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Titel: Affine reductive spaces

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Affine reductive spaces

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It is well known that a reductive homogeneous space yields a complete affine connection, namely the canonical connection of the second kind, on the underlying manifold. The main goal of this paper is to characterize intrinsically all manifolds with an affine connection which come from reductive homogeneous spaces in that way. (We shall call them *affine reductive spaces.*) Moreover, we shall establish a natural one-to-one correspondence between affine reductive spaces with a fixed origin and between special reductive homogeneous spaces called prime ones.
For these purposes we shall define *the group of transvections* (or displacements) of an arbitrary manifold with an affine connection. Let us remark that, for an affine globally symmetric space, our concept has the classical meaning.

1. We shall start with a short exposition of the theory of reductive homogeneous spaces. The results presented here are essentially known.
Let K be a connected Lie group and H its closed subgroup. Consider the homogeneous manifold K/H. Then $\pi: K \to K/H$ will denote the canonical projection and $o = \pi(H)$ the origin of K/H. Here K is acting on K/H to the left; we need not suppose that this action is effective.
Let $\mathfrak{k}$, $\mathfrak{h}$ be the Lie algebras of K and H respectively. Suppose that there is a subspace $\mathfrak{m}$ of $\mathfrak{k}$ such that $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ (direct sum) and $\mathrm{ad}\,(h)\,\mathfrak{m} = \mathfrak{m}$ for every $h \in H$. Then the homogeneous space K/H is called *reductive with respect to the decomposition* $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$. The underlying manifold of K/H will be usually denoted by M.
Let us consider the left-invariant distribution $\{\mathfrak{m}_g, g \in K\}$ on K generated by $\mathfrak{m}$. Then the property $\mathrm{ad}(h)\mathfrak{m} = \mathfrak{m}$ implies $\mathfrak{m}_h = (L_h)_* \,\mathfrak{m}_e = (R_h)_* \,\mathfrak{m}_e$ for each $h \in H$. We shall often identify $\mathfrak{m}$ with $\mathfrak{m}_e$ (and, naturally, $\mathfrak{k}$ with $\mathfrak{k}_e$).
Before going on, let us make a notational convention. Let K be a Lie group acting on a manifold M to the left. Then we define the action of the tangent bundle $T(K)$ on M as follows: for $X \in K_g$ and $p \in M$, we put $X \cdot p = \left.\frac{d}{dt}\right|_0 (\exp_g tX)\,(p)$, where $\exp_g = L_g \circ \exp \circ (L_{g^{-1}})_*$.
Again let $M = K/H$ be a reductive homogeneous space. Consider the frame bundle $L(M)$ over M; let $\bar{\pi}$ denote the bundle projection. The group K acts on $L(M)$ to the left and so does the tangent bundle $T(K)$. Moreover, the structural group $GL(n, R)$ acts on $L(M)$ to the right ($n = \dim M$).

Lemma. *The set of tangent subspaces* $Q_u \subset \big(L(M)\big)_u$ *along the fibre* $\bar{\pi}^{-1}(o) \subset L(M)$

given by $Q_u = \mathfrak{m} \cdot u$ *is* H*-invariant, and it satisfies* $\tilde{\pi}_*(Q_u) = M_0$, $Q_{us} = (R_s)_* Q_u$ *for* $u \in \tilde{\pi}^{-1}(o)$, $s \in GL(n, R)$.

Proof. For $h \in H$, $u \in \tilde{\pi}^{-1}(o)$ we have $(L_h)_*(Q_u = (L_h)^* (\mathfrak{m}_e \cdot u) = \mathfrak{m}_h \cdot u$ $= \big((R_h)_* \mathfrak{m}_e\big) \cdot u = \mathfrak{m}_e \cdot (h \cdot u) = Q_{h \cdot u}$. Further, $\tilde{\pi}_*(Q_u) = \tilde{\pi}_*(\mathfrak{m}_e \cdot u) = \mathfrak{m}_e \cdot o$ $= \pi_*(\mathfrak{m}_e) = M_0$. Finally, we have $g \cdot (us) = (g \cdot u)\, s$ for $g \in K, u \in \tilde{\pi}^{-1}(o)$, $s \in GL(n, R)$.

Because K acts transitively on the set of fibres of $L(M)$, we obtain from here:

Proposition A. *There is a unique* K*-invariant connection in* $L(M)$ *such that the horizontal subspaces along the fibre* $\tilde{\pi}^{-1}(o)$ *are given by the formula* $Q_u = \mathfrak{m} \cdot u$.

Definition. The connection Γ constructed in Proposition A is called *the canonical linear connection of the reductive homogeneous space* K/H. The corresponding covariant derivative ∇ on the underlying manifold $M = K/H$ is called *the canonical affine connection* (of the second kind) *on* M.

Obviously, there is no risk of confusion if we speak shortly about the canonical connection in either case.

Proposition B. *The canonical connection of a reductive homogeneous space* $M = K/H$ *is the unique* K*-invariant connection in* $L(M)$ *with the following property: for every frame* u *at* $o \in M$, *and for each* $X \in \mathfrak{m}$ *the orbit* $\exp(tX) \cdot u$ *is horizontal.*

Proof. We have $\dfrac{d}{dt}(\exp tX \cdot u) = \dfrac{d}{d\tau}\Big|_0 \big(\exp(t+\tau) X \cdot u\big) = (\exp tX)_* \dfrac{d}{d\tau}\Big|_0 (\exp \tau X \cdot u)$ $= (\exp tX)_* (X \cdot u)$. Now, let Q_z denote the horizontal subspace of the canonical connection at $z \in L(M)$. Then we have $X \cdot u \in Q_u$ and $\dfrac{d}{dt}(\exp tX \cdot u) \in (\exp tX)_* Q_u$ $= Q_{\exp tX \cdot u}$. Hence $\dfrac{d}{dt}(\exp tX \cdot u)$ is horizontal. The uniqueness part is obvious from Proposition 1.

Proposition C. *Consider the geometry on* $M = K/H$ *determined by the canonical connection. The following is true:*

(i) *For each* $X \in \mathfrak{m}$ *put* $x(t) = \exp(tX) \cdot o$ *in* M. *Then the parallel dispacement of tangent vectors at* o *along the curve* $x(t)$, $0 \leqq t \leqq s$, *coincides with the differential of* $\exp(sX)$ *acting on* M.

(ii) *For each* $X \in \mathfrak{m}$, *the curve* $x(t) = \exp(tX) \cdot o$ *is a geodesic. Conversely, every geodesic starting from* o *is of the form* $\exp(tX) \cdot o$ *for some* $X \in \mathfrak{m}$.

(iii) *The canonical connection on* M *is complete.*

Proof. (i) follows almost immediately from Proposition B. Indeed, choose $u \in L(M)$ at o. Since $\exp(tX) \cdot u$ is a horizontal curve which projects on the curve $x(t)$, $0 \leqq t \leqq s$, we can see that, for any $Y \in M_0$, $(\exp tX)_* (Y) = (\exp tX \cdot u) \cdot (u^{-1} Y)$ is parallel to Y along the curve $x(t)$, $0 \leqq t \leqq s$. (Here u^{-1} is considered as a map, $u^{-1}\colon M_0 \to R^n$.)

(ii) follows from (i) since the tangent vector $x^\cdot(t)$ is equal to $(\exp tX)_* \big(x^\cdot(0)\big)$.

(iii) follows immediately from (ii) since $\exp(tX)$ is defined for all $t \in R$. (Cf. [1].)

Proposition D. *The canonical connection is the unique* K*-invariant connection on* M *such that for every* $X \in \mathfrak{m}$ *and every vector field* $Y \in \chi(M)$ *we have* $(\nabla_{X^*} Y)_0 = [X^*, Y]_0$, *where* $X^* \in \chi(M)$ *denotes the vector field generated by the action of* X *on* M: $X^*_p = X \cdot p$ *for every* $p \in M$.

Proof. The infinitesimal version of (i), Proposition C, says that $(\nabla_{X^*} Y)_0 = (\mathscr{L}_{X^*} Y)_0$

(where $\mathscr{L}$ denotes the Lie derivative). On the other hand, the covariant derivatives $(\nabla_X\cdot)_0$, $X \in \mathfrak{m}$, and the K-invariance of the canonical connection determine ∇ uniquely on M.

Proposition E. *If a tensor field on M is invariant by K then it is parallel with respect to the canonical connection ∇.*

Proof. Let S be a K-invariant tensor field on M. Then $\mathscr{L}_X \cdot S = 0$ and hence $(\nabla_X \cdot S)_0 = (\mathscr{L}_X \cdot S)_0 = 0$ for every $X \in m$, i.e., $(\nabla S)_0 = 0$. (Here the covariant derivative $(\nabla_X\cdot)_0$ and the Lie derivative $(\mathscr{L}_X\cdot)_0$ have been extended to act on germs of the tensor fields at o. From the uniqueness of these extensions and from Proposition D we can see that both extended derivatives are equal.) Now, from the K-invariance of both ∇ and S we get $\nabla S = 0$ identically.

Corollary. *For the canonical connection of a reductive homogeneous space, both the curvature tensor field and the torsion tensor field are parallel*: $\nabla R = \nabla T = 0$.

Finally, let us recall the explicite formulas for the curvature and torsion of the canonical connection:

$$T(X \cdot o,\ Y \cdot o) = -([X,\ Y]_{\mathfrak{m}}) \cdot o \qquad \text{for } X,\ Y \in \mathfrak{m}, \tag{1}$$

$$R(X \cdot o,\ Y \cdot o)\,(Z \cdot o) = -\big[[X,\ Y]_{\mathfrak{h}},\ Z\big] \cdot o \qquad \text{for } X,\ Y,\ Z \in \mathfrak{m}. \tag{2}$$

Here the indices at the brackets indicate taking the $\mathfrak{m}$-component and the $\mathfrak{h}$-component of a vector of $\mathfrak{k}$.

The proof of these formulas is a bit long and it will be omitted (see [1], for instance).

We shall close this section with a result that will be useful in the main part of the paper:

Proposition F. *Let K/H be a reductive homogeneous space with respect to a decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$. Then the subspace $\mathfrak{l} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ is an ideal of $\mathfrak{k}$ and the corresponding connected normal subgroup $L \subset K$ is acting transitively on K/H to the left. Moreover, L is generated by the set* $\exp(\mathfrak{m})$, *where* $\exp\colon \mathfrak{k} \to K$ *is the exponential map at $e \in K$ (and $\mathfrak{k}$ is identified with the tangent space K_e).*

Proof. It is obvious from the relation $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$ that $\mathfrak{l}$ is an ideal of $\mathfrak{k}$; thus the corresponding connected Lie subgroup $L \subset K$ is normal. Identifying $\mathfrak{k}$ with K_e, we can easily see that $\pi_{*e}(\mathfrak{l}) = \pi_{*e}(\mathfrak{m}) = (K/H)_0$. Hence the transitivity of L follows easily by a standard theorem on implicite functions and from the fact that K/H is connected.

For the proof of the second statement, let L' be the subgroup of L generated by $\exp(\mathfrak{m})$. Put $A = \{x \in \mathfrak{l} \mid \exp tX \in L' \text{ for all } t \in R\}$. Let $\mathfrak{a}$ be the subspace of $\mathfrak{l}$ spanned by A. Then $\mathfrak{m} \subset A \subset \mathfrak{a}$. If $h \in L'$ and $X \in A$, we have $\mathrm{Ad}\,(h)\,(\exp tX) = \exp\big(t \cdot \mathrm{ad}\,(h)\,X\big) \in L'$, and hence $\mathrm{ad}\,(h)\,X \subset A$. It follows $\mathrm{ad}\,(h)\,(\mathfrak{a}) \subset \mathfrak{a}$. In particular, for $Y \in A$, $X \in \mathfrak{a}$, we have $e^{\mathrm{ad}(tY)}X = \mathrm{ad}\,(\exp tY) \cdot X \in \mathfrak{a}$, for each t, and hence $[Y, X] \in \mathfrak{a}$. By the linearity we get $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$; i.e., $\mathfrak{a}$ is a subalgebra of $\mathfrak{l}$. Now, $\mathfrak{l} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{a} + [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$, i.e., $\mathfrak{a} = \mathfrak{l}$. Thus A contains a basis $\{X_1, \ldots, X_m\}$ of $\mathfrak{l}$, and $\exp(t_1X_1) \cdots \exp(t_nX_n) \in L'$ for all t_i. This shows that L' contains a full neighbourhood of e in L and therefore coincides with L.

2. Now let (M, ∇) be a connected manifold with an affine connection. Thus, we have a connection Γ in the frame bundle $L(M)$. Let $u_0 \in L(M)$ be a fixed frame at a point $o \in M$, and $P(u_0)$ *the holonomy bundle through* u_0, i.e., the set of all points of $L(M)$ which can be joined to u_0 by a (piecewise differentiable) horizontal curve. Further,

let $\Phi(u_0)$ denote the holonomy group of Γ with the reference frame u_0. ($\Phi(u_0) \subset GL(n, R)$ is isomorphic to the holonomy group $\Psi(o)$ with the reference point o.)
The famous "Reduction Theorem" says that

(i) $P(u_0)$ is a differentiable subbundle of $L(M)$ with the structure group $\Phi(u_0)$,
(ii) the connection Γ is reducible to a connection in $P(u_0)$.

Let $f: M \to M$ be a diffeomorphism and $\tilde{f}: L(M) \to L(M)$ the induced automorphism of $L(M)$. Clearly, if $\tilde{f}$ preserves a fixed holonomy bundle $P(u_0)$ then it also preserves the holonomy bundle $P(u)$ for each $u \in L(M)$.

Definition 1. Let (M, ∇) be a connected manifold with an affine connection. The group of all affine transformations of M preserving the holonomy bundles $P(u)$ is called *the group of transvections of* (M, ∇), and it is denoted by $\mathrm{Tr}\,(M)$.

More geometrically, an affine transformation f belongs to $\mathrm{Tr}\,(M)$ if and only if it has the following property: for every point $p \in M$ there is a piecewise differentiable curve, starting at p and ending at $f(p)$, such that the tangent map $f_{*p}: M_p \to M_{f(p)}$ coincides with the parallel transport along this curve.

Remark. Every affine transformation of M preserves the foliation of $L(M)$ into the holonomy bundles but, in general, it interchanges the leaves. Hence $\mathrm{Tr}\,(M)$ is a *normal subgroup* of the whole affine group $A(M)$.

Now we can state our first main result:

Theorem 1. *Let (M, ∇) be a connected manifold with an affine connection. Then the following two conditions are equivalent:*

(i) *The transvection group* $\mathrm{Tr}\,(M)$ *acts transitively on each holonomy bundle $P(u)$.*
(ii) *M can be expressed as a reductive homogeneous space K/H with respect to a decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$, where K is effective on M, and ∇ is the canonical connection of K/H.*

Moreover, if (ii) *is satisfied, then* $\mathrm{Tr}\,(M)$ *is a connected Lie group, namely a normal Lie subgroup of K, and its Lie algebra is isomorphic to the ideal $\mathfrak{l} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ of $\mathfrak{k}$.*

Remark. The condition (i) in Theorem 1 means geometrically the following: for each piecewise differentiable curve in M there is an affine transformation $f \in \mathrm{Tr}\,(M)$ inducing the parallel transport along this curve from the initial point to the end point.

Proof of Theorem 1. (i) $\to$ (ii). Let $\mathrm{Tr}\,(M)$ act transitively on a holonomy bundle $P(u_0)$, where $u_0 \in L(M)$ is a frame at a point $o \in M$. Then $\mathrm{Tr}\,(M)$ is simply transitive on $P(u_0)$. Let $A(M)$ be the Lie group of all affine transformations of M, and $A(u_0) \subset L(M)$ the subbundle generated by the action of $A(M)$ on u_0. The map $g \mapsto g \cdot u_0$ is a diffeomorphism between $A(M)$ and $A(u_0)$.
According to the Reduction theorem, $P(u_0)$ is a differentiable subbundle of $L(M)$ and hence that of $A(u_0)$. If we identify $\mathrm{Tr}\,(M)$ with $P(u_0)$ and provide $\mathrm{Tr}\,(M)$ with the corresponding differentiable structure, then $\mathrm{Tr}\,(M)$ becomes a connected Lie subgroup of $A(M)$. In the following we shall denote $\mathrm{Tr}\,(M)$ by K and its Lie algebra by $\mathfrak{k}$.
Let $\{Q_u\}$ be the horizontal distribution on $L(M)$ corresponding to the linear connection Γ. Because Γ is reducible to $P(u_0)$, the subspaces Q_u are tangent to $P(u_0)$ for $u \in P(u_0)$. Also, the distribution $\{Q_u\}$ is K-invariant.
Let $H \subset K$ be the subgroup preserving the fibre of $P(u_0)$ over o, then $M \approx K/H$ and K is effective on M. Consider the linear isomorphism $\hat{u}_0: \mathfrak{k} \to \big(P(u_0)\big)_{u_0}$ given by the map $X \mapsto X \cdot u_0$ for $X \in \mathfrak{k}$. Then the decomposition of $\big(P(u_0)\big)_{u_0}$ into the vertical

subspace Φ_{u_0} and the horizontal subspace Q_{u_0} corresponds to a decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m} \subset \mathfrak{k}$ is a linear subspace. We obtain easily that $\mathrm{ad}(H)(\mathfrak{m}) = \mathfrak{m}$. Moreover, $Q_u = \mathfrak{m} \cdot u$ along the fibre of $L(M)$ over o. Thus K/H is a reductive homogeneous space and Γ is its canonical connection. Q.e.d.

Now suppose that the condition (ii) is satisfied and let L denote the Lie group from Proposition F. We shall show successively: $L \subset \mathrm{Tr}(M)$, L is transitive on a bundle $P(u_0)$, $L = \mathrm{Tr}(M)$.

Lemma 1. *For $p \in M$ denote by $\mathfrak{m}_p$ the subspace $\mathrm{ad}(g)\,\mathfrak{m} \subset \mathfrak{k}$, where $g \in K$ satisfies $g \in \pi^{-1}(p)$. Then $\mathfrak{m}_p$ is independent of the choice of g and $\mathfrak{l} = \mathfrak{m}_p + [\mathfrak{m}_p, \mathfrak{m}_p]$. Further, the group L is generated by the set $\exp(\mathfrak{m}_p)$.*

Proof. The reductivity of K/H means that $\mathrm{ad}(h)\,\mathfrak{m} = \mathfrak{m}$ for $h \in H$; hence $\mathfrak{m}_p$ is independent of $g \in \pi^{-1}(p)$. Further, $\mathfrak{l}$ is an ideal of $\mathfrak{k}$ and thus $\mathrm{ad}(g)\,\mathfrak{l} = \mathfrak{l}$ for $g \in K$. It implies $\mathfrak{m}_p + [\mathfrak{m}_p, \mathfrak{m}_p] = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] = \mathfrak{l}$. Finally, L is generated by $\exp(\mathfrak{m})$ (Proposition F) and hence $L = \mathrm{ad}(g)\,L$ is generated by $\mathrm{ad}(g)(\exp \mathfrak{m}) = \exp(\mathfrak{m}_p)$, q.e.d.

Lemma 2. *The global 1-parametric subgroup $f_t = \exp tX$, $X \in \mathfrak{m}_p$, has the property that the curve $x_t = f_t(p)$ is a geodesic and the tangent map $(f_s)_{*p}$ coincides with the parallel transport along the curve x_t, $0 \leq t \leq s$.*

Proof. For $p = o$ ($=$ the origin) Lemma 2 is equivalent to (i), Proposition C. For arbitrary $p \in M$ and $X \in \mathfrak{m}_p$ put $X = \mathrm{ad}(g)\,Y$, where $g \in K$, $Y \in \mathfrak{m}$. Then $(\exp tY)(o)$ is a geodesic and the tangent map $(\exp sY)_{*0}$ coincides with the parallel transport along $(\exp tY)(o)$, $0 \leq t \leq s$. Now, $f_t = g \cdot \exp tY \cdot g^{-1}$ and $x_t = [g \cdot \exp tY](o)$. Because g is an affine transformation of (M, ∇), then x_t is a geodesic and $(f_s)_{*p} = g_{*\,*}(\exp sY)_{*0} \circ g_{*p}^{-1}$ coincides with the parallel transport along x_t, $0 \leq t \leq s$, q.e.d.

Now, because K is effective on $M = K/H$, we can consider K as a group of affine transformations of (M, ∇). Denote by $\mathrm{Tr}^0(M)$ the group of all transformations $g \in K$ with the following property: For each $p \in M$ there is a broken geodesic γ from p to $g(p)$ such that the transformation g_{*p} coincides with the parallel transport along γ. Obviously, $\mathrm{Tr}^0(M) \subset \mathrm{Tr}(M)$. We shall show $L \subset \mathrm{Tr}^0(M)$; hence it will follow the inclusion $L \subset \mathrm{Tr}(M)$.

Let $\mathrm{Tr}^0(p)$ denote the set of all transformations $g \in K$ such that, for a broken geodesic γ from p to $g(p)$, g_{*p} induces the parallel transport along γ. Then $Tr^0(M) = \bigcap_{p \in M} \mathrm{Tr}^0(p)$. It suffices to show $L \subset \mathrm{Tr}^0(p)$ for each $p \in M$.

Now let $p \in M$ be given. According to Lemma 1, L is generated by the set $\exp(\mathfrak{m}_p)$. It means that $L = \bigcup_{i=1}^{\infty} L^i(p)$, where

$$L^i(p) = \{g \in L \mid g = \exp X_1 \cdots \exp X_i;\ X_1, \ldots, X_i \in \mathfrak{m}_p\}.$$

According to Lemma 2 we have $L^1(p) \subset \mathrm{Tr}^0(p)$. Now suppose $L^i(p) \subset \mathrm{Tr}^0(p)$ and let $g \in L^{i+1}(p)$, i.e., $g = g'h$ with $g' \in L^i(p)$ and $h = \exp X$, $X \in \mathfrak{m}_p$. Then there is a broken geodesic γ' from p to $g'(p)$ such that g' induces the parallel transport along γ'. On the other hand, we can write $g = h'g'$, where $h' = \mathrm{Ad}(g')\,h = \exp X'$ for $X' = \mathrm{ad}(g')\,X$. Here $X' \in \mathfrak{m}_{g'(p)}$. Using Lemma 2 once again for the new origin $g'(p)$ we can see that the transformation h' induces the parallel transport along a geodesic arc joining $g'(p)$ to $g(p) = (h'g')(p)$. Thus $g \in \mathrm{Tr}^0(p)$, which completes the induction step.

Now we shall need some more lemmas.

Lemma 3. *For each broken geodesic $\gamma = \widehat{pq}$ in M there is an element $g \in L$ such that the parallel transport along γ coincides with the differential g_{*p}.*

Proof. Let γ consist of geodesic arcs $\gamma_1, \ldots, \gamma_k$ starting at the points $p = p_1, \ldots, p_k$ respectively. Then the parallel transport along γ_i is given by the differential of a transformation $h_i = \exp X_i$, where $X_i \in \mathfrak{m}_{p_i} \subset \mathfrak{l}$. Thus the parallel transport along γ is given by the differential of $h = h_k \cdots h_1 \in L$.

Lemma 4. *Let $L(p)$ be the isotropy group of L at $p \in M$, and $L^0(p)$ its connected component. Then $L^0(p)$ is isomorphic to the restricted holonomy group $\Psi^0(p)$ of (M, ∇) with the reference point p.*

Proof. It is sufficient to show our assertion for $p = o$. The Lie algebra of $L^0(o)$ is isomorphic to $\mathfrak{l} \cap \mathfrak{h} = [\mathfrak{m}, \mathfrak{m}] \cap \mathfrak{h}$. Thus it is spanned by all elements of the form $[X, Y]_{\mathfrak{h}}$, $X, Y \in \mathfrak{m}$. According to a well-known theorem (see e.g. [1]), it is isomorphic to the holonomy algebra $\mathfrak{g}(o)$ with the reference point o. Hence the result follows.

Now we shall prove that $L \subset \mathrm{Tr}\,(M)$ is acting transitively on any holonomy bundle $P(u_0)$, $u_0 \in \bar{\pi}^{-1}(o)$. According to the Reduction theorem, if we consider the map $S \mapsto u_0S$ of $GL(n, R)$ onto the fibre $\bar{\pi}^{-1}(o) \subset L(M)$, then the holonomy group $\Phi(u_0)$ is mapped onto $\bar{\pi}^{-1}(o) \cap P(u_0)$ and the restricted holonomy group $\Phi^0(u_0)$ is mapped onto the connected component $C(u_0) \ni u_0$ of $\bar{\pi}^{-1}(o) \cap P(u_0)$. Now, the group L is acting freely on $P(u_0)$. By means of the map $g \mapsto g \cdot u_0$, the subgroup $L^0(o)$ obviously falls into $C(u_0)$. Consider the homomorphism $\lambda: L^0(o) \to \Phi^0(u_0)$ given by $g \cdot u_0 = u_0\lambda(g)$. Because λ is injective and further, $L^0(o)$ is isomorphic to $\Psi^0(o)$ according to Lemma 4, $\Phi^0(u_0)$ is isomorphic to $\Psi^0(o)$, and the Lie groups $L^0(o)$ and $\Phi^0(u_0)$ are both connected, hence we obtain $\lambda\big(L^0(o)\big) = \Phi^0(u_0)$. Hence $L^0(o)$ is transitive on $C(u_0)$.

Now, each homotopy class of (piecewise differentiable) loops at o can be represented by a closed broken geodesic. According to Lemma 3, the isotropy subgroup $L(o)$ acts transitively on the set of connected components of $\bar{\pi}^{-1}(o) \cap P(u_0)$. Because $L^0(o)$ was transitive on $C(u_0)$, $L(o)$ is transitive on the whole of $\bar{\pi}^{-1}(o) \cap P(u_0)$ and L is transitive on $P(u_0)$. As a consequence, we get the relation $L = \mathrm{Tr}\,(M)$. This completes the proof of our theorem.

Remark. Some partial results towards Theorem 1 can be found in [1], p. 194.

On the basis of the previous theorem we can introduce the following definition:

Definition 2. A connected manifold (M, ∇) with an affine connection is called *an affine reductive space* if the group $\mathrm{Tr}\,(M)$ acts transitively on each holonomy bundle $P(u_0)$.

Obviously, on an affine reductive space (M, ∇), the connection ∇ is always complete and it satisfies $\nabla R + \nabla T = 0$. Conversely, if (M, ∇) is connected and simply connected with ∇ complete and satisfying $\nabla R = \nabla T = 0$, then (M, ∇) is an affine reductive space. It follows from the well-known results about reductive homogeneous spaces and from Theorem 1.

Directly from Definition 2 we obtain: *On an affine reductive space (M, ∇), a tensor field is parallel if and only if it is invariant with respect to the group* $\mathrm{Tr}\,(M)$.

We shall close the paper with a different (and perhaps, more systematic) version of Theorem 1. A reductive homogeneous space K/H with the given decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$ will be called *prime* if K acts effectively on K/H and $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} = \mathfrak{h}$. A connected manifold (M, ∇) with an affine connection will be called *pointed* if there is given a fixed point (origin) $o \in M$.

Theorem 2. *There is a one-to-one correspondence between the pointed affine reductive spaces* (M, ∇, o) *and the prime reductive homogeneous spaces* K/H *with a given decomposition. This correspondence is given by the formulas* $K = \mathrm{Tr}\,(M)$, $H = K_0$ *in one direction, and by the formulas* $M \approx K/H$, ∇ = *the canonical connection of* K/H, *in the other direction.*

The proof is obvious from that of Theorem 1.

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