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DER KÖNIGLICHEN GESELLSCHAFT DER WISSENSCHAFTEN ZU GÖTTINGEN.
MATHEMATISCH-PHYSIKALISCHE KLASSE.
NEUE FOLGE BAND II. Nro. 4.

Constitution of Matter and Analytical Theories of Heat.

An Essay in four Parts.

By

G. Prasad.

Berlin.
Weidmannsche Buchhandlung.
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By

Ganesh Prasad of Balia (India).

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Preface.

„Mathematical analysis has no marks to express confused notions." J. Fourier.

The following essay was undertaken with the object of utilising the recent advances in mathematical analysis for working out analytical theories of heat, each theory being based on definite suppositions as to the constitution of matter.

In carrying on the investigation embodied in this essay I consulted a very large number of authorities. I am chiefly indebted to the writings of G. Cantor, Fourier, Du Bois Reymond, Lord Kelvin, Larmor, Maxwell, Poisson, Brodén, Karl Pearson, Poincaré and Weierstrass. To these I must add Dini-Lüroth's „Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse", from which my knowledge of continuous analysis is chiefly derived. I am also indebted to Fitzgerald's scientific papers[1]) and Sommerfeld's dissertation[2]). Finally, I must mention that in the early stage of this investigation I received great encouragement from the views of F. Klein as given in his Evanston Colloquium and „Gutachten der Göttinger philosophischen Facultät betreffend die Beneke-Preisaufgabe für 1901" (Göttinger Nachrichten 1901, Geschäftliche Mitteilungen; Math. Ann. Bd. 55).

The essay in its present form was completed in the year 1902, with the exception of certain purely verbal alterations and a few unimportant corrections.

1) „The Scientific Writings of George Francis Fitzgerald", Dublin 1902.

2) „Die willkürlichen Functionen in der Mathematischen Physik", Inaugural-Dissertation, Königsberg 1891.

Introduction.

(1)

The object of this essay is to show how the recent advances in mathematical analysis may be used for working out analytical theories of the linear conduction of heat in a homogeneous solid; each theory being based on definite suppositions as to the constitution of the solid. The essay is divided into four parts. In Part I. we give a theory which treats the solid as a continuum with the same properties in all its points. Part II. contains a carefully worked out theory which treats the solid as molecular in structure but takes no account of the constitution of the molecules; at the end of this part a criticism of Fourier's theory is given. In Part III. is worked out a theory which regards the solid as *improperly continuous*, i. e., as a continuum in which an everywhere dense but enumerable aggregate of points is marked out to distinguish the solid from all other solids; this part concludes with a discussion of the question of the „uniqueness of the solution." Part IV. contains a brief discussion of the theories expounded in the previous parts.

(2)

We restrict ourselves throughout this essay to the simple problem of linear conduction in an infinite slab bounded by two parallel planes impermeable to heat and at distance 2π from each other, the initial temperature being an even function of the distance from the central plane of the slab. The first step towards the solution of this problem is to define the conditions of the phenomenon. We take the fundamental condition to be the satisfaction of the principle of the conservation of energy; the other condition is that if with varying *time* any physical quantity passes from one value to another it assumes all the intermediate values. The next step is to transform the problem from a physical into an analytical one; and this is done by translating, with the help of suitable hypotheses, the above two conditions, and the supposition of the impermeability of the faces, into analytical language. We then find without difficulty an extensive class of solutions in terms of functions of the type

$$V(x, t) = \tfrac{1}{2}a_0 + \sum_1^{\infty} a_m \cos mx\, e^{-m^2 t}, \qquad t > 0,$$

where $a_m = \dfrac{2}{\pi}\int_0^{\pi} f(x') \cos mx'\, dx'$, $f(x)$ being any integrable function; such functions $V(x, t)$ we call, for the sake of convenience, functions of *Fourier's type.*

(3)

In Part I. we first find the analytical representations of the conditions of the phenomenon. These are the conditions (A), (B) and (C) of Art. 4. In Art. 5 attention is drawn to the fact that there may be solutions of the problem other than functions of Fourier's type. Throughout the remainder of this part we discuss solutions of Fourier's type. We begin by expressing (A), (B) and (C) in terms of $V(x, t)$, $f(x)$ being the initial temperature. We thus obtain in Art. 10 the group of conditions which is necessary and sufficient in order that $V(x, t)$ be the solution of the problem. These conditions are the following:

i. For every value of x or, at least, for an everywhere dense aggregate of its values, there exists a finite constant P such that, for any value of t, however small, $\left|\dfrac{\partial}{\partial x} V(x, t)\right| < P, \ -\pi < x < \pi.$

ii. $\lim\limits_{t=+0} V(x, t) = f(x)$ if the limit exists; or, the limit does not exist, and then $f(x)$ is contained in the aggregate of values assumed by $V(x, t)$ as t approaches zero.

In order to find the nature of the restrictions which these conditions impose on $f(x)$, we carefully investigate in Arts. 11—21 the behaviour of $V'(x, t)$ and $V(x, t)$ when t approaches zero. The final results of this investigation, in which we make use of Du Bois Reymond's Infinital Calculus (Infinitärcalcül), are given in Arts. 19 and 21 [1]). Making use of these results, we give in Art. 22

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1) It would be convenient to explain the notation which we use: If $A(y)$ and $B(y)$ be two functions of y such that $\dfrac{A(y)}{B(y)}$ is positive and monoton in the neighbourhood of y_0; then it is said that $A(y) \gtrsim\!\!\!< B(y)$, as y approaches y_0, according as $\dfrac{A(y)}{B(y)}$ has a limit which is infinite, finite or zero. This is Du Bois Reymond's notation. We have found it convenient to introduce four new symbols, viz., $\succ\!\!\bullet$, $\backsim\!\!\bullet$, $\bullet\!\!\prec$ and $\pm\!\succ$. By $L(y) \; \overset{\succ\bullet}{\underset{\bullet\prec}{\backsim}} \; M(y)$, we mean $L(y) \gtrsim\!\!\!< M(y)N(y)$ where $N(y)$ is finite and positive (different from zero) but *not* necessarily monoton. For example, $\dfrac{1}{y^2} \succ\!\!\bullet\; l\left(\dfrac{1}{y}\right)\left(2 + \cos\dfrac{1}{y}\right)$ as y approaches zero. If, with the approach of y to y_0, $C(y)$ makes an indefinitely large number of oscillations, from positive to negative values, with indefinitely increasing amplitude; then we say that $C(y) \pm\!\succ 1$ as y approaches y_0. For example, $\dfrac{1}{y}\cos\left(\dfrac{1}{y}\right) \pm\!\succ 1$ as y approaches zero.

certain necessary and sufficient conditions for $f(x)$, which are of extensive applicability. These conditions at once indicate the classification of initial states as stable or unstable, non-oscillatory or oscillatory, admissible or inadmissible. This classification is discussed in Art. 23. We conclude this part with Art. 24 in which five examples[1]) of initial states are given; the solution of the problem corresponding to each of these being of Fourier's type. In the first the initial state is continuous, stable and non-oscillatory; in the second, discontinuous but stable and non-oscillatory; in the third, discontinuous, stable and oscillatory; in the fourth as well as in the fifth, discontinuous and unstable.

(4)

In Part II. we give a carefully worked out theory which treats the solid as molecular in structure but takes no account of the constitution of the molecule. The scheme of our exposition is as follows: —

In Art. 25 we begin by specifying clearly and in detail the molecular oscillations in the solid. The solid is thus supposed to contain rows of molecules parallel to the axis of x; the molecular oscillations in each row being the same. Each row contains r assemblages, the number of molecules in each assemblage being s. Now, with each assemblage in a row is associated a quantity which is a function of time and which depends only on the molecular oscillations in the assemblage; this quantity we call the *temperature* of the assemblage. At the end of Art. 25 is given the following formulation of the problem: —

The initial temperatures, $T_1(0)$, $T_2(0)$, ... $T_r(0)$, of the assemblages A_1, A_2, ... A_r in any particular row being given, find their subsequent temperatures.

The investigation embodied in Arts. 26—31 requires very delicate considerations and is the most difficult portion of the exposition. In these Articles we show how, by means of the hypotheses (α_1), (β_1) and (γ_1) of Art. 27, approximate analytical representations of the actual conditions of the phenomenon can be obtained in terms of a continuous function $Y(x, t)$ which we call the *auxiliary function* of the problem; this function is such that $Y(x_{\varrho,t}, t) = T_\varrho(t)$ where $x_{\varrho,t}$ is the x-coordinate of the centre, and $T_\varrho(t)$ the temperature, of the assemblage A_ϱ at time t. As the first step to this end, we find in Art. 28 an approximation to the quantity of heat which flows across a unit area, placed at right angles to the axis of x, in any interval $(t, t+\tau)$: the result is given in the equation (I). As the next step we obtain in Art. 29 an approximation to the quantity of heat absorbed by a cylinder, with its axis parallel to the axis of x and its faces, of unit area, $x = x_1$, $x = x_2$, in any interval: the result is given in the equation (II).

1) It is hardly necessary to mention that the functions $f(x)$ used in the examples (iii) and (iv) belong to a new type which was suggested to the present writer by the procedure in Arts. 110*—11* of Dini-Lüroth's „Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse.“ The investigation given there is, with evident modifications, applicable to these functions.

In Art. 30 we discuss these two results and attention is drawn to the fact that in order that approximations be at all possible it is necessary that the periphery of the unit area be of restricted size and shape. For example, when the area is a rectangle $\left(\frac{\lambda}{2}\times\frac{2}{\lambda}\right)$ no approximation is possible, λ being the length of the molecule. Making use of (I) and (II) we obtain in Art. 31 the approximate conditions of the phenomenon. These are (A_1), (B_1) and (C_1). (A_1) corresponds to the principle of energy; (B_1), to the condition that as $T_Q(t)$ passes from one value to another it assumes all the intermediate values; and (C_1), to the supposition of the impermeability of the faces. The remaining portion of the exposition is much simpler; in it we find what conditions $f(x)$ must satisfy in order that the auxiliary function be of Fourier's type, where $f(x) = Y(x, 0)$ and $f(x)$, $f'(x)$ and $f''(x)$ are finite and continuous. We show in Art. 32 that, as (A_1), (B_1) and (C_1) are in form the same as (A), (B) and (C), the only conditions to be satisfied are those which are necessary in order that (A_1), (B_1) and (C_1) be approximations to the actual conditions. These necessary conditions are given in the end of the article.

The final result, regarding the condition to be satisfied by $f(x)$, may be thus stated:

If it be supposed that

$$\lambda_1 < 10^{-2},$$
$$\lambda_2 \leq 10^{-2}\lambda_1,$$
$$\lambda_1 \geq \frac{2}{s};$$

then it is sufficient that

$$\frac{\lambda_1}{2\pi\bar{f}}\left\{43\,|f'(\pi)| + 67\,b + \tfrac{3}{5}\sum_1^\infty \frac{|b_m|}{m}\right\}$$

be negligible. This is the condition $(\mathfrak{P}_1)$ given at the end of Art. 37. Here $2\lambda_1$ is the quantity which first appears on page 26 and may be called the radius of the sphere of influence of any assemblage, λ_2 is the greatest length that an assemblage can have, $\bar{f} = \frac{1}{\pi}\left|\int_0^\pi f(x')\,dx'\right|$, $b_m = \frac{2}{\pi}\int_0^\pi f''(x')\cos mx'\,dx'$ and b is the greatest value of $|b_m|$. In Art. 38 attention is drawn to the fact that, in a professedly inexact theory like the present one, the important thing is the *nature of the restriction* imposed on $f(x)$; and, as the quasi-necessary condition $(\mathfrak{P}_1)$ clearly shows, this is purely arithmetical. We conclude the exposition of this theory with Art. 39 in which illustrative examples are given; in each example we start with *definite* suppositions as to the magnitudes of λ, λ_2, λ_1, and find out the corresponding superior limit of error. The table given in the fourth example draws attention to the fact that the error increases very rapidly as the change of initial temperature from assemblage to assemblage increases; in the fifth example this change is so great that the theory fails to give any approximation to the solution.

In Arts. 40—43 we give a brief criticism of Fourier's theory. We begin by pointing out that Fourier's theory is a continuous one; we then find in Art. 41 the conditions which are necessary and sufficient in order that the conditions given by Fourier's theory may have any meaning and, further, follow from the conditions (A), (B) and (C) of our theory. In Art. 42 we consider the solutions of Fourier's type and prove the following result: —

In order that the conditions given by Fourier's theory viz.,

$$\frac{\partial}{\partial t} T(x, t) = \frac{\partial^2}{\partial x^2} T(x, t),$$

$T(x, t)$ is continuous in t, and $T'(\pi, t) = 0$, $T'(-\pi, t) = 0$, $-\pi \leq x \leq \pi$, $0 \leq t$, may have meaning and necessarily follow from (A), (B) and (C), it is necessary that $f''(x)$ exist and be finite; and it is sufficient that $f''(x)$ be finite and continuous, and $f'(\pi)$, $f'(-\pi)$ be zero.

We conclude this part with Art. 43 which contains seven examples specially selected to illustrate the limited scope of Fourier's theory.

(5)

Part III. contains a theory which regards the solid as improperly continuous; to work out this theory we have to employ an *improperly continuous analysis*, i. e., an analysis in which one of the independent variables has for its domain an everywhere dense but enumerable aggregate. We begin by carefully specifying in Art. 44 the notation which we employ. In Art. 45 we formulate the following improperly continuous theory of solids:

With the slab is associated an enumerable aggregate G, of positive numbers, which is everywhere dense in the interval $(0, \pi)$ and contains π: we call G the *discriminating aggregate* of the slab. Also, two slabs differ only in this that they have different discriminating aggregates. Taking ξ for a variable with the aggregates $-G$ and G as its domain, we define the temperature $T(x, t)$ at any point x by the equations

$$T(x, t) = \frac{1}{2\mathfrak{l}} \int_{x-\mathfrak{l}}^{x+\mathfrak{l}} C(\xi, t)\, d\xi, \quad 0 \leq x \leq \pi - \mathfrak{l},$$

$$T(x, t) = \frac{1}{(\pi - x + \mathfrak{l})} \int_{x-\mathfrak{l}}^{\pi} C(\xi, t)\, d\xi, \quad \pi - \mathfrak{l} \leq x \leq \pi,$$

where $\mathfrak{l}$ is dependent on G and is less than, say, 10^{-7}.

These are the equations $(\mathfrak{A}_2)$ of page 50: $C(\xi, t)$ is called the *characteristic function* of the slab. At the end of Art. 45 the problem of linear conduction is thus formulated:

Given the initial characteristic function $C(\xi, 0)$, find the characteristic function at any subsequent time.

We then go on to obtain the analytical representations of the conditions of the phenomenon; these are the conditions (A_2), (B_2), (C_2) and (D_2) of Art. 47. In Art. 48 attention is drawn to the definition of temperature, as given by $(\mathfrak{A}_2)$, which indicates the classification of initial temperatures as possible or impossible. In Arts. 49—53 we discuss characteristic functions of Fourier's type. We begin by expressing (A_2), (B_2), (C_2) and (D_2) in terms of $V(x, t)$ and in the notation of continuous analysis; we thus obtain in Art. 51 the group of conditions which is necessary and sufficient in order that $C(\xi, t) = \chi(\xi, t)$, $C(\xi, 0)$ being $\varphi(\xi)$: here $\chi(\xi, t) = V(\xi, t)$ and $\varphi(\xi) = f(\xi)$.

These conditions are as follows:

i. For every value of ξ, there exists a finite constant P such that, for any value of t, however small, $\left|\left\{\frac{\partial}{\partial x} V(x, t)\right\}_{x=\xi}\right|$ is less than P, $-\pi < \xi < \pi$.

ii. $\lim\limits_{t=+0} V(\xi, t) = \varphi(\xi)$ if the limit exists; or, the limit does not exist, and then $\varphi(\xi)$ is contained in the aggregate of values assumed by $V(\xi, t)$ as t approaches zero.

Making use of the results of Arts. 19 and 21 we find in Art. 52 necessary and sufficient conditions, for $\varphi(\xi)$, of an extensive applicability. These conditions at once indicate the classification of initial states as stable or unstable, oscillatory or non-oscillatory, admissible or inadmissible. This classification is discussed in Art. 53. In Arts. 54 and 55 we discuss approximations to impossible initial temperatures. We conclude the exposition of the theory with Art. 56 in which nine examples are given to illustrate the salient features of the theory.

This part ends with Arts. 57—59 in which we give a careful discussion of the question of „the uniqueness of the solution“, specially bearing in mind the fact that there *may* be solutions other than functions of Fourier's type.

(6)

In Part IV. we discuss briefly the theories, expounded in the previous parts, in so far as they illustrate the nature of the relation of mathematical analysis to physics.