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On a problem of M. F. Janowitz[1])

MANFRED STERN

§ 1. Introduction

For the following definitions and notions cf. [3]. A lattice L with 0 is called atomistic if every element of L is the join of a family of atoms. In a lattice L with 0 let $F(L)$ denote the set of elements that may be expressed as the join of a finite (possibly empty) family of atoms. We say that in a lattice L the element a ($\in L$) covers the element b ($\in L$) and write $b \prec a$ in case $b < a$ and $b \leqq x \leqq a$ implies $x = b$ or $x = a$. The covering property (C) is introduced as follows:

(C) If p is an atom and $p \nleqq a$ then $a \prec a \cup p$.

We call L an AC-lattice if it is an atomistic lattice with the covering property (C). A matroid-lattice may then be defined to be an algebraic AC-lattice. In a lattice L the pair (a, b) $(a, b \in L)$ is called a modular pair, denoted by $(a, b)\ M$, *if* $c \leqq b$ implies $(c \cup a) \cap b = c \cup (a \cap b)$. A lattice L with 0 is called finite-modular if $a \in F(L)$ implies $(x, a)\ M$ for every $x \in L$.
Following the terminology of GRÄTZER and SCHMIDT (cf. [1], p. 30) we call an ideal S of a lattice L standard if

$$I \wedge (S \vee K) = (I \wedge S) \vee (I \wedge K)$$

holds for any pair of ideals I, K of L. An ideal I of a lattice L will be called a p-ideal if L has a zero element and I is closed under perspectivity. Without proof we state the following well-known

Lemma 1. *Let L be a lattice with zero element. Then every standard ideal of L is a p-ideal.*

M. F. JANOWITZ has proved the following

Theorem 2 (cf. [2], Theorem 4.6). *If L is a finite-modular AC-lattice, then F(L) is a standard ideal.*

Likewise in [2] the following problem was raised: Is $F(L)$ a standard ideal for L an arbitrary AC-lattice? What is if L is a matroid-lattice?

[1]) To the memory of my teacher Professor ANDOR KERTÉSZ.

Our aim is to show that there are AC-lattices L (even matroid-lattices) with the property that $F(L)$ is not standard. More exactly, we prove that in a kind of matroid-lattices L the ideal $F(L)$ is not a p-ideal. After this, from Lemma 1 we get the assertion.[1])

§ 2. Preliminaries

In the sequel we need the notion of parallel elements:

Definition 3 (cf. [3], Def. 17.1, p. 72). Let L be a lattice with 0 and let $a, b \in L$. We write $a <| b$ when

$$a \cap b = 0 \quad \text{and} \quad b \prec a \cup b.$$

When $a <| b$ and $b <| a$ we say that a and b are parallel and write $a \parallel b$.

Next we consider under which conditions in a complete AC-lattice an atom is perspective to a dual atom. We have

Lemma 4. *Let L be a complete AC-lattice of length $h(1) > 2$. Then there exists an atom $p \in L$ which is perspective to a dual atom $m \in L$ if and only if there exists a dual atom $n \in L$ such that $m \parallel n$.*

Proof. If we have two dual atoms $m, n \in L$ such that $m \parallel n$, then we take an atom $p \in L$ for which $p < m$. Then $p \sim m$. Conversely, let $p \in L$ be an atom, $m \in L$ be a dual atom with $p \neq m$. If $p \sim m$, then there exists an $x \in L$ such that

$$p \cup x = m \cup x \quad \text{and} \quad p \cap x = m \cap x = 0.$$

From this and from the fact that $x \neq 0, 1$ we get $x \nleqq m$ and therefore $m \cup x = 1 = p \cup x$ Here x is a dual atom, because if there were any $y \in L$ with

$$x < y \leqq x \cup p = 1$$

then by the covering property (C) we get $y = 1$ and the lemma is proved.

§ 3. An example

After this we give now an example of a matroid-lattice L, in which $F(L)$ is not standard: let L be an affine matroid-lattice (cf. [3], Def. 18.3, p. 78) of infinite length satisfying Euclid's strong parallel axiom (cf. [3], Axiom 18.1, p. 78). Then it follows that every line of L is incomplete (cf. [3], Remark 18.10, p. 80).
Next we show that in L there are two dual atoms m_ν, n such that $m_\nu \parallel n$. Let

$$1 = \bigcup (a_\nu \mid \nu \in \Gamma)$$

where $\langle \ldots, a_\nu, \ldots \rangle_{\nu \in \Gamma}$ is a maximal independent set of atoms in L. We define

$$m_\nu \overset{\text{def}}{=} \bigcup (a_\mu \mid \mu \in \Gamma, \mu \neq \nu).$$

[1]) As M. F. JANOWITZ remarked, he obtained another solution to his problem. More precisely, he obtained necessary and sufficient conditions for $F(L)$ to be a p-ideal, where L is an atomistic Wilcox lattice.

Then $m_\nu \cup a_\nu = 1$ and $m_\nu \cap a_\nu = 0$, therefore $a_\nu \nleqq m_\nu$ and $m_\nu < 1$. By the covering property (C) we get from $m_\nu < y \leqq m_\nu \cup a_\nu = 1$ that $y = 1$. Therefore m_ν is a dual atom in L.

Since in L every line is incomplete, every in m_ν contained line ist incomplete. This means that m_ν is an incomplete element of L (cf. [3], Def. 18.5, p. 78). Now we apply

Lemma 5 (cf. [3], Corollary 18.13, p. 81). *In an affine matroid-lattice L let $a \in L$ be an element of height $\geqq 2$. Then the following two statements are equivalent:*

(i) *$a \in L$ is an incomplete element and $a \neq 1$;*
(ii) *there exists an element $b \in L$ with $a \parallel b$.*

We know that m_ν fulfils condition (i). Then according to (ii) there exists an $n \in L$ such that $m_\nu \parallel n$. By Definition 3 we have $n \prec m_\nu \cup n = 1$. Hence n is a dual atom.

Theorem 6. *There exists a matroid-lattice L such that $F(L)$ is not a standard ideal.*

Proof. In the above affine matroid-lattice L there exist two dual atoms m_ν, n such that $m_\nu \parallel n$. Then, by Lemma 4, there exists an atom $p \in L$ such that $p \sim m_\nu$. We have evidently $p \in F(L)$. However $m_\nu \notin F(L)$ since L was supposed to be of infinite length. Hence $F(L)$ is not a p-ideal. Then, according to Lemma 1, $F(L)$ is not standard, which finishes the proof.

Remarks. (i) There is a concrete example for a matroid-lattice of the above kind: Let E be a vector space over a division ring. The set $L_A(E)$ of all affine subsets of E forms a lattice, ordered by set-inclusion. If the dimension of E is not less than 3, then $L_A(E)$ is an affine matroid-lattice satisfying Euclid's strong parallel axiom (cf. [3], Theorem 22.9, p. 106).

(ii) Since an affine matroid-lattice is weakly modular (cf. [3], Def. 1.10, p. 3) we have the following corollary: in a weakly modular AC-lattice L, in general $F(L)$ is not a standard ideal.

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VERFASSER:

Manfred Stern, Sektion Mathematik der Martin-Luther-Universität Halle—Wittenberg