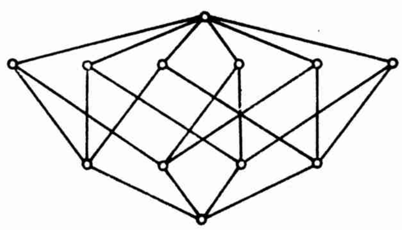
Beiträge zur Algebra und Geometrie
7 (1978), 13–19

On perspectivities in incidence geometries of grade n

Manfred Stern

1. In projective geometries perspectivities behave in a regular manner in the following sense: they are transitive and perspective elements have the same dimension. Both properties do not hold in affine geometries; this is shown immediately by the following affine geometry of order 2.



In [7] we have proved that with respect to perspective elements in affine geometries an extremal case is possible: there exists an affine geometry of infinite dimension such that an atom (point) is perspective to a dual atom (hyperplane). This is why one is interested e.g. in characterizing those affine geometries (of infinite dimension) in which an element of finite dimension can be perspective only to an element of finite dimension.
In [9] we have given such a characterization for weakly modular matroid lattices which include the affine geometries. M. F. Janowitz has proposed to extend these results on incidence geometries of grade n which have been introduced by R. Wille in [11].
For the non upper-continuous case (that is, for certain generalized finite-modular AC-lattices which need not be matroid lattices) this was done in [10]. In this note we extend the result of [9] and [10] to incidence geometries of grade n.

Acknowledgements. We thank Dr. M. F. Janowitz for his valuable remarks.

2. By $b \prec a$ we mean that the element a covers the element b. If $0 \prec p$ holds in a lattice with 0, then p is called an *atom*. A lattice with 0 is called *atomistic*, if each element ($\neq 0$) can be expressed as a union of atoms. An *AC-lattice* is an atomistic

lattice in which the following implication holds: if p is an atom and $a \wedge p = 0$, then $a \prec a \vee p$. *Matroid lattices* can be defined as upper continuous AC-lattices. For the theory of AC-lattices and matroid lattices we refer to [5].
In [9, Hilfssatz 3.2] we have shown that there exists to each element x of a matroid lattice a uniquely determined cardinal number $r(x)$ which is called the *rank* of x.[1]) This notion of rank is a natural generalization of the notion of height as used in [5].
By [5, Remark 13.2, p. 56] an interval $[a, b]$ of a matroid lattice L is itself a matroid lattice. By $r[a, b]$ we denote the rank of $[a, b]$ that is, the rank of b with respect to $[a, b]$. Instead of $r[0, 1]$ we write $r(L)$. A matroid lattice L is said to be *of infinite length* if $r(L) \geqq \aleph_0$.
In [9, Folgerung 3.4] we have shown that for a matroid lattice L of infinite length the set

$$F_{\aleph}(L) = \{a \mid a \in L \text{ and } r(a) < \aleph\}$$

$\left(\aleph_0 \leqq \aleph \leqq r(L)\right)$ is an ideal in L. $F_{\aleph_0}(L)$ coincides with the set $F(L)$ as defined in [5, Def. 8.1, p. 35]. $F(L)$ consists of 0 and of all those elements which can be expressed as a union of finitely many atoms; this is why $F(L)$ is also called the ideal of the finite elements of L.

According to R. WILLE [11] we introduce

Definition 1. A lattice L is said to be an *incidence geometry* of grade n if L is a matroid lattice and if for every element $a \in L$ with $r(a) = n$ the interval $[0, a]$ is distributive and the principal dual ideal $[a)$ is modular.

We note that we do not use the distributivity of the interval $[0, a]$ in this paper.
Incidence geometries of grade n are far-reaching generalizations of projective geometries: for $n = 0$ one obtains just the projective geometries; $n = 1$ gives the strongly planar geometries (cf. [6] and [4]) which include the affine geometries. The Möbius geometries (cf. [2]) are examples of incidence geometries of grade 2.

3. In a lattice with 0, we introduce perspectivity as follows: two elements a, b are said to be *perspective*, and we write $a \sim b$ if there exists an element x for which

$$a \vee x = b \vee x \qquad \text{and} \qquad a \wedge x = b \wedge x = 0$$

(cf. [5, Def. 6.1, p. 26]). Furthermore we need

Definition 2. Let L be an AC-lattice and $y, z \in L$. If $r(y \wedge z) \leqq n$ (n fixed non-negative integer) and $z \prec z \vee y$, then we write $y <|_n z$. Instead of $y <|_0 z$ we write simply $y <| z$.
For the relation $<|$ which abstracts a property of parallelity, we refer to [5, Def. 17.1, p. 72]. The relation $<|_n$ (cf. also [10]) is a natural generalization of $<|$.
According to [1] we call an ideal S of a lattice L a *standard ideal* if $J \wedge (S \vee K) = (J \wedge S) \vee (J \wedge K)$ holds for arbitrary ideals J, K of L.
An ideal S of a lattice L is said to be *projective* (cf. [3]) if $a \in S$, $b, x \in L$, $a \vee x \geqq b \vee x$ and $a \wedge x \geqq b \wedge x$ imply $b \in S$. An ideal S of a lattice L with 0 is called *p-ideal* if S is closed with respect to perspectivity, that is, if $a \in S$, $b \sim a$ imply $b \in S$.

---

[1]) Cf. also S. MACLANE, A lattice formulation for transcendence degrees and p-bases, Duke Math. J. 4 (1938), 455–468.

4. With respect to the ideal $F_{\aleph}(L)$ of a matroid lattice L, we shall consider the following conditions:

(i) $F_{\aleph}(L)$ is a standard ideal;
(ii) $F_{\aleph}(L)$ is a projective ideal;
(iii) $F_{\aleph}(L)$ is a p-ideal;
(iv) $y < |_n z \Rightarrow y \in F_{\aleph}(L)$;
(v) $y < | z \Rightarrow y \in F_{\aleph}(L)$.

Among these conditions, we have the following interdependence relations:

$$\begin{array}{lcl} \text{(i)} & & \\ \Downarrow & & \\ \text{(ii)} & \Rightarrow & \text{(iv)} \\ \Downarrow & & \Downarrow \\ \text{(iii)} & \Rightarrow & \text{(v)}. \end{array}$$

It is known that the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold for arbitrary ideals of a lattice with 0.
In [8] we proved (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i) in case $\aleph = \aleph_0$ for strongly planar AC-lattices (without assuming upper continuity); this result was generalized in [10]. In the presence of upper continuity we could prove the same for arbitrary $\aleph$ (cf. [9, Satz 4.1]).

These results are here extended to incidence geometries of grade n, for which we prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i) (cf. Theorem 6). In the case of strongly planar geometries we can even show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) (cf. Corollary 8).

5. We shall need three results on matroid lattices which are given in this section. The first one is a necessary and sufficient condition for $F_{\aleph}(L)$ to be a standard ideal of a matroid lattice L.

Theorem 3 (cf. [9, Satz 3.6]). *Let L be a matroid lattice of infinite length and let $\aleph_0 \leqq \aleph \leqq r(L)$. Then the following two conditions are equivalent:*

(a) *$F_{\aleph}(L)$ is a standard ideal of L;*

(b) *if $[a, a \vee b]$ and $[a \wedge b, b]$ are transposed intervals and if $r[a, a \vee b] < \aleph$, then $r[a \wedge b, b] < \aleph$.*

Next we show how to gain in a matroid lattice special elements y, z for which $y < |_n z$ holds.

Proposition 4. *Let L be a matroid lattice and let $a, b \in L$ with $r(b) > n$. If $r(a \wedge b) < n$ then there exists a maximal element c with $a \leqq c < a \vee b$ and $r(b \wedge c) \leqq n$. Moreover, if p is an atom of L with $p \nleqq c$ and $p < a \vee b$, then $b \wedge (c \vee p) < |_n c$.*

Proof. Consider the set

$$C = \{c_\nu \in L \mid a \leqq c_\nu < a \vee b \text{ and } r(b \wedge c_\nu) \leqq n\}.$$

The set C ist not empty since $a \in C$. Let now K denote an arbitrary chain in C and let $c_\mu \in K$ $(\mu \in \Gamma)$. Since L is upper continuous, we have

$$b \wedge \bigvee(c_\mu \mid \mu \in \Gamma) = \bigvee(b \wedge c_\mu \mid \mu \in \Gamma). \tag{1}$$

By assumption $r(b \wedge c_\mu) \leqq n$ $(\mu \in \Gamma)$. Since L is an AC-lattice, there exists a $d \in L$ with $r(d) = n$ and $b \wedge c_\mu \leqq d$ $(\mu \in \Gamma)$. From this it follows that $\bigvee(b \wedge c_\mu \mid \mu \in \Gamma) \leqq d$, that is $r\big(\bigvee(b \wedge c_\mu \mid \mu \in \Gamma)\big) \leqq n$. Because of (1), we obtain therefore

$$r\big(b \wedge \bigvee(c_\mu \mid \mu \in \Gamma)\big) \leqq n,$$

that is, $\bigvee(c_\mu \mid \mu \in \Gamma) \in C$. Thus every chain K of C has an upper bound in C. By the Lemma of KURATOWSKI-ZORN then there exists an element $c \in L$ which is maximal with respect to the properties $a \leqq c < a \vee b$ and $r(b \wedge c) \leqq n$.

Let now p be an atom of L with $p \nleqq c$ and $p < a \vee b$. Then $c \prec c \vee p \leqq a \vee b$ and by the maximality of c we get $r\big(b \wedge (c \vee p)\big) > n$. Furthermore the relation $b \wedge (c \vee p) \nleqq c$ holds; for, if $b \wedge (c \vee p) \leqq c$, then $b \wedge (c \vee p) = b \wedge (c \vee p) \wedge c = b \wedge c$ and it follows that $r\big(b \wedge (c \vee p)\big) = r(b \wedge c)$, a contradiction to our conditions on the ranks. Thus we obtain

$$[b \wedge (c \vee p)] \vee c = c \vee p$$

and

$$b \wedge (c \vee p) \wedge c = b \wedge c,$$

that is

$$b \wedge (c \vee p) <|_n c$$

holds (cf. Definition 2), which finishes the proof.

Finally we need

Proposition 5. *Let L be a matroid lattice. Consider the following three conditions:*

(a) *$F_\aleph(L)$ is a projective ideal;*

(b) *if $a \in F_\aleph(L)$, $a \wedge x = b \wedge x$ and $a \vee x = b \vee x$, then $b \in F_\aleph(L)$;*

(c) *$y <|_n z$ implies $y \in F_\aleph(L)$.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b): this follows from the definition of the projective ideal.

(b) $\Rightarrow$ (c): let $y <|_n z$ for $y, z \in L$. By [5, Lemma 8.18, p. 39] $[z \wedge y, z \vee y]$ is also an AC-lattice and thus there exists an $a \rightarrowtail z \wedge y$ such that

$$z \wedge a = z \wedge y \qquad \text{and} \qquad z \vee a = z \vee y.$$

Since $r(z \wedge y) \leqq n$, we get $z \wedge y \in F(L)$ and therefore $a \in F(L) \subset F_\aleph(L)$. By (b) this yields $y \in F_\aleph(L)$, and the proposition is proved.

We remark that in [10] we have been able to prove the equivalence fo the three conditions of the preceding proposition at least for $\aleph = \aleph_0$ (without assuming upper continuity).

6. We are now ready to prove the main result of this paper.

Theorem 6. *Let L be an incidence geometry of grade n. Then the following three conditions are equivalent:*

(a) *$F_\aleph(L)$ is a standard ideal* ((i) *in Section* 4);

(b) *$F_\aleph(L)$ is a projective ideal* ((ii) *in Section* 4);

(c) $y <|_n z \Rightarrow y \in F_\aleph(L)$ ((iv) *in Section* 4).

Proof. (a) $\Rightarrow$ (b): this is known for arbitrary ideals.
(b) $\Rightarrow$ (c): this follows from Proposition 5.
(c) $\Rightarrow$ (a): because of Theorem 3 it is sufficient to prove the following implication

$$r[a, a \vee b] < \aleph \Rightarrow r[a \wedge b, b] < \aleph. \tag{2}$$

Assume that $r[a, a \vee b] < \aleph$. If $b \in F_\aleph(L)$, then also $a \wedge b \in F_\aleph(L)$ and $r[a \wedge b, b] < \aleph$. Hence we may suppose that $b \notin F_\aleph(L)$. With respect to the rank of $a \wedge b$ we distinguish two cases: $r(a \wedge b) \geqq n$ and $r(a \wedge b) < n$.
If $r(a \wedge b) \geqq n$, then there exists a $d \in L$ with the properties $d \leqq a \wedge b$ and $r(d) = n$, and the principal dual ideal $[d)$ is a modular lattice (cf. Definition 1). This immediately yields $r[a \wedge b, b] < \aleph$, since transposed intervals in a modular lattice are isomorphic. Thus (2) is proved in this case.
Let now $r(a \wedge b) < n$. By Proposition 4 there exists an $(a \leqq)\, c\, (< a \vee b)$ which is maximal with respect to the property $r(b \wedge c) \leqq n$; moreover, if p is an atom with $p \nleqq c$ and $p \leqq a \vee b$ (such an atom exists since L is an AC-lattice), then $c \prec c \vee p \leqq a \vee b$. Because of the maximality of c it follows that $r\big(b \wedge (c \vee p)\big) > n$. Then one can find an element $e \in L$ such that $r(e) = n$ and $b \wedge c \leqq e < b \wedge (c \vee p)$ since L is an AC-lattice. The principal dual ideal $[e)$ is a modular lattice (cf. Definition 1). Moreover

$$b \wedge (c \vee p),\ c \vee p,\ b,\ b \vee (c \vee p) = a \vee b \in [e)$$

and

$$r[c \vee p, a \vee b] < \aleph$$

because of $a \leqq c \vee p < a \vee b$ and $r[a, a \vee b] < \aleph$. This yields

$$r[b \wedge (c \vee p), b] < \aleph \tag{3}$$

since $[c \vee p, a \vee b]$ and $[b \wedge (c \vee p), b]$ are transposed intervals in the modular lattice $[e)$. By Proposition 4 we get further

$$b \wedge (c \vee p) <|_n\, c.$$

Because of (c), this yields $b \wedge (c \vee p) \in F_\aleph(L)$. From this it follows that $r[0, b \wedge (c \vee p)] < \aleph$ and therefore

$$r[a \wedge b, b \wedge (c \vee p)] < \aleph \tag{4}$$

since $0 \leqq a \wedge b < b \wedge (c \vee p)$. It is now easy to see that (3) and (4) together give

$$r[a \wedge b, b] < \aleph.$$

Thus implication (2) holds also in the case $r(a \wedge b) < n$ which proves the theorem.

Corollary 7. *Let L be an incidence geometry of grade n and of infinite length. Let $\aleph_0 \leqq \aleph \leqq r(L)$. Then the following two conditions are equivalent:*
(a) *$F(L)$ is a standard ideal in L;*
(b) *$F_\aleph(L)$ is a standard ideal in L for all $\aleph$ $\big(\aleph_0 \leqq \aleph \leqq r(L)\big)$.*

Proof. (b) $\Rightarrow$ (a): this implication holds trivially.
(a) $\Rightarrow$ (b): Let $F(L)$ be a standard ideal in L. Then by Theorem 6 the implication

$$y <|_n\, z \Rightarrow y \in F(L)$$

holds. Since $F(L) \subset F_{\aleph}(L)$ for all $\aleph$ $(\aleph_0 \leqq \aleph \leqq r(L))$, we get that

$$y < |_n z \Rightarrow y \in F_{\aleph}(L)$$

holds. Again by Theorem 6, $F_{\aleph}(L)$ is then a standard ideal in L, and the corollary is proved.

For the special case of weakly modular matroid lattices we proved this in [9].

Corollary 8. *Let L be a strongly planar geometry (that is, an incidence geometry of grade 1). Then then following three conditions are equivalent:*

(a) *$F_{\aleph}(L)$ is a standard ideal ((i) in Section 4);*

(b) *$F_{\aleph}(L)$ is a p-ideal ((iii) in Section 4);*

(c) *$y < | z \Rightarrow y \in F_{\aleph}(L)$ ((v) in Section 4).*

Proof. (a) $\Rightarrow$ (b): as we already remarked, this holds for arbitrary ideals of a lattice with 0.

(b) $\Rightarrow$ (c): assume that $y < | z$ holds in L, but $y \notin F_{\aleph}(L)$. For a $p < y$ then we obtain

$$z \vee y = z \vee p \quad \text{and} \quad z \wedge y = z \wedge p = 0,$$

that is, $p \sim y$. Because of $p \in F_{\aleph}(L)$, then $F_{\aleph}(L)$ cannot be a p-ideal.

(c) $\Rightarrow$ (a): consider elements $y, z \in L$ for which $y < |_n z$ holds, that is $z \prec z \vee y$ and $r(y \wedge z) \leqq n$ (cf. Definition 2).

If $y \wedge z = 0$, then $y < | z$ holds (cf. Definition 2) and we get $y \in F_{\aleph}(L)$ by (c).

If $y \wedge z > 0$, then $[y \wedge z)$ is a modular lattice (cf. Definition 1) and it follows that $y \wedge z \prec y$. Because of $r(y \wedge z) \leqq n$, this yields $y \in F(L) \subset F_{\aleph}(L)$.

Thus in either case the implication

$$y < |_n z \Rightarrow y \in F_{\aleph}(L)$$

holds in L. By Theorem 6 it follows that $F_{\aleph}(L)$ is a standard ideal which was to be proved.

For the special case of an affine geometry the equivalence of these an of other conditions has been proved in [9, Satz 5.2].

REFERENCES

[1] GRÄTZER, G., and E. T. SCHMIDT: Standard ideals in lattices. Acta Math. Acad. Sci. Hung. *12* (1961), 17–86.

[2] HOFFMAN, A. J.: On the foundations of inversion geometry. Trans. Amer. Math. Soc. *71* (1951), 218–242.

[3] JANOWITZ, M. F.: A characterization of standard ideals, Acta Math. Acad. Sci. Hung. *16* (1965), 289–301.

[4] JÓNSSON, B.: Lattice-theoretic approach to projective and affine geometry. In: The Axiomatic Method (ed. by L. HENKIN, P. SUPPES, A. TARSKI). Studies in Logic, North-Holland Publishing Comp., Amsterdam 1959, pp. 188–203.

[5] MAEDA, F., and S. MAEDA: Theory of Symmetric Lattices, Springer-Verlag, Berlin—Heidelberg—New York 1970.

[6] SASAKI, U.: Lattice-theoretic characterization of geometries satisfying „Axiome der Verknüpfung". J. Soc. Hiroshima, Ser. A, *16* (1952/53), 417–423.

[7] STERN, M.: On a problem of M. F. Janowitz. Beiträge zur Algebra und Geometrie *4* (1975), 89–91.
[8] STERN, M.: Strongly planar AC-lattices in which the ideal of the finite elements is standard. Acta Math. Acad. Sci. Hung. *26* (1975), 229–232.
[9] STERN, M.: Standard ideals in matroid lattices. Publ. Math. Debrecen (to appear).
[10] STERN, M.: О конечо-модулярных AC-решетках n-ой ступени. Упорядоченные множества и решетки (to appear).
[11] WILLE, R.: Verbandstheoretische Charakterisierung n-stufiger Geometrien, Arch. Math. *18* (1967), 465–468.

Manuskripteingang: 2. 2. 1976

VERFASSER:

MANFRED STERN, Sektion Mathematik der Martin-Luther-Universität
Halle—Wittenberg