Beiträge zur Algebra und Geometrie
6 (1977), 101–104

Atomistic Wilcox lattices in which the ideal of the finite elements is standard

MANFRED STERN

§ 1. Introduction

This paper has its roots in [3] and is a continuation of [5, 6], and [7].
In [3] M. F. JANOWITZ proved that in a finite-modular AC-lattice the ideal $F(L)$ of the finite elements of L is always standard ([3, Theorem 4.6]) and posed the question: is $F(L)$ standard where L is an arbitrary AC-lattice? This turned out to be not true in general: a counterexample was given in [5]. Now the following problem arises: give characterizations for $F(L)$ to be a standard ideal of an AC-lattice of infinite length. This problem was answered in [7]. M. F. JANOWITZ gave equivalent conditions for $F(L)$ to be a p-ideal where L is an atomistic Wilcox lattice. This result was not published. It is the aim of this paper to extend this result, which is done in § 4. We frequently use here our results [7].

Acknowledgements. We are thankful to Dr. M. F. JANOWITZ for communicating his unpublished results and for his helpful remarks.

§ 2. Basic notions

Let L be a lattice and $a, b \in L$. We say that (a, b) is a modular pair and we write $(a, b)M$ if $(c \cup a) \cap b = c \cup (a \cap b)$ for every $c \leqq b$. If $(a, b)M$ implies $(b, a)M$ in L, then we call L an M-symmetric lattice. A lattice L with 0 is called weakly modular if in L

$$a \cap b \neq 0 \quad \text{implies} \quad (a, b)M.$$

Now let Λ be a given complemented modular lattice with the lattice operations $\sqcup$ and $\sqcap$. Let S be a fixed subset of $\Lambda - \{0, 1\}$ with the following two properties:

$$a \in S \quad \text{and} \quad 0 < b \leqq a \quad \text{imply} \quad b \in S$$

and

$$a, b \in S \quad \text{implies} \quad a \sqcup b \in S.$$

If in the set $L \equiv \Lambda - S$ we give the same order as in Λ, then L is a weakly modular M-symmetric lattice (cf. [4, Theorem 3.11, p. 12]). If a weakly modular M-symmetric lattice L arises from a complemented modular lattice in the manner described above, then we call L a Wilcox lattice.

According to G. GRÄTZER and E. T. SCHMIDT (cf. [1, p. 30]) we call an ideal R of a lattice L standard ideal if $I \wedge (R \vee K) = (I \wedge R) \vee (I \wedge K)$ holds for any pair of ideals I, K of L.
Let L be a lattice with 0. We say that $a \in L$ and $b \in L$ are *perspective* and write $a \sim b$, if $a \cup x = b \cup x$ and $a \cap x = b \cap x = 0$ für some $x \in L$. An ideal R of a lattice L with 0 is called a p-ideal if $a \in R$ and $b \sim a$ imply $b \in R$. Without proof we state

Lemma 2.1. *Let L be a lattice with 0. Then every standard ideal of L is a p-ideal.*

In a lattice L we write $b \prec a$ $(a, b \in L)$ if $b < a$ and $b \leqq x \leqq a$ implies $x = b$ or $x = a$. If $0 \prec p$ in a lattice with 0, then p is called an *atom*. A lattice L is called *atomistic*, if every element of L is the join of atoms. The covering property is defined as follows:

if p is an atom and $p \nleqq a$ $(a, p \in L)$, then $a \prec a \cup p$.

A lattice is called *AC-lattice* if it is an atomistic lattice with covering property.

§ 3. The ideal $F(L)$ of an AC-lattice L

Lemma 3.1 (cf. [4, Lemma 8.8, p. 37]). *In every AC-lattice L the set $F(L)$ is an ideal of L (by $F(L)$ we denote the set of those elements of L which can be written as a union of finitely many atoms).* This ideal is called the *ideal of the finite elements of L.*

In [7] the following characterization for $F(L)$ to be a standard ideal was given:

Theorem 3.2 (cf. [7, Theorem 3.2]). *Let L be an AC-lattice (of infinite length). Then the following conditions are equivalent:*

(i) *$F(L)$ is a standard ideal for L;*

(ii) *if $[x, b \cup x]$ and $[b \cap x, b]$ are transposed intervals of L and if the interval $[x, b \cup x]$ is of finite length, then the interval $[b \cap x, b]$ is of finite length, too.*

The following definition will be needed in the sequel.

Definition 3.3 (cf. [4, Definition 17.1, p. 72]). Let L be a lattice with 0 and let $y, z \in L$ $(y \neq 0,\ z \neq 0)$. We write $y <| z$ if $y \cap z = 0$ and $z \prec z \cup y$. If $y <| z$ and $z <| y$, then we say that y and z are *parallel elements* and write $y \parallel z$.

Lemma 3.4. *Let L be an AC-lattice (of infinite length). Consider the following conditions:*

(i) *$F(L)$ is a standard ideal of L;*
(ii) *$F(L)$ is a p-ideal of L;*
(iii) *$y <| z$ implies $y \in F(L)$;*
(iv) *$y \parallel z$ implies $y \in F(L)$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof. For (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) cf. [7, Lemma 3.4].

(iii) $\Rightarrow$ (iv): this is clear from Definition 3.3.

§ 4. Atomistic Wilcox lattices

We refer to [4] for the notions not define here.

By [4, Remark 20.1, p. 91] every atomistic Wilcox lattice is an AC-lattice. Furthermore, if $L \equiv \Lambda - S$ is an atomistic Wilcox lattice, then Λ is likewise an AC-lattice by [4, Lemma 20.3, p. 91] and by the modularity of Λ. Hence it is clear what is meant by $F(\Lambda)$. At several places we need

Remark 4.1. Let $L \equiv \Lambda - S$ be an atomistic Wilcox lattice. Then $a \in F(L)$ if and only if $a \in F(\Lambda)$. Moreover, the height $h(a)$ of a in L coincides with that in Λ (cf. [4, Remark 20. 4, p. 92]).

For weakly modular AC-lattices we proved

Theorem 4.2 (cf. [6, Corollary 8] or [7, Corollary 5.2]). *Let L be a weakly modular AC-lattice (especially: an atomistic Wilcox lattice). Then the following conditions are equivalent:*

(i) *$F(L)$ is a standard ideal of L;*

(ii) *$y <| z$ implies $y \in F(L)$.*

If L is actually an atomistic Wilcox lattice (with or without imaginary unit) one can give further equivalent conditions:

Theorem 4.3. *Let $L \equiv \Lambda - S$ be an atomistic Wilcox lattice. Consider the conditions:*

(i) *$F(L)$ is a standard ideal of L;*

(ii) *$F(L)$ is a p-ideal of L;*

(iii) *$y <| z$ implies $y \in F(L)$;*

(iv) *$y \,||\, z$ implies $y \in F(L)$;*

(v) *$S \subseteqq F(\Lambda)$;*

(vi) *$i \in F(\Lambda)$ (i denotes the imaginary unit).*

Then the conditions (i)—(v) *are equivalent. If L has an imaginary unit, then all six conditions are equivalent.*

Remark. The equivalence of the conditions (ii)—(v) was first proved by M. F. Janowitz whose result was not published as we remarked in § 1. New is that "$F(L)$ is a standard ideal" can be added as a further equivalent condition. If L is relatively complemented (i.e. if L is an affine matroid lattice) then the equivalence of conditions (i) and (ii) also follows from [2, Theorem 4.2].

Proof of the theorem. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv): this follows from Lemma 3.4. (iv) $\Rightarrow$ (v): for the set S of the imaginary elements of Λ we have $S \equiv \Lambda - \{0, 1\}$ (cf. § 2). Therefore, if $u \in S$, then

$$u \neq 0, 1. \tag{1}$$

Choose an atom $p \in L$ for which

$$p \nleq u. \tag{2}$$

By (1) this is always possible, since L is an AC-lattice. Consider now the element

$$a = p \cup u = p \sqcup u.^{1)} \tag{3}$$

[1]) According to [4, Theorem 3.11, p. 12] the union of two elements in L coincides with that in Λ.

By (1) and (2) we get $h(a) \geqq 2$ in L and in Λ (cf. Remark 4.1). The element $a \in L$ is singular by [4, Definition 21.1, p. 96). We distinguish two cases: $a \neq 1$ and $a = 1$. If $a \neq 1$, then by [4, Lemma 21.7, p. 97] there exists an element $b \in L$ such that $a \parallel b$. Then by condition (iv) it follows that $a \in F(L)$ and from this $a \in F(\Lambda)$ by Remark 4.1. By (3) we have $u \leqq a$ in Λ and therefore $u \in F(\Lambda)$.

If $a = 1$, then by [4, Lemma 21.7, p. 97) there exist two singular elements $a_1, a_2 \neq 1$ such that $1 = a_1 \cup a_2$. Proceeding as above, we get $a_1, a_2 \in F(L)$ and hence $a_1 \cup a_2 = 1 \in F(L)$. This means that L is of finite length. Then by Remark 4.1, Λ is of finite length, too. Therefore $1 \in F(\Lambda) = \Lambda$.

(v) $\Rightarrow$ (iii): let $y <\!| \, z$. By [4, Lemma 17.6, p. 72] $y <\!| \, z$ implies $(z, y)\overline{M}$ (i.e. (z, y) is not a modular pair) provided that y is not an atom. Furthermore, by [4, 3.11.5, p. 12] $(z, y)\ \overline{M}$ holds if and only if $z \sqcap y \in L$. Hence $z \sqcap y \in S$ and by condition (v) it follows that

$$z \sqcap y \in S \subseteqq F(\Lambda). \tag{4}$$

Now since $z \prec z \cup y = z \sqcup y$ in Λ we get by the modularity of Λ that $z \sqcap y \prec y$. Hence by (4) it follows that $y \in F(\Lambda)$. Therefore $y \in F(L)$ by Remark 4.1.

(iii) $\Rightarrow$ (i): this follows from Theorem 4.2.

Let now $L \equiv \Lambda - S$ be an atomistic Wilcox lattice with imaginary unit i. Then the implication (v) $\Rightarrow$ (vi) holds since $i \in S$. On the other hand, the implication (vi) $\Rightarrow$ (v) holds since $u \leqq i$ for all $u \in S$. This proves the theorem.

REFERENCES

[1] GRÄTZER, G., and E. T. SCHMIDT: Standard ideals in lattices. Acta Math. Acad. Sci. Hung. *12* (1961), 17–86.

[2] JANOWITZ, M. F.: A characterization of standard ideals. Acta Math. Acad. Sci. Hung. *16* (1965), 289–301.

[3] JANOWITZ, M. F.: On the modular relation in atomistic lattices. Fund. Math. *66* (1969/70), 337–346.

[4] MAEDA, F., and S. MAEDA: Theory of Symmetric Lattices. Springer-Verlag, Berlin—Heidelberg—New York 1970.

[5] STERN, M.: On a problem of M. F. Janowitz. Beiträge zur Algebra und Geometrie *4* (1975), 89–91.

[6] STERN, M.: Strongly planar *AC*-lattices in which the ideal of the finite elements is standard. Acta Math. Acad. Sci. Hung. *26* (1975), 229–232.

[7] STERN, M.: On *AC*-lattices in which the ideal of the finite elements is standard. Beiträge zur Algebra und Geometrie *5* (1976), 15–21.

Manuskripteingang: 22. 9. 1975

VERFASSER:

MANFRED STERN, Sektion Mathematik der Martin-Luther-Universität Halle—Wittenberg