mathscr{H}$ yielding $g \in H$. Now, $b_i \in \mathrm{SP}\mathscr{A}$ proves $a \in \mathrm{HLSP}\mathscr{A}$.

ad (iii). First we shall prove the following inclusions:

$$\mathrm{LSP}\mathscr{A} \subseteq \mathrm{SPUp}\mathscr{A} \subseteq \mathrm{SP}^r\mathscr{A} \subseteq \mathrm{SLP}\mathscr{A} \subseteq \mathrm{LSP}\mathscr{A}.$$

1. By corollary 4 $\mathrm{LSP}\mathscr{A} \subseteq \mathrm{Inj}\, K_{01}^{1sS}\mathscr{A} \subseteq \mathrm{SPUp}\mathscr{A}$ since validity of K_{01}^{1sS}-cones is easily seen to be preserved under direct limits.
2. $\mathrm{PUp}\mathscr{A} \subseteq P^r P^r \mathscr{A}$ and we proved in ANDRÉKA-MARKI-NÉMETI [1] that $P^r P^r = P^r$.
3. $P^r\mathscr{A} \subseteq \mathrm{LP}\mathscr{A}$ by definition.
4. (i) states, that LSP is a closure operation, and therefore $\mathrm{SLP}\mathscr{A} \subseteq \mathrm{LSP}\mathscr{A}$.

All these obviously mean HLSP = HSLP = HSPUp = HSP^r.

ad (iv). The proof is a not very ingenious computation, and therefore we omit it.

5. Examples

(1) Some well-known theorems of model theory and universal algebra are special cases of corollary 4. E.g.:

- The model classes closed w.r.t. SUp are exactly the universally axiomatisable ones (LOS, MAL'CEV and TARSKI);
- The quasivarieties are exactly the SPUp = SP^r closed classes (MAL'CEV);
- Birkhoff's characterisation of varieties;
- Preservation theorems, for finitary as well as infinitary model theory, the syntactic characterisation of the formulas preserved by HS, S, SP, SP^r, HSP for different choices of H and S. (Some of these are due to KEISLER and LYNDON, cf. the recent CHANG-KEISLER monograph on model theory, Sec. 5.2.)

Since cones represent formulas, we shall speak about formulas here.

(2) Consider the category of *partial algebras* of a fixed similarity type. In the first order language of partial algebras if τ and σ are two terms, then $\exists\, \tau$ means that τ is defined, while $\tau = \sigma$ means that both τ and σ are defined, and they are equal (cf. ANDRÉKA-NÉMETI [2], NÉMETI [16] or NÉMETI [17]).

Let H_w, H_s and H_c stand for weak homomorphic images (i.e. onto homomorphisms), strong homomorphic images and closed homomorphic images respectively (cf. HÖFT [12] or ANDRÉKA-NÉMETI [2] etc.).

Let S_w, S_r and S_s stand for weak subalgebras, relative ones and strong ones (cf. as above).

All these concepts are defined in the monograph GRÄTZER [8], p. 80—81, with the following changes in terminology: S_s is called "subalgebras", H_s is called "full homomorphic images" and H_c is called "strong ones" there.

Any combination of the above H and S choices ($H_s S_w$, $H_s S_s$, etc.) satisfy the conditions of corollary 4. Thus we have 48 different axiomatisability results, which imply preservation theorems also. (Cf. ANDRÉKA-NÉMETI [2], NÉMETI [16] or NÉMETI [17].)

For example:

- A class (of partial algebras) is closed w.r.t. $H_s S_r$Up iff it can be axiomatised by formulas of the form:

$$[\exists\, m_1(\bar{v}_1) \wedge \cdots \wedge \exists\, m_j(\bar{v}_j)] \rightarrow [\tau_1 = f_1(\sigma_{11} \cdots) \vee \cdots \vee \tau_n = f_n(\sigma_{n1} \cdots)],$$

where τ_i, σ_{ij} must occure on the left: m_i, f_i are operation symbols: and $\bar{v}_1\bar{v}_2 \ldots \bar{v}_j$ is a repetitionless sequence of variables.

— A class is closed w.r.t. H_sS_sUp iff it is axiomatisable by formulas of the form

$$[\exists\, m_1(\bar{v}_1) \wedge \cdots] \rightarrow [\tau_1 = \sigma_1 \vee \cdots],$$

where the m_i's are operation symbols, and $\bar{v}_1 \ldots \bar{v}_j$ is a repetitionless sequence of variables.

— A class is closed w.r.t. H_cS_rUp iff it is axiomatisable by formulas of the form

$$[\exists\, v_1 \wedge \cdots] \rightarrow [\tau_1 = f_1(\sigma_{11} \ldots) \vee \cdots],$$

where the f_i's are operation symbols, and τ_i, σ_{ij} must occure on the left.

— A class is closed w.r.t. H_wS_rUp iff it is axiomatisable by formulas of the form

$$[x_1 = f_1(\bar{y}_1) \vee \cdots \vee x_n = f_n(\bar{y}_n)].$$

Similarly. A formula is preserved under both H_s und S_s iff it is elementarily equivalent to a finite conjunction of formulas of the form mentioned above:

$$[\exists\, m_1(\bar{v}_1) \wedge \cdots] \rightarrow [\tau_1 = f_1(\sigma_{11} \ldots) \vee \cdots].$$

If P is present, then the right side has exactly one element. A list of all cases can be found in NÉMETI [16].

(3) Models. Consider the category of relational structures, with H_s, H_w, S_s, S_w defined similarly to the above.
Here again corollary 4 yields axiomatisability results, e.g.:

— A class is closed w.r.t. H_sS_sUp iff it is axiomatisable by formulas of the form

$$[R_1(\bar{v}_1) \wedge \cdots \wedge R_j(\bar{v}_j)] \rightarrow [P_1 \vee \cdots],$$

where P_i are prime formulas, R_i are predicate symbols and $\bar{v}_1 \ldots \bar{v}_j$ is repetitionless.

— To test generality or applicability of corollary 4, let us try some ad-hoc choice of H. Let there be only one binary relation symbol R. Let $\mathfrak{A} \xrightarrow{f} \mathfrak{B} \in H_1$ iff

$$[f(a)Rb \quad \text{implies the existence of} \quad f(b') = b,\, aRb'].$$

A class is closed w.r.t. H_1S_sUp iff it is axiomatisable by formulas of the form

$$[R_1(\bar{v}_1) \wedge \cdots \wedge R_j(\bar{v}_j)] \rightarrow [P_1 \vee \cdots],$$

where the relational structure defined over the set of variable symbols by the left side is a disjoint union of trees each grown from one point.

(4) Commutative cancellative monoids possibly with zero. Consider the category of those commutative monoids, in which every element is cancellable except the zero element ∞ (if it exists).
Let H_c, H_s, S_s, S_w be defined in the usual category theoretical sense. H_c has been defined by PÁSZTOR [20] and investigated in distrubutive lattices in NÉMETI [18]. H_c is the class of epimorphisms extremal w.r.t. the bimorphisms.
In this example P_i stand for formulas of the form $\tau = \sigma$ or $\exists\, x\,(y + x) = 0$.

— A class is closed w.r.t. S_sUp iff it is axiomatisable by formulas of the form

$$[P_1 \wedge \cdots] \rightarrow [P_{j+1} \vee \cdots].$$

— A class is closed w.r.t. H_cS_sUp iff it is axiomatisable by formulas of the form

$$[\exists\, x_1\,(y_1 + x_1 = 0) \wedge \cdots] \rightarrow [P_1 \vee \cdots].$$

— A class is closed w.r.t. H_cS_wUp iff it is axiomatisable by formulas of the form

$$[\exists x_1 (y_1 + x_1 = 0) \wedge \cdots] \to [\tau_1 = \sigma_1 \vee \cdots \vee \tau_n = \sigma_n].$$

(The class of Abelian groups possibly extended with zero is closed w.r.t. H_sS_s but not w.r.t. S_w.)

Consider the cone (N, f_1, f_2), where N is the semigroup of natural numbers, $N \xrightarrow{f_1} E$ is its natural embedding into that of the integers, while f_2 takes N into the one-element group extended with a zero $\{0, \infty\}$, taking 0 into 0 and all the others into ∞. Clearly $(N, f_1, f_2) \in K^{H,S}$, and Inj $\{(N, f_1, f_2)\}$ = "Abelian groups possibly extended with zero".

6. On the conditions of the main theorem

Proposition (Independence of conditions of Theorem 1). Consider the following conditions:

(i) The category is s.algebroidal.
(ii) $(\mathscr{H}, S)$ is a factorisation system.
(iii) $\mathscr{H} \subseteq \mathrm{Epi}'$.
(iv) S is inductive.
(v) The category is $\mathscr{H}$-arrow-algebroidal.

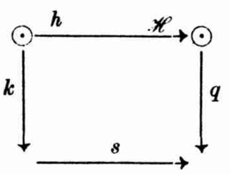
These five conditions are independent of each other in the following sense: no one of them is implied by the rest.

The proof consists of lengthy constructions of five "counterexamples" and therefore we omit it.

Note, that conditions (i)—(v) are the ones required for Inj $K_0^{IsS}\mathscr{A} = SUp\mathscr{A}$.

Lemma 7. *Let the category be s.algebroidal, $\mathscr{H}$-arrow-algebroidal and let $(\mathscr{H}, S)$ be a factorisation system, $\mathscr{H} \subseteq$ Epi.*
Now, conditions (i) *and* (ii) *below are equivalent.*

(i) *S is inductive.*
(ii) *K_{01}^{IsS} defines S in the following sense:*
$s \in S$ iff for every (commutative) square



there exists a "diagonal fill-in" δ such that $h\delta = k$ and $\delta s = q$.

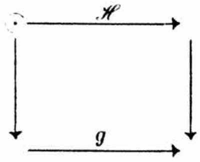
Remark. Condition (ii) in the above lemma is equivalent to the following:

$$s \in S \quad \textit{iff} \quad \textit{for every "formula"} \odot \xrightarrow{f\ \mathscr{H}} \odot \textit{ and "valuation" } k,$$
$$\mathrm{cod}\,(s) \models f[ks] \quad \textit{implies} \quad \mathrm{dom}\,(s) \models f[k].$$

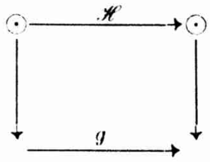
That is, S is the class of those morphisms, which "preserve" a special class (K_{01}^{IsS}) of formulas "backward". It is interesting to note, that every subalgebra-concept introduced so far e.g. in the theory of partial algebras has been defined this way.

Proof. In the proof we shall use the following observation: By the $\mathscr{H}$-arrow-algebroidalness it is not difficult to see that for every arrow g:

(*) every commutative square



has a diagonal fill-in
iff
every commutative square

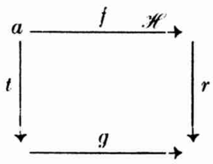


has a diagonal fill-in.

We leave the proof of this to the reader.

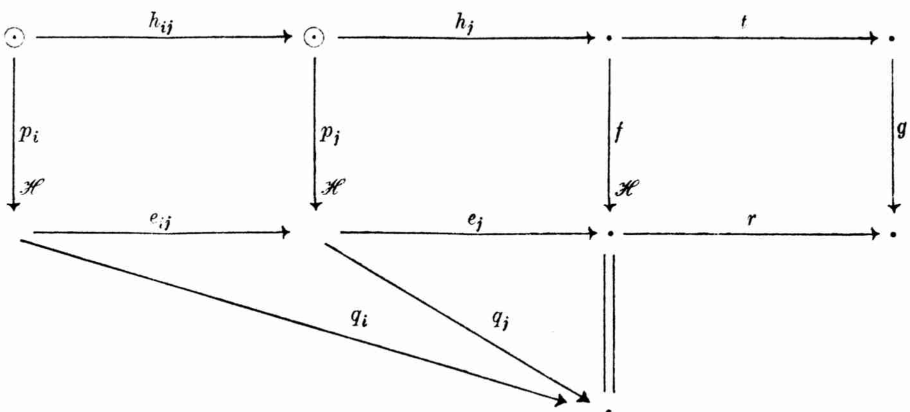
Now, let us prove (i) $\Rightarrow$ (ii): Let S be inductive and suppose that g satisfies (*). We show that $g \in S$.

Let



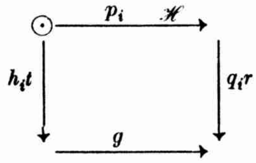
be an arbitrary (commutative) square over g.

By factorisation system, it is enough to prove, that this also has a diagonal fill-in. By algebroidalness, there is a direct system $(\odot \xrightarrow{h_{ij}} \odot : i, j, \in I)$ with colimit $(\xrightarrow{h_i} a : i \in I)$. Now Lemma 2 can be applied to the cone $(h_i f : i \in I)$ yielding



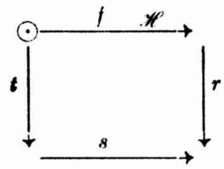
everywhere commuting as in the statement of Lemma 2.

By hypothesis, the squares



have diagonal fill-ins, say k_i. By epiness of p_i, these form a commuting cocone over $(e_{ij}: i, j \in I)$ and induce an arrow $e_i k = k_i$. Thus $h_i f k = h_i t$ and since $(h_i: i \in I)$ is a colimit, this implies $fk = t$.

(ii) $\Rightarrow$ (i): Let $(h_{ij}: i, j \in I)$ be a direct system with colimit $(h_i: i \in I)$ and cocone $\left(\xrightarrow{s_i \ S}: i \in I\right)$ commuting over it. Let the induced arrow be s. By (ii) we have to show that every square



has a diagonal fill-in.

By smallness there is a t' and i such that $t'h_i = t$. Since $s_i \in S$, there is a diagonal fill-in $fk = t'$. Therefore $f(kh_i) = t$ completing the proof.

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