Radicals Coinciding with the Jacobson Radical on Linearly Compact Rings[1])

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To Prof. O.-H. Keller on his 65th birthday

§ 1. Introduction

It is the purpose of this paper to approximate or to localize the Jacobson radical class by upper and lower radical classes in the following sense: we shall construct two radical properties which will coincide with the Jacobson radical on the class of linearly compact rings such that if a radical coincides with the Jacobson radical on the linearly compact rings, then this radical class lies between the given radical classes. In [2] DIVINSKY has given boundaries for radicals coinciding with the Jacobson radical on the class of artinian rings. Similar investigations were done by F. SZÁSZ [9] on the class of the so called MHR-rings. Since the class of MHR-rings is a biger one than that of the artinian rings, so the boundaries are closer. Here we present similar investigations according to the class of linearly compact rings, and it will turn out that our approximation is better than those made earlier, it will be proved that neither the Brown-McCoy radical, nor Koethe's nil-radical (and so also Baer's lower radical) do not coincide with the Jacobson radical on the class of linearly compact rings. At the end of the paper we give some suggestions for further researches in these topics.

§ 2. Preliminaries

Let **R** be a property that associative rings may possess. **R** is associated with a class of rings which can be also denoted by **R**. The rings having property **R** will be called **R**-rings.

By a radical we shall always mean a Kurosh-Amitsur radical property. Let us recall its definition (cf. DIVINSKY [3] and F. SZÁSZ [10]). A property **R** of rings is a *radical property*, if

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(i) a homomorphic image of an **R**-ring is an **R**-ring;
(ii) every ring A contains an **R**-ideal $\mathbf{R}(A)$ which contains every other **R**-ideal of A;
(iii) the factor ring $A/\mathbf{R}(A)$ has not non-zero **R**-ideals.

The ideal $\mathbf{R}(A)$ of the ring A is called the **R**-*radical of* A.

A class of rings is called a *semisimple class* with respect to a radical property **R**, if it consists of all **R**-semisimple rings, i.e. of rings having 0 **R**-radical. To any class **X** of rings one can construct the *lower radical class* $\mathscr{L}\mathbf{X}$ which is the smallest radical class containing the class **X**. If **Y** is a class of rings, then there exists the *upper radical class* $\mathscr{U}\mathbf{Y}$, this is the biggest radical class according to which every **Y**-ring is $\mathscr{U}\mathbf{Y}$-semisimple. For details about the theory of general radicals, Jacobson radical and other concrete radicals we hint to the books [3, 4, 5, 10].

We shall need

Proposition 1 ([1], Corollary 2 of Theorem 1). *Every semisimple class* **S** *is hereditary* (i.e. if $A \in \mathbf{S}$ and J is an ideal of A, then also $J \in \mathbf{S}$ holds).

A filter $\mathfrak{F}$ is a system of subsets of a topological space such that if $F_\mu, F_\nu \in \mathfrak{F}$, then there exists a subset $F_\lambda \in \mathfrak{F}$ with $\emptyset \neq F_\lambda \subseteqq F_\mu \cap F_\nu$. An A-leftmodule M is called *linerarly compact* if M has a basisfilter consisting of A-submodules, and for every filter $\mathfrak{F} = \{F_\lambda\}$ consisting of cosets of closed A-submodules, the intersection $\bigcap_\lambda F_\lambda$ is not empty. A ring A is said to be linearly compact if it is linearly compact as an A-leftmodule. The class of all linearly compact rings will be denoted by **L**. An A-leftmodule is called *linearly compact in a narrow sense* if it is the inverse limit of discrete A-modules satisfying the descending chain condition on submodules; a ring A is linearly compact in a narrow sense, if it is such an A-module. The class of all in a narrow sense linearly compact rings will be denoted by $\mathbf{L}^*$. $\mathbf{L}^*$ is a subclass of **L**, moreover, every artinian ring belongs to $\mathbf{L}^*$.

The Jacobson-semisimple linearly compact rings are characterized by LEPTIN's Structure Theorem.

Proposition 2 ([6], Sätze 12 and 13). *A linearly compact Jacobson-semisimple ring* A *is a complete direct sum* $\sum_\alpha^* A_\alpha$ *of rings* A_α *which are full rings of linear transformations of vector spaces over division rings, moreover, the linearly compact Jacobson-semisimple rings are in a narrow sense linearly compact.*

Let A be a topological ring, and consider the following ideals of A:

$$A_1 = A; \quad A_{\mu+1} = \overline{A_\mu \cdot A_\mu}; \quad A_\lambda = \bigcap_{\mu<\lambda} A_\lambda \text{ if } \lambda \text{ is a limit ordinal.}$$

If there exists an ordinal ζ such that $A_\zeta = 0$ then the ring A is called *transfinite nilpotent.*

Proposition 3 ([11], Satz 7). *A ring* A *of* $\mathbf{L}^*$ *is a Jacobson-radicalring if and only if it is transfinite nilpotent.*

§ 3. The radical properties $\mathscr{L}\mathbf{Q}$ and $\mathscr{U}\mathbf{H}$

We shall say that a radical property $\mathbf{R}$ *coincides with the Jacobson radical* $\mathbf{J}$ *on the class* $\mathbf{L}$, if for any ring $A \in \mathbf{L}$ we have $\mathbf{R}(A) = \mathbf{J}(A)$. Let us observe that if $\mathbf{R}$ coincides with $\mathbf{J}$ on $\mathbf{L}$, then $\mathbf{R} \cap \mathbf{L} = \mathbf{J} \cap \mathbf{L}$ is valid, however, $\mathbf{R} \cap \mathbf{L} = \mathbf{J} \cap \mathbf{L}$ does not imply the coinciding of $\mathbf{R}$ with $\mathbf{J}$ on $\mathbf{L}$.

Let $\mathbf{Q}$ denote the class of all Jacobson radical rings which are Jacobson radicals of linearly compact rings, and consider the lower radical property $\mathscr{L}\mathbf{Q}$ determined by $\mathbf{Q}$.

Consider the class $\mathbf{T}$ of all full rings of linear transformations of vector spaces over division rings. The rings of $\mathbf{T}$ can be equipped with a linearly compact topology, moreover, $\mathbf{T} \subset \mathbf{L}^*$ holds. We shall say that a subring B of a ring A is an *accessible subring*, if there exists a finite set of subrings $A_1, \ldots, A_n$ of A such that

$$B = A_n \subset A_{n-1} \subset \cdots \subset A_1 \subseteqq A_0 = A$$

holds where A_i is an ideal of A_{i-1} for $i = 1, 2, \ldots, n$. Consider the class $\mathbf{H}$ consisting of every ring $\mathbf{H}$ which is an accessible subring of a ring $T \in \mathbf{T}$. Hence $\mathbf{H}$ is the hereditary closure of $\mathbf{T}$. Thus the hereditary class $\mathbf{H}$ determines an upper radical property $\mathscr{U}\mathbf{H}$, and all the $\mathbf{H}$-rings are $\mathscr{U}\mathbf{H}$-semisimple.
The following theorems will localize the Jacobson radical by the radicals $\mathscr{L}\mathbf{Q}$ and $\mathscr{U}\mathbf{H}$.

Theorem 1. *The Jacobson radical property coincides with the* $\mathscr{L}\mathbf{Q}$ *and* $\mathscr{U}\mathbf{H}$ *radical properties on the class* $\mathbf{L}$ *of linearly compact rings, moreover, if* $\mathbf{R}$ *is a radical property coinciding with the Jacobson radical property on* $\mathbf{L}$, *then* $\mathscr{L}\mathbf{Q} \subseteqq \mathbf{R} \subseteqq \mathscr{U}\mathbf{H}$, *and conversely, if* $\mathbf{R}$ *is a radical class with* $\mathscr{L}\mathbf{Q} \subseteqq \mathbf{R} \subseteqq \mathscr{U}\mathbf{H}$, *then* $\mathbf{R}$ *coincides with the Jacobson radical property on* $\mathbf{L}$.

Proof. Let $A \in \mathbf{L}$ be an arbitrary ring. By definition we have $\mathbf{J}(A) \in \mathbf{Q} \subseteqq \mathscr{L}\mathbf{Q}$ which implies $\mathbf{J}(A) \subseteqq \mathscr{L}\mathbf{Q}(A)$. On the other hand from $\mathbf{Q} \subseteqq \mathbf{J}$ it follows $\mathscr{L}\mathbf{Q}(A) \subseteqq \mathbf{J}(A)$ trivially.
Let $\mathbf{R}$ be a radical property coinciding with $\mathbf{J}$ on $\mathbf{L}$. Now $\mathbf{Q} \subseteqq \mathbf{R}$ is obviously true, hence also $\mathscr{L}\mathbf{Q} \subseteqq \mathbf{R}$ is valid.
Next, we show $\mathbf{J} \subseteqq \mathscr{U}\mathbf{H}$. Suppose this is not true, now there exists a ring $A \in \mathbf{J}$ with $A \notin \mathscr{U}\mathbf{H}$. By the construction of the upper radical, the ring A can be mapped by a non-zero homomorphism onto a ring $A' \in \mathbf{H}$. Hence A' is an accessible subring of a ring $T \in \mathbf{T}$, and T is also Jacobson-semisimple. By Proposition 1 semisimplicity is hereditary, so also A' is Jacobson-semisimple. On the other hand, $A \in \mathbf{J}$ implies $0 \neq A' \in \mathbf{J}$ which contradicts the semisimplicity of A'.

Now we are going to prove that the Jacobson radical property coincides with the $\mathscr{U}\mathbf{H}$-radical property on $\mathbf{L}$. By $\mathbf{J} \subseteqq \mathscr{U}\mathbf{H}$ it follows $\mathbf{J}(A) \subseteqq \mathscr{U}\mathbf{H}(A)$ for every ring A. Hence it is sufficient to prove $\mathscr{U}\mathbf{H}(A) \subseteqq \mathbf{J}(A)$ for linearly compact rings. Suppose $\mathscr{U}\mathbf{H}(A) \nsubseteq \mathbf{J}(A)$ for an $A \in \mathbf{L}$, and consider the factor ring $A' = A/\mathbf{J}(A)$ which differs from 0 by $\mathbf{J}(A) \subset \mathscr{U}\mathbf{H}(A) \subseteqq A$. Applying Proposition 2, A' is a complete direct sum $\sum_{\alpha}^{*} A_\alpha$ of rings $A_\alpha \in \mathbf{T} \subset \mathbf{H}$. Since $\mathbf{J}(A) \nsubseteq \mathscr{U}\mathbf{H}(A)$, therefore A' has a non-zero ideal $J' \in \mathscr{U}\mathbf{H}$, and according to the decomposition $A' = \sum_{\alpha}^{*} A_\alpha$, J' can be mapped homomorphically onto a non-zero ideal J_{α_0} of A_{α_0} for some index α_0. J_{α_0} belongs at one hand to $\mathbf{H}$, so it is $\mathscr{U}\mathbf{H}$-semisimple. On the other hand, J_{α_0} as a homomorphic image of the $\mathcal{U}\mathbf{H}$-radical ring J' belongs to $\mathcal{U}\mathbf{H}$. This contradicts the fact $J_{\alpha_0} \neq 0$.

Suppose that the radical property $\mathbf{R}$ coincides with the Jacobson radical property on $\mathbf{L}$, but there exists a ring $A \in \mathbf{R}$ with $A \notin \mathcal{U}\mathbf{H}$. Now A can be mapped homomorphically onto a non-zero ring $H \in \mathbf{H}$. By definition, H is an accessible subring of a ring $T \in \mathbf{T} \subset \mathbf{L}$, moreover $\mathbf{R}(T) = \mathbf{J}(T) = \mathcal{U}\mathbf{H}(T) = 0$ holds. Applying Proposition 1, we get also $\mathbf{R}(H) = 0$. On the other hand H is a homomorphic image of $A \in \mathbf{R}$, which implies $H \in \mathbf{R}$ contradicting $H \neq 0$. Hence also $\mathbf{R} \subseteqq \mathcal{U}\mathbf{H}$ is proved.

The last assertion of Theorem 1 follows immediately from the definitions of $\mathcal{L}\mathbf{Q}$ and $\mathcal{U}\mathbf{H}$ radicals.

The next theorem shows us the efficiency of the approximation of the Jacobson radical property by the $\mathcal{L}\mathbf{Q}$ and $\mathcal{U}\mathbf{H}$ radicals.

Theorem 2. *Neither the Brown-McCoy radical property* $\mathbf{G}$ *nor Koethe's nil-radical property* $\mathbf{N}$ *coincide with the Jacobson radical on* $\mathbf{L}$, *in addition* $\mathbf{N} \cap \mathbf{L}^*$ *is a proper subclass of* $\mathbf{J} \cap \mathbf{L}^*$.

Proof. If W is the ring of all linear transformations of a countable dimensional vector space V over a division ring, and Z is the set of all finite-valued linear transformations of V, then Z is the only proper ideal of W, moreover, W is Jacobson semisimple, but its Brown-McCoy radical is obviously Z (cf. DIVINSKY [3], Example 11). This simple example shows that the Brown-McCoy radical does not coincide with the Jacobson radical on $\mathbf{L}$ (not even on $\mathbf{L}^*$).

Next we present a ring $R \in \mathbf{J} \cap \mathbf{L}^*$ which is not a nil-ring. For this aim let us consider the ring P of p-adic numbers equipped with the p-adic topology, we may assume that P is the full endomorphism ring of the quasicyclic group $C(p^\infty)$. Denote the additive group of P by P^+ and write $P_i = p^i P$ $(i = 1, 2, \ldots)$. Define the multiplication on the group theoretical direct sum $P_1^+ \oplus C(p^\infty)$ by the rule

$$(a + \alpha)(b + \beta) = ab + a\beta \qquad \big(a, b \in P; \alpha, \beta \in C(p^\infty)\big).$$

As it was shown in [11], we have obtained a ring R moreover R is in a narrow sense linearly compact, if the left ideals P_n $(n = 1, 2, \ldots)$ are chosen as a basisfilter. In [11] it was also shown that R is transfinite nilpotent and so by Proposition 3 it is a Jacobson radical ring. Hence we have $R \in \mathbf{J} \cap \mathbf{L}^*$ on the other hand, by

$$(a + \alpha)^n = a^n + a^{n-1}\alpha.$$

R is not a nil-ring, so $R \notin \mathbf{N} \cap \mathbf{L}^*$ follows.
By the view of $\mathbf{N} \subset \mathbf{J}$, we have obtained $\mathbf{N} \cap \mathbf{L}^* \subset \mathbf{J} \cap \mathbf{L}^*$ and thus Theorem 2 is proved.

In [2] DIVINSKY has given radical classes $\mathbf{D}$ and $\mathbf{U}$ such that if a radical $\mathbf{R}$ coincides with the Jacobson radical on the class of artinian rings, then $\mathbf{D} \subseteqq \mathbf{R} \subseteqq \mathbf{U}$ follows, further, as it is well-known, for Baer's lower radical class $\mathbf{B}$ Koethe's nil-radical class $\mathbf{N}$, and the Brown-McCoy radical class $\mathbf{G}$ the relation

$$\mathbf{D} \subset \mathbf{B} \subset \mathbf{N} \subset \mathbf{J} \subset \mathbf{G} \subset \mathbf{U}$$

holds. F. SZÁSZ [9] has given boundaries in a similar manner for radicals coinciding with the Jacobson radical on MHR-rings (those are rings with descending chain condition on principal right ideals), and it turned out that Baer's lower radical as well as Koethe's nil-radical coincide with the Jacobson radical on the class of MHR-rings. By the view of Theorem 2, our boundaries approximate better the Jacobson radical property.

A difficult and open question is to identify the Jacobson radical as an upper or lower radical property associated with some classes of rings. SĄSIADA and SULIŃSKI [8] have shown that the Jacobson radical is not the upper radical property determined by all simple primitive rings; this is a negative answer of a problem raised by KUROSH. Now we can ask

Problem 1. Is $\mathbf{J} = \mathscr{U}\mathbf{H}$ valid?

To disprove this conjecture, it would be sufficient to give a primitive ring P which can not be mapped homomorphically onto any accessible subring of a ring of $\mathbf{T}$. Namely, if P is such a ring, then it is, of course, Jacobson-semisimple but at the same time also an $\mathscr{U}\mathbf{H}$-radical ring (for the structure of primitive rings we refer to [3, 4, 5]).

Problem 2. Does $\mathscr{L}\mathbf{Q} = \mathbf{J}$ hold?

Since the class $\mathbf{B}$ of Baer's lower radical rings is a subclass of $\mathbf{J}$, so this conjecture will be disproved, if one will show that $\mathbf{B}$ is not contained in $\mathscr{L}\mathbf{Q}$, e.g. the zero-ring $Z(\infty)$ over the infinite cyclic group is a $\mathbf{B}$-ring, but may be it is not an $\mathscr{L}\mathbf{Q}$-ring.

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