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On endomorphisms and quasi-endomorphisms of torsionfree groups

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Introduction

The problem of characterizing all abelian torsionfree groups which have a commutative endomorphism ring, is an old one ([5], Problem 46a) and seems far from being completely solved.
In the present paper we use the concept of quasi-endomorphism, introduced by BEAUMONT and PIERCE after basic work of JÓNSSON, and applied to torsionfree groups by BEAUMONT and REID ([1, 9]).
For special classes of torsionfree groups of finite rank we obtain some results with the aid of the ring of quasi-endomorphisms (Theorems 3.1 and 3.3). The problem of the commutativity of the endomorphism ring of a torsionfree abelian group G is closely connected with that of characterizing the groups G for which $G \simeq \text{End } G$. A conjecture of PH. SCHULTZ ([10]) is that $G \simeq \text{End } G$ implies that $E(G)$ is commutative. In theorem 4.2 we show that this conjecture is true for a wide class of torsionfree groups of finite rank. The structure of the groups in this class is known to a certain extent (Remark 4.3). Again using additional requirements, besides $\text{End } G \simeq G$, one can prove that $E(G)$ is commutative (Theorems 4.4 and 4.5).
The word "group" will always mean abelian group and, unless otherwise stated, torsionfree group. We consider only torsionfree groups of finite rank. If G is a group, then End G is the group of all endomorphisms of G and $E(G)$ is the endomorphism ring of G. If R is a ring, then R^+ denotes the underlying abelian group, so $E(G)^+ = \text{End } G$. By $QE(G)$ we mean the minimal Q-algebra containing $E(G)$.
Here Q ist the group (ring) of rationals. Also Z is the group of rational integers.
If A, B are groups (rings), then $A + B$ denotes the direct sum of the groups (rings) A and B.

1. Reduction to reduced torsionfree groups

Let G be an arbitrary torsionfree abelian group. Then G can be written as a direct sum: $G = D + H$, where D is a divisible group and H is a reduced group, i.e. a group which has no divisible subgroups $\neq 0$.
Now it can easily be shown that the endomorphism ring $E(G)$ is commutative if and only if $E(D)$ and $E(H)$ are both commutative and $\text{Hom}\,(D, H) = \text{Hom}\,(H, D) = 0$ ([6], p. 23). Since D is divisible and H is reduced, it follows that $\text{Hom}\,(D, H) = 0$.

Now suppose that $D \neq 0$. Since D is torsionfree divisible, we have $D \cong \sum Q$, a direct sum of copies of Q. Hence $E(D) \cong E(\sum Q)$ and $E(D)$ is commutative if and only if $D \cong Q$ ([6], p. 24).
Then $\text{Hom}(H, D) \cong \text{Hom}(H, Q) = 0$ if and only if $H = 0$. Hence under the assumption that $D \neq 0$ we find that $E(G)$ is commutative if and only if $H = 0$, $D \cong Q$, i.e. $G \cong Q$.
From now on we suppose that $D = 0$ and we restrict ourselves to reduced torsionfree groups.

2. The ring of quasi-endomorphisms

Let G be a torsionfree group of rank 1, i.e. G is isomorphic to a subgroup of Q. Then, as is well-known, $E(G)$ is isomorphic to a subring of Q, hence $E(G)$ is commutative.
For torsionfree groups G of rank n, $n \geqq 2$, one does not have a nice classification. Instead of trying to find conditionsfor $E(G)$, the endomorphism ring of G, we look at another ring now.
We suppose that G is a torsionfree group of finite rank. Let G^* be the [uniquely determined] minimal divisible group containing G. Then G^* is a vector space over Q and the dimension of G^* over Q, which is denoted by $[G^*:Q]$, equals the rank of G. So $[G^*:Q]$ is finite.
Let $E(G)$ be the endomorphism ring of G. Then $E(G)$ is a torsionfree ring, i.e. the underlying additive group End G is torsionfree. Now $QE(G)$ is defined as the minimal Q-algebra containing $E(G)$. The underlying additive group of $QE(G)$ is the minimal divisible group containing End G.
$QE(G)$ is called the *quasi-endomorphism ring* of G. It is easy to show that $QE(G)$ is that subring of the ring of all linear transformations of the vector space G^*, that consists of all linear transformations Φ with the property, that there exists an integer $n \neq 0$ with $n\Phi(G) \subseteq G$ ([1], p. 47). Now we claim that $E(G)$ is commutative if and only if $QE(G)$ is commutative. Since $E(G)$ is a subring of $QE(G)$, one part of the assertion is trivial. Assume that $E(G)$ is commutative and let Φ, Ψ be two elements of $QE(G)$. Then there exist integers $n \neq 0$, $m \neq 0$ such that $n\Phi \in E(G)$, $m\Psi \in E(G)$. Hence $(n\Phi)(m\Psi) = (m\Psi)(n\Phi)$ or $nm(\Phi\Psi - \Psi\Phi) =$ zero mapping of G^*. Since $QE(G)^+ = (\text{End } G)^*$, it follows that this additive group of $QE(G)$ is torsionfree. Hence $\Phi\Psi = \Psi\Phi$ or $QE(G)$ is commutative.
Since $QE(G)$ is a rational algebra of finite dimension ($[QE(G):Q] \leqq n^2$, if rank $G = n$), we may represent $QE(G)$, with the aid of a base, by means of $n \times n$-matrices having entries in Q.
$QE(G)$ reflects many interesting properties of G. It is possible, in special cases, to determine $QE(G)$ explicitly. For instance, if G has rank 1, then $QE(G) \cong Q$. For groups G of rank 2 the algebras $QE(G)$ also have been computed ([2], p. 31). In the last case, the rings $QE(G)$ have been used to give a classification of torsionfree groups of rank 2 in terms of quasi-equality. Let G and H be groups. Then G is said to be *quasi-equal* to H, denoted by $G \doteq H$, in case there exist integers $n \neq 0$, $m \neq 0$ such that $nG \subseteq H$, $mH \subseteq G$.
One can easily see that $G \doteq H$ implies that rank $G =$ rank H. Also, if G and H are torsionfree groups of rank 1, $G \doteq H$ if and only if $G = H$.
The importance of quasi-equality with respect to the rings $QE(G)$ is a consequence of the following: $G \doteq H$ implies $QE(G) = QE(H)$.

Another concept that plays a role in this theory is that of quasi-decomposability. A group G is said to be *quasi-decomposable* if there exist independent groups A and B such that $G \doteq A + B$ (direct sum). If such groups do not exist, then G is strongly indecomposable.

Any torsionfree group G of finite rank has a quasi-decomposition into strongly indecomposable summands and the number of these summands is an invariant of G. Applying Theorem 7.1 of REID ([9], p. 64), one sees directly that a torsionfree group G of rank 2 has a commutative $E(G)$ if and only if G is either strongly indecomposable or $G \doteq G_1 + G_2$ (direct sum) where G_i is a group of rank 1 and the types of G_i are incomparable ($i = 1, 2$).

3. Irreducible and strongly indecomposable groups

Definition. A group G is *irreducible* if G has no non-trivial pure fully invariant subgroups $\neq 0$. Of course any torsionfree group of rank 1 is irreducible. Let G be an irreducible torsionfree group of finite rank. Then $QE(G) = \Gamma_m$, i.e. a ring of $m \times m$-matrices over Γ, Γ a division algebra, where m is the number of strongly indecomposable summands in a quasi-decomposition of G and $m[\Gamma : Q] = \text{rank } G$ ([9], Theorem 5.5).

Theorem 3.1. *An irreducible torsionfree group G of finite rank has a commutative $E(G)$ if and only if $QE(G)$ is a field.*

Proof. Suppose $E(G)$ is commutative, hence $QE(G)$ is commutative. Since $QE(G) = \Gamma_m$ in general, we must have $m = 1$ and Γ is a field.

Conversely, if $QE(G)$ is a field, it is clear that $E(G)$, being a subring of $QE(G)$, is commutative.

Remark 3.2. A group G satisfying the conditions of the theorem has the following properties:

a) G is strongly indecomposable,

b) $[QE(G) : Q] = \text{rank } G$.

So $QE(G)$ is an algebraic number field and $E(G)$ is a subring of such a field. In a previous paper ([7], Theorem 2) I have shown that supposing that G is an irreducible torsionfree group of prime rank we have:

$E(G)$ is commutative if and only if G is strongly indecomposable.

This might raise the question whether it is sufficient, for a group G of finite rank $\neq$ prime number, to require that G is irreducible and strongly indecomposable in order that $E(G)$ be commutative.

This is not the case. We will give an example of a torsionfree group G of finite rank $\neq$ prime number satisfying a) G is strongly indecomposable and b) G is irreducible, such that $QE(G)$, and hence $E(G)$, is not commutative.

Let

$$R = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_i \in Z\}$$

be the ring of integer quaternions. R is a reduced, countable torsionfree ring and R has a base $\{1, i, j, k\}$ over Z. Then, by a result of ZASSENHAUS ([12], p. 180), $R \simeq E(G)$, where G is a torsionfree abelian group with rank $G = 4$. Here $QE(G)$ is the field of quaternions. By REID ([9], p. 56), G is strongly indecomposable. Also $[QE(G):Q] = 4 = \text{rank } G$. Hence G is irreducible. But $E(G) \cong R$ is not commutative.

However, there is a special class of torsionfree groups of finite rank which are both strongly indecomposable and irreducible and have commutative endomorphism rings.

Let G be a torsionfree group. A subgroup B of G is a full subgroup of G if G/B is a torsion group. G is called a *quotient-divisible group* if G contains a full subgroup B such that B is free and G/B is divisible. The quotient-divisible groups G of rank 1 are exactly the groups of non-nil type, i.e. if G is given by the characteristic $(k_1, k_2, \ldots, k_j, \ldots)$, then $k_j = 0$ or ∞ for almost all k_j. Clearly any torsionfree group of rank 1 is irreducible and strongly indecomposable. As not every group of rank 1 is non-nil, it follows that not every torsionfree, irreducible and strongly indecomposable group is quotient-divisible.

BEAUMONT and PIERCE [3] have shown the equivalence of the following statements for a torsionfree group G of finite rank:

1. G is irreducible, quotient-divisible with $QE(G) \cong$ field K.
2. G is strongly indecomposable and G is isomorphic to the additive group of a full subring R of K.

Here a subring R of K is said to be *full* in case R^+ is a full subgroup of K^+.

Groups G satisfying either 1. or 2. have commutative $E(G)$. In this case $K \cong QE(G)$ is an algebraic number field. Hence by 2., G is isomorphic to the additive group of a full subring R of K, where K is an algebraic number field.

Now we can show:

Theorem 3.3. *Let G be a torsionfree group of finite rank and assume that G is isomorphic to the additive group of a full subring R of K, where K is an algebraic number field. Then $E(G)$ is commutative if and only if G is strongly indecomposable.*

Proof. If G is strongly indecomposable, then G satisfies 2., so by the result of BEAUMONT and PIERCE, $QE(G) \cong$ field K and hence $E(G) \subset QE(G)$ is commutative.

Conversely, let $E(G)$ be commutative, hence $QE(G)$ is commutative. Now BEAUMONT also proved: $QE(G) \cong$ full matrix ring $M_m(F)$ (as a rational algebra), where F is the smallest field of definition of G and $m = [K:F]$. As $QE(G)$ is commutative, $m = 1$ and $K = F$. Also $QE(G) \cong K$ $(= F)$, and since K is a field, we get that G is strongly indecomposable ([9], p. 56).

If the conditions of theorem 3.3. are fulfilled, and R has an identity, then we get $E(G) \cong R$, hence End $G \cong G$ $(= R^+)$. We will show this in the next section.

4. *E*-groups

A [torsionfree] group G is called an *E-group* if there exists a ring R with identity over G $(R^+ = G)$ such that $E(G)$ is the ring of left multiplications in R (cf. [10], Definition, p. 134).

FUCHS ([5], Problem 45) has posed the problem of characterizing the groups G for which $G \cong$ End G. Since this problem is closely connected with the commutativity of $E(G)$, we discuss it here. In a recent paper [10], PH. SCHULTZ proves that G is an E-group if and only if $G \cong$ End G and $E(G)$ is commutative (Corollary 6, p. 134). The problem is whether one can prove:

G is an E-group $\Leftrightarrow G \cong$ End G. If this is true, then $G \cong$ End G implies that $E(G)$ is commutative.

Lemma 4.1. *Let $G \cong \text{End}\, G = H$. Then the following statements are equivalent:*

(1) *Any non-zero endomorphism of G is a monomorphism.*

(2) *G is an E-group and $E(G)$ has no divisors of zero.*

Proof. (1) $\to$ (2). We prove that H is an E-group and since $G \cong H$, this proves that G is an E-group. Let $\varphi \in \text{End}\, H$ and suppose that $\varphi(1_G) = \pi$, where $1_G(g) = g$ for all $g \in G$. The left multiplication π_l, defined by $\pi_l \varrho = \pi \cdot \varrho$ for any $\varrho \in H$, is an endomorphism of H, hence $\pi_l \in \text{End}\, H$. Now $\pi_l(1_G) = \pi \cdot 1_G = \pi = \varphi(1_G)$, hence $(\pi_l - \varphi)(1_G) = 0$. If $\pi_l \neq \varphi$, then since $G \cong H$ and by (1) we get: 1_G = zero-endomorphism of G, which is impossible. Hence $\pi_l = \varphi$. So any endomorphism of H is a left multiplication endomorphism and $E(H)$ is the ring of left multiplications in $E(G)$. Hence H is an E-group. Now assume $\Phi\chi = 0$, $\chi \neq 0$ for $\Phi, \chi \in E(G)$. Then $\chi(g) \neq 0$ for at least one $g \in G$. Hence Ker $\Phi \neq 0$, which implies $\Phi = 0$ by (1).

(2) $\to$ (1). Let $\varphi \in \text{End}\, G$ and let φ_H be the corresponding endomorphism of H in $\text{End}\, G \cong \text{End}\, H$. Since H is an E-group it follows that φ_H is a left multiplication endomorphism of H, say $\varphi_H = \pi_l$. Suppose $\varphi_H \neq 0$ and let $\varrho \in \text{Ker}\, \varphi_H$, then $\varphi_H(\varrho) = \pi_l(\varrho) = \pi \cdot \varrho = 0$ in $E(G)$. Since $E(G)$ has no divisors of zero, it follows that $\varrho = 0$, as $\pi \neq 0$. Hence Ker $\varphi_H = 0$ and φ_H is a monomorphism.
This proves (1).

Theorem 4.2. *Let G be a torsionfree group of finite rank n such that $QE(G)$ is semi-simple i.e. the Jacobson radical of $QE(G) = 0$.*
Then G is an E-group $\Leftrightarrow G \cong \text{End}\, G$.

Proof. If G is an E-group then $G \cong \text{End}\, G$ wihtout any further conditions [10], so one part is clear.
Conversely, assume that $G \cong \text{End}\, G$. Since G is a torsionfree group of finite rank, it has a quasi-decomposition into a finite number of strongly indecomposable summands.
First suppose that this number equals one, i.e. G is strongly indecomposable. Then, by Corollary 4.3 [9], $QE(G)$ is a division algebra. It follows that every non-zero endomorphism of G is monic ([4], Remark 2.13, p. 19). Then G is an E-group (Lemma 4.1).
Secondly assume that G is quasi-decomposable. Then $QE(G)$ is a semi-simple ring which is not a division ring, since G is quasi-decomposable. If $QE(G)$ is simple it must be therefore a full matrix ring over a division ring of degree n (since G has rank n) by the Artin-Wedderburn theorem.
Since $G \cong \text{End}\, G$, we get $[QE(G) : Q] = [(\text{End}\, G)^* : Q]$ as a vector space and then

$$[(\text{End}\, G)^* : Q] = [G^* : Q] = \text{rank}\, G,$$

so $[QE(G) : Q] = \text{rank}\, G = n$. Here $[QE(G) : Q] = n^2$, hence $n^2 = n$ or $n = 1$. But then G is strongly indecomposable, which is a contradiction. Thus $QE(G)$ is not simple.
Now $QE(G)$ is the minimal Q-algebra containing $E(G)$. Hence G admits multiplication of algebra type $QE(G)$ since there exists a ring $E(G)$ with $E(G)^+ = \text{End}\, G \cong G$ and such that $QE(G)$ is the algebra type of $E(G)$ [1]. So G admits multiplication of semi-simple algebra type. Hence

(i) G is a quotient divisible group,

(ii) G is quasi-isomorphic to $\sum_{i=1}^{r} B_i$ (direct sum), where B_i is a strongly indecomposable group, $QE(B_i)$ is an algebraic number field with

$$[QE(B_i):Q] = \text{rank } B_i \quad (i = 1, \ldots, r)$$

(cf. [1], p. 48).

Hence $QE(G) \simeq QE\left(\sum_{i=1}^{r} B_i\right)$. Clearly $\sum_{i=1}^{r} E(B_i)$ (direct sum) may be considered as a subring of $E\left(\sum_{i=1}^{r} B_i\right)$ (isomorphical embedding). Hence

$$Q\left(\sum_{i=1}^{r} E(B_i)\right) \subseteq QE\left(\sum_{i=1}^{r} B_i\right). \tag{1}$$

On the other hand for a direct sum $\sum_{i=1}^{r} (B_i)$ one has:

$$Q\left(\sum_{i=1}^{r} E(B_i)\right) = \sum_{i=1}^{r} QE(B_i).$$

Hence it follows that

$$\begin{aligned}\left[QE\left(\sum_{i=1}^{r} B_i\right):Q\right] &= [QE(G):Q] = \text{rank } G = \text{rank}\left(\sum_{i=1}^{r} B_i\right)\\ &= \sum_{i=1}^{r} (\text{rank } B_i) = \sum_{i=1}^{r} [QE(B_i):Q]\\ &= \left[\sum_{i=1}^{r} QE(B_i):Q\right] = \left[Q\left(\sum_{i=1}^{r} E(B_i)\right):Q\right].\end{aligned} \tag{2}$$

From (1) and (2) we infer:

$$QE\left(\sum_{i=1}^{r} B_i\right) = Q\left(\sum_{i=1}^{r} E(B_i)\right) = \sum_{i=1}^{r} QE(B_i)$$

is commutative.

Hence $QE(G)$ is commutative. So $E(G)$ is commutative and G is an E-group.

Remark 4.3. It may be remarked that the proof shows that $QE(G)$ is simple and End $G \simeq G$ imply G is strongly indecomposable, irreducible and $QE(G)$ is a field (algebraic number field).

If G is quasi-decomposable, $QE(G)$ cannot be simple, but End $G \simeq G$ implies that $QE(G)$ is a finite direct sum of fields. Hence G has a quasi-decomposition: $G \doteq G_1 + G_2 + \cdots + G_m$ and one can show that each of the groups G_i is a strongly indecomposable, irreducible group with $QE(G_i)$ a field.

Theorem 4.4. *Let G be a torsionfree group of finite rank. Then the following statements are equivalent:*

(1) *G is an E-group and G is irreducible.*

(2) End $G \simeq G$ *and $QE(G)$ is simple.*

(3) *G is strongly indecomposable and isomorphic to the additive group of a full subring R of F, where F is a simple algebra with identity* 1 *such that* $1 \in R$.

Proof. (1) → (2). G is an E-group implies that End $G \cong G$. Also, as G is irreducible, we get $QE(G) = \Gamma_m$, where Γ is a division algebra. Hence $QE(G)$ is simple. Actually, $m = 1$ and Γ is a field.

(2) → (3). Since $QE(G)$ has an unit element, it cannot be J-radical (J = Jacobson radical). So $QE(G)$ is semisimple. Then $QE(G)$ is simple and End $G \cong G$ imply that G is strongly indecomposable, as we have seen. Also G is irreducible and $QE(G)$ is a field. Since G admits multiplication of semisimple algebra type, it follows that G is isomorphic to the additive group of a full subring R of $QE(G)$. Here $QE(G)$ is simple and has an identity. Now $E(G)$ is a full subring of $QE(G)$, since $E(G)^+ =$ End $G \cong G$ is a full subgroup of $QE(G)^+$. Also $1 \in E(G)$.

(3) → (1). Since G is strongly indecomposable, F must be a field ([4], Corollary 1.14) and $QE(G) = F$.
By the result of Beaumont and Pierce we get that G is irreducible. Since F is simple with 1 and G is strongly indecomposable, it follows that R, as a full subring of F, is isomorphic to $E(G)$ ([4], Corollary 1.16).
Hence $R^+ = G \cong$ End G. Then $E(G)$ is commutative implies that G is an E-group.

Theorem 4.5. *Let G be a torsionfree group of finite rank n and let $QE(G)$ be semisimple. Then the following statements are equivalent:*

(1) *End $G \cong G$ and G is irreducible.*

(2) *G is an E-group and G is strongly indecomposable.*

For groups G satisfying either (1) *or* (2) *one has:*
If rank (Hom $(G, Z)) = 1$, *then* $G = Z$.
If rank (Hom $(G, Z)) \neq 1$, *then* Hom $(G, Z) = 0$.

Proof. (1) → (2). Since G is irreducible, $QE(G) = \Gamma_m$, where Γ is a division algebra, m is the number of strongly indecomposable summands in a quasi-decomposition of G and $m[\Gamma : Q] =$ rank G ([9], Theorem 5.5). As End $G \cong G$, the Q-dimension of $\Gamma_m = m^2[\Gamma : Q] =$ rank $G = m[\Gamma : Q]$. Hence $m^2 = m$ and $m = 1$. So G is strongly indecomposable. This also implies that G is an E-group (see the proof of theorem 4.2).

(2) → (1). G is an E-group implies End $G \cong G$. As the Jacobson radical of $QE(G) = 0$, it follows that $QE(G)$ is a division ring ([9], Corollary 4.3).
Since End $G \cong G$, we get $[QE(G) : Q] =$ rank G. Now since G is strongly indecomposable, we have G is irreducible ([9], Theorem 5.5).
Hence (1) and (2) are equivalent.
As we have seen in the proof of Theorem 4.2, G is a quotient divisible group which means: G is an extension of a free group B by a divisible torsion group. Here rank $B = n$, since G has rank n. As $0 \to B \to G \to T \to 0$ is exact, where T is a divisible torsion group, it follows that $0 \to$ Hom $(G, B) \to$ Hom $(G, G) \cong G$ is exact.
So G contains an isomorphic copy of

$$\sum_1^n \text{Hom}\,(G, Z) \cong \text{Hom}\,(G, B).$$

Suppose that Hom (G, Z) has rank t, t finite, then rank (Hom $(G, B)) = tn \leqq n$, hence $t = 0$ or $t = 1$. The exact sequence $0 \to B \to G \to T \to 0$ also implies that

$$0 \to \text{Hom}\,(G, Z) \to \text{Hom}\,(B, Z) \cong \sum_1^n Z$$