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On AC-lattices in which the ideal of the finite elements is standard

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§ 1. Introduction

Let L denote an AC-lattice and $F(L)$ the ideal of the finite elements of L.
D. SACHS defined modulated matroid lattices and proved for a certain class of these lattices that $F(L)$ is a homomorphism kernel (cf. [6, Lemma 20]). In [4] there is a remark that in a finite-modular AC-lattice $F(L)$ is a p-ideal (cf. [4, Remark 35.8, p. 162]). M. F. JANOWITZ has proved more: In a finite-modular AC-lattice $F(L)$ is a standard ideal (cf. [3, Theorem 4.6]). Likewise in [3] the following question was raised: Is $F(L)$ a standard ideal where L is an arbitrary AC-lattice? The answer turned out to be negative: In a non-modular affine matroid lattice L of infinite length satisfying EUCLID's strong parallel axiom $F(L)$ is not standard (cf. [7]). As M. F. JANOWITZ remarked, another counter-example is provided by a partition lattice of infinite length: this is simple by a result of O. ORE in [5] and hence $F(L)$ cannot be standard.
All the above mentioned results naturally suggest to give conditions for $F(L)$ to be a standard ideal of an AC-lattice L of infinite length. A first step in this direction was done by M. F. JANOWITZ who obtained necessary and sufficient conditions for $F(L)$ to be a p-ideal where L is an atomistic Wilcox lattice. This result, which was not published, is cited in § 5.
In the present work we extend some of our results in [8]. However, the knowledge of [8] is not necessary for reading this paper.
In § 2 we give some basic notions. In § 3 equivalent conditions are given for $F(L)$ to be a standard ideal. In §§ 4—5 applications are given to section-complemented AC-lattices and to strongly planar AC-lattices.

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§ 2. Basic notions and theorems

In a lattice L we say that an element $a \in L$ *covers* the element $b \in L$ and write $b \prec a$ if $b < a$ and $b \leqq x \leqq a$ implies $x = b$ or $x = a$. If $0 \prec p$ in a lattice with 0, then p is called an *atom*. A lattice with 0 is called *atomistic* if every element $\neq 0$ is a join of atoms. In a lattice with 0, the following property is called *covering property*:

If p is an atom and $a \cap p = 0$, then $a \prec a \cup p$.

L is called an *AC-lattice* if it is an atomistic lattice with covering property. A *matroid lattice* can be defined as an upper continuous AC-lattice.
In a lattice L we say that (a, b) is a *modular pair* and write $(a, b)M$ when

$$c \leqq b \quad \text{implies} \quad (c \cup a) \cap b = c \cup (a \cap b).$$

An element $a \in L$ is called *modular* when $(x, a)M$ for every $x \in L$. A lattice with 0 is called *weakly modular* when in L

$$a \cap b \neq 0 \quad \text{implies} \quad (a, b)M.$$

A lattice L with 0 is called *finite-modular* if every finite element of L is modular (by a finite element we mean either 0 or the join of a finite number of atoms).

Lemma 2.1 ([4, Theorem 9.5, p. 42]). *An AC-lattice is finite-modular if and only if $z \prec z \cup y$ implies $z \cap y \prec y$ in L.*

Now we can define strongly planar AC-lattices.

Definition 2.2 [(4, Lemma 14.4, p. 59]). An AC-lattice is called *strongly planar* if the principal dual ideal $[p)$ is finite-modular for every atom p of L.

In a lattice L with 0 perspectivity is defined as follows: We say that $a, b \in L$ are *perspective* and write $a \sim b$ when $a \cup x = b \cup x$ and $a \cap x = b \cap x = 0$ for some $x \in L$.

The following definition plays an important role in this paper.

Definition 2.3. Let L be a lattice with 0 and let $a, b \in L$ $(a \neq 0, b \neq 0)$. We write $a <| b$ when

$$a \cap b = 0 \quad \text{and} \quad b \prec a \cup b.$$

According to G. GRÄTZER and E. T. SCHMIDT (cf. [1, p. 30]) we call an ideal S of a lattice L *standard* if

$$I \cap (S \cup K) = (I \cap S) \cup (I \cap K)$$

holds for any pair of ideals I, K of L. An ideal S of a lattice L with 0 is called a *p-ideal* if S is closed under perspectivity. Without proof we state the following well-known

Lemma 2.4. *In a lattice with 0 every standard ideal is a p-ideal.*

In an AC-lattice L, the set $F(L)$ of the finite elements of L is always an ideal (cf. [4, Lemma 8.8, p. 37]); further every non-zero $a \in F(L)$ has a unique rank $r(a)$ by [4, Theorem 8.4, p. 36]. We define $r(0) = 0$ and $r(a) = \infty$ if $a \notin F(L)$. Every interval $[a, b]$ of an AC-lattice is likewise an AC-lattice by [4, Lemma 8.18, p. 39]. By $r[a, b]$ we mean the rank of b in $[a, b]$. By $r[a, b] < \aleph_0$ we mean that b has a finite rank in $[a, b]$.

§ 3. Equivalent conditions for $F(L)$ to be a standard ideal

In this paragraph we give necessary and sufficient conditions for $F(L)$ to be a standard ideal of an arbitrary AC-lattice L. To do this we need

Theorem 3.1 ([1, Theorem 2, p. 30]). *An ideal S of a lattice L is standard if and only if*

$$S \cup (x] = \{s \cup x_1 \mid s \in S \quad \text{and} \quad x_1 \leqq x\}$$

for every principal ideal $(x]$ of L.

Now we are able to prove

Theorem 3.2. *Let L be an AC-lattice. Then the following two conditions are equivalent:*
(i) *$F(L)$ is a standard ideal of L;*
(ii) *if $[x, b \cup x]$ and $[b \cap x, b]$ are transposed intervals of L and $r[x, b \cup x] < \aleph_0$ then $r[b \cap x, b] < \aleph_0$.*

Proof. (i) $\Rightarrow$ (ii): Let $F(L)$ be a standard ideal of L and let $r[x, b \cup x] < \aleph_0$. Since L is atomistic, there is an $a \in L$ such that $x \leq b \cup x = a \cup x$ and $r(a) < \aleph_0$. Hence $a \in F(L)$. Then $b \in F(L) \cup (x]$. By Theorem 3.1 there are elements

$$x_1 \leqq x \tag{1}$$

and

$$a_1 \in F(L) \tag{2}$$

such that

$$b = x_1 \cup a_1. \tag{3}$$

By (2) we have $r(a_1) < \aleph_0$ and hence

$$r[x_1, x_1 \cup a_1] = r[x_1, b] < \aleph_0. \tag{4}$$

By (1) and (3) we get $x_1 \leqq x \cap b \leqq b$. From this it follows by (4) that $r[x \cap b, b] < \aleph_0$ which was to be proved.

(ii) $\Rightarrow$ (i): Let $b \leqq x \cup a$ and $a \in F(L)$. Then $r[x, x \cup a] < \aleph_0$. Since $x \leqq x \cup b \leqq x \cup a$, we get $r[x, x \cup b] < \aleph_0$. Hence $r[x \cap b, b] < \aleph_0$ by (ii) and there exists (since L is atomistic) an $a_1 \in F(L)$ such that $b = (x \cap b) \cup a_1$. By Theorem 3.1 it follows that $F(L)$ is standard.

Corollary 3.3. *Let L be an AC-lattice. Then the following two conditions are equivalent:*
(i) *$F(L)$ is a standard ideal of L;*
(ii) *$z \prec z \cup y$ implies $r[z \cap y, y] < \aleph_0$.*

Proof. (i) $\Rightarrow$ (ii): This follows from Theorem 3.2.
(ii) $\Rightarrow$ (i): By Theorem 3.2 it suffices to prove that

$$r[x, x \cup b] < \aleph_0 \quad \text{implies} \quad r[b \cap x, b] < \aleph_0, \tag{5}$$

where $[x, x \cup b]$ and $[x \cap b, b]$ are two transposed intervals of L. To prove (5) let

$$r[x, x \cup b] < \aleph_0. \tag{6}$$

This means that there exists a finite maximal chain

$$x \cup b = m_0 \succ m_1 \succ \cdots \succ m_i \succ m_{i+1} \succ \cdots \succ m_n = x$$

between x and $x \cup b$ (cf. [4, Remark 8.6, p. 36 and Remark 8.9, p. 37]). Let M denote the number of the intervals $[m_{i+1}, m_i]$. By (6) we get

$$M < \aleph_0. \tag{7}$$

Now consider the intervals of the form

$$[b \cap m_{i+1}, b \cap m_i] \tag{8}$$

and denote by N the number of these intervals. Since

$$b \cap m_{i+1} = b \cap m_i \tag{9}$$

is possible, we get

$$N \leqq M < \aleph_0. \tag{10}$$

Now we ask: What can be said about the rank of an interval (8)? If (9) holds, then $r[b \cap m_{i+1}, b \cap m_i] = 0$. Let now

$$b \cap m_{i+1} < b \cap m_i. \tag{11}$$

Then

$$b \cap m_i \nleqq m_{i+1} \tag{12}$$

because if $b \cap m_i \leqq m_{i+1}$ then $b \cap m_i \leqq b \cap m_{i+1}$ which contradicts (11). From (12) we get

$$m_{i+1} \prec m_i = m_{i+1} \cup (b \cap m_i). \tag{13}$$

Hence by condition (ii) it follows that

$$r[b \cap m_{i+1}, b \cap m_i] = r[b \cap m_i \cap m_{i+1}, b \cap m_i] < \aleph_0. \tag{14}$$

Summarizing we have

$$r[b \cap x, b] = N \cdot r[b \cap m_{i+1}, b \cap m_i]$$

and from this by (10) und (14)

$$r[b \cap x, b] < \aleph_0 \cdot \aleph_0 = \aleph_0.$$

Therefore implication (5) holds in L which proves the theorem.

The following lemma shows the connection between standard ideals, p-ideals and the relation $<|$ of Definition 2.3.

Lemma 3.4. *Let L be an AC-lattice. Consider the following conditions:*

(i) *$F(L)$ is a standard ideal of L;*

(ii) *$F(L)$ is a p-ideal of L;*

(iii) *$y <| z$ implies $y \in F(L)$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii): This follows from Lemma 2.4.

(ii) $\Rightarrow$ (iii): Assume that there are elements $y, z \in L$ such that $y <| z$ but $y \notin F(L)$. Take an atom $p < y$. Then $p \sim y$ and hence $F(L)$ is not a p-ideal which proves the lemma.

Now we pose the question: For which AC-lattices can one prove that (iii) implies (i) in the preceding lemma? In the following paragraphs we give an answer to this question for some classes of AC-lattices.

§ 4. Applications to section-complemented AC-lattices

We apply Corollary 3.3 to section-complemented AC-lattices (an AC-lattice is *section-complemented* if every interval of the form $[0, a]$ is complemented). Since a matroid lattice is relatively complemented and hence section-complemented, the following corollary holds for matroid lattices as well.

Corollary 4.1. *Let L be a section-complemented AC-lattice. Then the following two conditions are equivalent:*

(i) *$F(L)$ is a standard ideal of L;*

(ii) *$y <| z$ implies $y \in F(L)$.*

Proof. (i) $\Rightarrow$ (ii): This follows from Lemma 3.4.

(ii) $\Rightarrow$ (i): By Corollary 3.3 it suffices to show that the implication

$$z \prec z \cup y \quad \text{implies} \quad r[z \cap y, y] < \aleph_0 \tag{15}$$

holds in L. To prove (15) let

$$z \prec z \cup y. \tag{16}$$

Let $\bar{y}$ be a complement of $z \cap y$ in $[0, y]$. Then $z \cup y = z \cup (y \cap z) \cup \bar{y} = z \cup \bar{y}$. Hence from (16) we get

$$z \prec z \cup \bar{y}. \tag{17}$$

Since $\bar{y} \leqq y$, we have

$$z \cap \bar{y} = z \cap y \cap \bar{y} = 0. \tag{18}$$

(17) and (18) mean that $\bar{y} <| z$. Hence it follows from conditions (ii) that $\bar{y} \in F(L)$. From this we get

$$r[y \cap z, y] = r[y \cap z, (y \cap z) \cup \bar{y}] < \aleph_0$$

which proves the implication (15) and thereby the corollary.

§ 5. Applications to strongly planar AC-lattices

We give equivalent conditions for $F(L)$ to be a standard ideal of a strongly planar AC-lattice, of a weakly modular AC-lattice and apply the results to finite-modular AC-lattices.

Corollary 5.1 ([8, Theorem 7]). *Let L be a strongly planar AC-lattice. Then the following two conditions are equivalent:*

(i) *$F(L)$ is a standard ideal of L;*

(ii) *$y <| z$ implies $y \in F(L)$.*

Proof. (i) $\Rightarrow$ (ii): This follows from Lemma 3.4.

(ii) $\Rightarrow$ (i): We show that the implication

$$z \prec z \cup y \quad \text{implies} \quad r[z \cap y, y] < \aleph_0 \tag{19}$$

holds in L. Then the assertion follows from Corollary 3.3. To prove (19) let

$$z \prec z \cup y. \tag{20}$$

If $z \cap y > 0$ then there exists an atom $p \leqq z \cap y$ such that $[p)$ is a finite-modular AC-lattice (cf. Definition 2.2). Then $z \cap y, z, y, z \cup y \in [p)$. Since $[p)$ is finite-modular, we get by Lemma 2.1 that $r[z \cap y, y] = 1$ which proves (19).
Let now

$$z \cap y = 0. \tag{21}$$

Then by (20) and (21) we have $y <| z$. Applying condition (ii) it follows that $y \in F(L)$ and hence

$$r[z \cap y, y] = r[0, y] < \aleph_0.$$

This proves (19) and thereby the corollary.

Corollary 5.2. *Let L be a weakly modular AC-lattice (specially: an atomistic Wilcox lattice). Then the conditions* (i) *and* (ii) *of Corollary* 5.1 *are equivalent.*

Proof. A weakly modular AC-lattice is strongly planar and an atomistic Wilcox lattice is a weakly modular AC-lattice (cf. [4, Remark 20.1, p. 91]). Hence one can apply Corollary 5.1.

We remark at this place that M. F. JANOWITZ obtained the following result:

Theorem.[1]) *Let $L \equiv \Lambda - S$ be an atomistic Wilcox lattice. Then the following conditions are equivalent:*

(i) *$F(L)$ is a p-ideal of L;*

(ii) *$a \parallel b$ implies $a \in F(L)$;*

(iii) *$S \subseteqq F(\Lambda)$.*

By [4, Lemma 21.7, p. 97] $y <| z$ holds in an atomistic Wilcox lattice for some z if and only if $y \parallel z$ holds for some z. Hence the equivalence of conditions (i) and (ii) of JANOWITZ's Theorem can be deduced with the aid of Corollary 5.2. The whole Theorem, of course, cannot be deduced with the aid of Corollary 5.2. On the other hand we are able to strengthen the Theorem of JANOWITZ by adding "$F(L)$ is a standard ideal" as a further equivalent condition. In case L is relatively complemented (e.g. if L is an affine matroid lattice) this follows already from [2, Theorem 4.2].

We conclude this paragraph by

Corollary 5.3. ([3, Theorem 4.6]). *Let L be a finite-modular AC-lattice. Then $F(L)$ is a standard ideal of L.*

Proof. Since a finite-modular AC-lattice is strongly planar we have only to check condition (ii) of Corollary 5.1: Let $y <| z$ in L. Then by Lemma 2.1 it follows that $0 = z \cap y \prec y$, i.e. y is an atom and hence $y \in F(L)$. This proves the corollary.

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