# Nova methodus, aequationes differentiales partiales primi ordinis inter numerum variabilium quemcunque propositas integrandi.

(Ex ill. *C. G. J. Jacobi* manuscriptis posthumis in medium protulit *A. Clebsch.*)

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*Reductio problematis generalis in formam simpliciorem* *).

1.

Sit $V$ functio quaesita, sint $q_1, q_2, \dots q_m$ variabiles independentes atque $p_1, p_2, \dots p_m$ differentialia partialia ipsius $V$ secundum $q_1, q_2, \dots q_m$. Problema de integratione aequationum differentialium partialium primi ordinis inter numerum quemcunque variabilium hoc est:

*Data aequatione inter quantitates $V, q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$, ipsam $V$ ut functionem ipsarum $q_1, q_2, \dots q_m$ determinare.*

Supponam aequationem propositam ipsam functionem quaesitam $V$ non continere. Quoties enim continet, problema ad aliud revocari potest, in quo numerus variabilium independentium unitate auctus est, sed functio ipsa incognita ex aequatione differentiali evasit. Introducta enim nova variabili $t$, sit

$$W = t.V,$$

erit

$$V = \frac{\partial W}{\partial t}, \quad p_1 = \frac{\partial V}{\partial q_1} = \frac{1}{t}\frac{\partial W}{\partial q_1}, \quad p_2 = \frac{\partial V}{\partial q_2} = \frac{1}{t}\frac{\partial W}{\partial q_2}, \quad \dots \text{**)}.$$

Quibus valoribus substitutis in aequatione inter $V$ et quantitates $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ proposita, prodibit aequatio inter variabiles independentes $t, q_1, q_2, \dots q_m$, atque differentialia partialia functionis $W$ secundum variabiles illas sumta, ipsam functionem $W$ non continens. Hinc, quia numerum variabilium independentium *m quemcunque* assumsimus, concessa est suppositio, aequationem differentialem propositam functionem incognitam non continere.

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*) Epitomae paragraphorum in ipso manuscripto praeter paragraphos 66, 67 non inveniuntur. Quae tamen in usum lectoris, ut longioris commentationis decursus facilius perspiceretur, adjiciendae videbantur. *C.*

**) Significandis differentialibus partialibus signum characteristicum — $\partial$ —, significandis completis signum — $d$ — adhibebo. Quod bene tenendum est.

*Problema sub ea, qua in sequentibus utatur, forma proponitur.*

2.

Si functio incognita ipsa aequationem differentialem partialem propositam non ingreditur, problema maxima generalitate sic enuntiari potest:

*Proposita expressione*

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m,$$

*si data est aequatio inter quantitates* $q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m$, *invenire* $m-1$ *alias aequationes inter easdem quantitates, e quibus quantitates* $p_1, p_2, \ldots p_m$ *tales prodeant functiones ipsarum* $q_1, q_2, \ldots q_m$, *ut expressio proposita*

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$$

*evadat differentiale completum* $dV$.

Ut expressio

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$$

sit differentiale completum, satisfieri debet $\frac{m(m-1)}{2}$ aequationibus conditionalibus hoc schemate contentis:

$$\left(\frac{\partial p_i}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_i}\right),$$

in qua aequatione indicibus $i$ et $k$ valores $1, 2, 3, \ldots m$ tribui possunt, vel ut aequationes tantum inter se diversae obtineantur, indici $i$ tribuantur valores $1, 2, 3, \ldots m-1$ et pro singulis ipsius $i$ valoribus tribuantur indici $k$ valores tantum ipso $i$ maiores.

In aequationibus praecedentibus quantitates $p_1, p_2, \ldots p_m$ ut functiones ipsarum $q_1, q_2, \ldots q_m$ consideratae sunt. Quod quoties fit, differentialia partialia illarum quantitatum uncis includam, sicuti antecedentibus factum est.

*Conditionum integrabilitatis forma prima exhibetur.*

3.

Negotium, quod suscipiam, primum est transformatio aequationum conditionalium. Quippe quas ita exhibeamus, quales fiunt, si non ut antea omnes $p_1, p_2, \ldots p_m$ ut ipsarum $q_1, q_2, \ldots q_m$ functiones considerantur, sed

$p_1$ ut ipsarum $p_2, p_3, p_4, p_5, \ldots p_m, q_1, q_2, \ldots q_m$,
$p_2$ ut ipsarum $p_3, p_4, p_5, \ldots p_m, q_1, q_2, \ldots q_m$,
$p_3$ ut ipsarum $p_4, p_5, \ldots p_m, q_1, q_2, \ldots q_m$,
. . . . . . . . . . . . . . . . . . . . .
$p_{m-1}$ ut ipsarum $p_m, q_1, q_2, \ldots q_m$,
$p_m$ ut ipsarum $q_1, q_2, \ldots q_m$

functiones. Ad quam suppositionem referam sequentibus differentiationes per partes instituendas, nisi aliud disertis verbis statutum sit, aut differentialia uncis inclusa reperias, quo facto semper innuitur, omnes $p_1, p_2, \dots p_m$ tamquam ipsarum $q_1, q_2, \dots q_m$ functiones spectari.

Systema *primum* aequationum conditionalium, quod respondet valori $i = 1$, hoc est:

$$\left(\frac{\partial p_1}{\partial q_2}\right) = \left(\frac{\partial p_2}{\partial q_1}\right), \quad \left(\frac{\partial p_1}{\partial q_3}\right) = \left(\frac{\partial p_3}{\partial q_1}\right), \quad \dots \quad \left(\frac{\partial p_1}{\partial q_m}\right) = \left(\frac{\partial p_m}{\partial q_1}\right).$$

Quod e supra statutis sic representari potest:

$$\begin{aligned}
\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_2}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_3}{\partial q_2}\right) + \dots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_m}{\partial q_2}\right) + \frac{\partial p_1}{\partial q_2} &= \left(\frac{\partial p_2}{\partial q_1}\right), \\
\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_2}{\partial q_3}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_3}{\partial q_3}\right) + \dots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_m}{\partial q_3}\right) + \frac{\partial p_1}{\partial q_3} &= \left(\frac{\partial p_3}{\partial q_1}\right), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_2}{\partial q_m}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_3}{\partial q_m}\right) + \dots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_m}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_m} &= \left(\frac{\partial p_m}{\partial q_1}\right).
\end{aligned}$$

Quae aequationes per aequationes conditionales in has transformari possunt:

$$\begin{aligned}
\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_2}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_2}{\partial q_3}\right) + \dots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_2}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_2} &= \left(\frac{\partial p_2}{\partial q_1}\right), \\
\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_3}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_3}{\partial q_3}\right) + \dots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_3}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_3} &= \left(\frac{\partial p_3}{\partial q_1}\right), \\
\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_4}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_4}{\partial q_3}\right) + \dots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_4}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_4} &= \left(\frac{\partial p_4}{\partial q_1}\right), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{\partial p_1}{\partial p_2}\left(\frac{\partial p_m}{\partial q_2}\right) + \frac{\partial p_1}{\partial p_3}\left(\frac{\partial p_m}{\partial q_3}\right) + \dots + \frac{\partial p_1}{\partial p_m}\left(\frac{\partial p_m}{\partial q_m}\right) + \frac{\partial p_1}{\partial q_m} &= \left(\frac{\partial p_m}{\partial q_1}\right).
\end{aligned}$$

Multiplicemus aequationem $2^{\text{am}}$, $3^{\text{am}}$, $\dots$ $(m-1)^{\text{tam}}$ per. $\frac{\partial p_2}{\partial p_3}$, $\frac{\partial p_2}{\partial p_4}$, $\dots$ $\frac{\partial p_2}{\partial p_m}$ et productarum summam deducamus a prima; multiplicemus aequationem $3^{\text{am}}$, $4^{\text{tam}}$, $\dots$ $(m-1)^{\text{tam}}$ per $\frac{\partial p_3}{\partial p_4}$, $\frac{\partial p_3}{\partial p_5}$, $\dots$ $\frac{\partial p_3}{\partial p_m}$ et productarum summam subducamus de secunda; multiplicemus aequationem $4^{\text{tam}}$, $5^{\text{tam}}$, $\dots$ $(m-1)^{\text{tam}}$ per $\frac{\partial p_4}{\partial p_5}$, $\frac{\partial p_4}{\partial p_6}$, $\dots$ $\frac{\partial p_4}{\partial p_m}$, et productarum summam deducamus de tertia; et ita porro. Quibus patratis aliud eruimus systema aequationum, systemati primo aequivalens, hoc:

$$
(A.)\left\{\begin{array}{ll}
(1.) & \frac{\partial p_1}{\partial p_2}\frac{\partial p_2}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_2}{\partial q_3}+\frac{\partial p_1}{\partial p_4}\frac{\partial p_2}{\partial q_4}+\frac{\partial p_1}{\partial p_5}\frac{\partial p_2}{\partial q_5}+\dots+\frac{\partial p_1}{\partial p_m}\frac{\partial p_2}{\partial q_m} \\
& \quad +\frac{\partial p_1}{\partial q_2}-\frac{\partial p_2}{\partial p_3}\frac{\partial p_1}{\partial q_3}-\frac{\partial p_2}{\partial p_4}\frac{\partial p_1}{\partial q_4}-\frac{\partial p_2}{\partial p_5}\frac{\partial p_1}{\partial q_5}-\dots-\frac{\partial p_2}{\partial p_m}\frac{\partial p_1}{\partial q_m}=\frac{\partial p_2}{\partial q_1} \\
(2.) & \frac{\partial p_1}{\partial p_2}\frac{\partial p_3}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_3}{\partial q_3}+\frac{\partial p_1}{\partial p_4}\frac{\partial p_3}{\partial q_4}+\frac{\partial p_1}{\partial p_5}\frac{\partial p_3}{\partial q_5}+\dots+\frac{\partial p_1}{\partial p_m}\frac{\partial p_3}{\partial q_m} \\
& \quad +\frac{\partial p_1}{\partial q_3}-\frac{\partial p_3}{\partial p_4}\frac{\partial p_1}{\partial q_4}-\frac{\partial p_3}{\partial p_5}\frac{\partial p_1}{\partial q_5}-\dots-\frac{\partial p_3}{\partial p_m}\frac{\partial p_1}{\partial q_m}=\frac{\partial p_3}{\partial q_1} \\
(3.) & \frac{\partial p_1}{\partial p_2}\frac{\partial p_4}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_4}{\partial q_3}+\frac{\partial p_1}{\partial p_4}\frac{\partial p_4}{\partial q_4}+\frac{\partial p_1}{\partial p_5}\frac{\partial p_4}{\partial q_5}+\dots+\frac{\partial p_1}{\partial p_m}\frac{\partial p_4}{\partial q_m} \\
& \quad +\frac{\partial p_1}{\partial q_4}-\frac{\partial p_4}{\partial p_5}\frac{\partial p_1}{\partial q_5}-\dots-\frac{\partial p_4}{\partial p_m}\frac{\partial p_1}{\partial q_m}=\frac{\partial p_4}{\partial q_1} \\
\dots & \dots \\
(m-2.) & \frac{\partial p_1}{\partial p_2}\frac{\partial p_{m-1}}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_{m-1}}{\partial q_3}+\dots+\frac{\partial p_1}{\partial p_{m-1}}\frac{\partial p_{m-1}}{\partial q_{m-1}}+\frac{\partial p_1}{\partial p_m}\frac{\partial p_{m-1}}{\partial q_m} \\
& \quad +\frac{\partial p_1}{\partial q_{m-1}}-\frac{\partial p_{m-1}}{\partial p_m}\frac{\partial p_1}{\partial q_m}=\frac{\partial p_{m-1}}{\partial q_1} \\
(m-1.) & \frac{\partial p_1}{\partial p_2}\frac{\partial p_m}{\partial q_2}+\frac{\partial p_1}{\partial p_3}\frac{\partial p_m}{\partial q_3}+\dots+\frac{\partial p_1}{\partial p_{m-1}}\frac{\partial p_m}{\partial q_{m-1}}+\frac{\partial p_1}{\partial p_m}\frac{\partial p_m}{\partial q_m} \\
& \quad +\frac{\partial p_1}{\partial q_m}=\frac{\partial p_m}{\partial q_1}.
\end{array}\right.
$$

E quibus aequationibus differentialia partialia uncis inclusa evaserunt.

## 4.

Systema *secundum* aequationum conditionalium, quod respondet valori $i=2$, hoc est:

$$\left(\frac{\partial p_2}{\partial q_3}\right)=\left(\frac{\partial p_3}{\partial q_2}\right),\quad \left(\frac{\partial p_2}{\partial q_4}\right)=\left(\frac{\partial p_4}{\partial q_2}\right),\quad \dots\quad \left(\frac{\partial p_2}{\partial q_m}\right)=\left(\frac{\partial p_m}{\partial q_2}\right).$$

Designante $k$ quemlibet e numeris 3, 4, ... $m$, aequatio

$$\left(\frac{\partial p_2}{\partial q_k}\right)=\left(\frac{\partial p_k}{\partial q_2}\right)$$

sic etiam exhiberi potest:

$$\frac{\partial p_2}{\partial p_3}\left(\frac{\partial p_3}{\partial q_k}\right)+\frac{\partial p_2}{\partial p_4}\left(\frac{\partial p_4}{\partial q_k}\right)+\dots+\frac{\partial p_2}{\partial p_m}\left(\frac{\partial p_m}{\partial q_k}\right)+\frac{\partial p_2}{\partial q_k}=\left(\frac{\partial p_k}{\partial q_2}\right)$$

quae adhibendo aequationes

$$\left(\frac{\partial p_{k'}}{\partial q_k}\right)=\left(\frac{\partial p_k}{\partial q_{k'}}\right),$$

in sequentem abit:

$$\frac{\partial p_2}{\partial p_3}\Big(\frac{\partial p_k}{\partial q_3}\Big)+\frac{\partial p_2}{\partial p_4}\Big(\frac{\partial p_k}{\partial q_4}\Big)+\cdots+\frac{\partial p_2}{\partial p_m}\Big(\frac{\partial p_k}{\partial q_m}\Big)+\frac{\partial p_2}{\partial q_k}=\Big(\frac{\partial p_k}{\partial q_2}\Big).$$

Si aequationem $1^{\text{am}}$, $2^{\text{am}}$, ... $(m-2)^{\text{tam}}$ vocamus, quae prodeunt ex aequatione praecedente loco $k$ respective ponendo valores 3, 4, ... $m$, multiplicemus aequationem $2^{\text{am}}$, $3^{\text{am}}$, ... $(m-2)^{\text{tam}}$ per $\frac{\partial p_3}{\partial p_4}$, $\frac{\partial p_3}{\partial p_5}$, ... $\frac{\partial p_3}{\partial p_m}$ et productarum summam deducamus de prima; multiplicemus $3^{\text{am}}$, $4^{\text{tam}}$, ... $(m-2)^{\text{tam}}$ per $\frac{\partial p_4}{\partial p_5}$, $\frac{\partial p_4}{\partial p_6}$, ... $\frac{\partial p_4}{\partial p_m}$ et productarum summam deducamus de secunda; et ita porro. Eruetur his transactis systema aequationum hoc:

$$(\text{B.})\left\{\begin{array}{ll}
(1.) & \frac{\partial p_2}{\partial p_3}\frac{\partial p_3}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial p_3}{\partial q_4}+\frac{\partial p_2}{\partial p_5}\frac{\partial p_3}{\partial q_5}+\ \cdots\ +\frac{\partial p_2}{\partial p_m}\frac{\partial p_3}{\partial q_m}\\
& \quad +\frac{\partial p_2}{\partial q_3}-\frac{\partial p_3}{\partial p_4}\frac{\partial p_2}{\partial q_4}-\frac{\partial p_3}{\partial p_5}\frac{\partial p_2}{\partial q_5}-\ \cdots\ -\frac{\partial p_3}{\partial p_m}\frac{\partial p_2}{\partial q_m}=\frac{\partial p_3}{\partial q_2}\\
(2.) & \frac{\partial p_2}{\partial p_3}\frac{\partial p_4}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial p_4}{\partial q_4}+\frac{\partial p_2}{\partial p_5}\frac{\partial p_4}{\partial q_5}+\ \cdots\ +\frac{\partial p_2}{\partial p_m}\frac{\partial p_4}{\partial q_m}\\
& \quad +\frac{\partial p_2}{\partial q_4}-\frac{\partial p_4}{\partial p_5}\frac{\partial p_2}{\partial q_5}-\ \cdots\ -\frac{\partial p_4}{\partial p_m}\frac{\partial p_2}{\partial q_m}=\frac{\partial p_4}{\partial q_2}\\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
(m-3.) & \frac{\partial p_2}{\partial p_3}\frac{\partial p_{m-1}}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial p_{m-1}}{\partial q_4}+\ \cdots\ +\frac{\partial p_2}{\partial p_{m-1}}\frac{\partial p_{m-1}}{\partial q_{m-1}}+\frac{\partial p_2}{\partial p_m}\frac{\partial p_{m-1}}{\partial q_m}\\
& \quad +\frac{\partial p_2}{\partial q_{m-1}}-\frac{\partial p_{m-1}}{\partial p_m}\frac{\partial p_2}{\partial q_m}=\frac{\partial p_{m-1}}{\partial q_2}\\
(m-2.) & \frac{\partial p_2}{\partial p_3}\frac{\partial p_m}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial p_m}{\partial q_4}+\ \cdots\ +\frac{\partial p_2}{\partial p_{m-1}}\frac{\partial p_m}{\partial q_{m-1}}+\frac{\partial p_2}{\partial p_m}\frac{\partial p_m}{\partial q_m}\\
& \quad +\frac{\partial p_2}{\partial q_m}=\frac{\partial p_m}{\partial q_2}.
\end{array}\right.$$

Quod aequationum systema e praecedente (A.) eruitur, si indices omnes unitate augentur, quantum fieri per limites indicum potest.

## 5.

Prorsus eadem ratione demonstratur *generalis* aequatio:

$$(a.)\left\{\begin{array}{l}
\frac{\partial p_i}{\partial p_{i+1}}\frac{\partial p_k}{\partial q_{i+1}}+\frac{\partial p_i}{\partial p_{i+2}}\frac{\partial p_k}{\partial q_{i+2}}+\frac{\partial p_i}{\partial p_{i+3}}\frac{\partial p_k}{\partial q_{i+3}}+\cdots+\frac{\partial p_i}{\partial p_m}\frac{\partial p_k}{\partial q_m}\\
\quad +\frac{\partial p_i}{\partial p_k}-\frac{\partial p_k}{\partial p_{k+1}}\frac{\partial p_i}{\partial q_{k+1}}-\frac{\partial p_k}{\partial p_{k+2}}\frac{\partial p_i}{\partial q_{k+2}}-\cdots-\frac{\partial p_k}{\partial p_m}\frac{\partial p_i}{\partial q_m}=\frac{\partial p_k}{\partial q_i},
\end{array}\right.$$

in qua $i$ designare potest unumquemque e numeris $1, 2, 3, \dots m-1$ atque pro singulis ipsius $i$ valoribus designare potest $k$ numerum unumquemque ipso $i$ majorem usque ad valorem $k = m$. Quae igitur aequatio generalis amplectitur numerum $\frac{m(m-1)}{2}$ aequationum inter se diversarum, quae e totidem aequationibus

$$\left(\frac{\partial p_i}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_i}\right)$$

derivatae sunt.

*De forma usitata conditionem integrabilitatis ex ea quae proponitur derivanda.*

6.

Vice versa ex aequationibus ($a$.) deduci possunt aequationes conditionales initio propositae

$$\left(\frac{\partial p_i}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_i}\right),$$

sive demonstrari potest theorema sequens:

Theorema I.

*Supponatur:*

$p_1$ *functio quantitatum* $p_2, p_3, p_4, p_5, \dots p_m, q_1, q_2, \dots q_m,$
$p_2$ – – $p_3, p_4, p_5, \dots p_m, q_1, q_2, \dots q_m,$
$p_3$ – – $p_4, p_5, \dots p_m, q_1, q_2, \dots q_m,$
. . . . . . . . . . . . . . . . . . . . . .
$p_{m-1}$ – – $p_m, q_1, q_2, \dots q_m,$
$p_m$ – – $q_1, q_2, \dots q_m,$

*quae tales sint functiones, ut habeatur identice:*

$$(a.)\quad \begin{cases} 0 = -\dfrac{\partial p_k}{\partial q_i} + \dfrac{\partial p_i}{\partial p_{i+1}}\dfrac{\partial p_k}{\partial q_{i+1}} + \dfrac{\partial p_i}{\partial p_{i+2}}\dfrac{\partial p_k}{\partial q_{i+2}} + \dots + \dfrac{\partial p_i}{\partial p_m}\dfrac{\partial p_k}{\partial q_m} \\ \qquad + \dfrac{\partial p_i}{\partial q_k} - \dfrac{\partial p_k}{\partial p_{k+1}}\dfrac{\partial p_i}{\partial q_{k+1}} - \dfrac{\partial p_k}{\partial p_{k+2}}\dfrac{\partial p_i}{\partial q_{k+2}} - \dots - \dfrac{\partial p_k}{\partial p_m}\dfrac{\partial p_i}{\partial q_m}, \end{cases}$$

*designante $i$ unumquemque e numeris $1, 2, 3, \dots m-1$ et pro singulis ipsius $i$ valoribus designante $k$ unumquemque e numeris $i+1, i+2, \dots m$, unde numerus totus aequationum est $\frac{m(m-1)}{2}$; erunt aequationes illae numero $\frac{m(m-1)}{2}$ conditiones quum necessariae tum sufficientes, ut expressis* *omnibus* $p_1, p_2, \dots p_m$ *per quantitates* $q_1, q_2, \dots q_m$, *expressio*

$$p_1 dq_1 + p_2 dq_2 + \dots + p_m dq_m$$

*evadat differentiale completum.*

*Forma secunda conditionum integrabilitatis.*

7.

Conditiones illas esse necessarias antecedentibus comprobavi, quippe quas locum habere demonstravi, quoties expressio

$$p_1 dq_1 + p_2 dq_2 + \dots + p_m dq_m$$

differentiale completum sit. Iam demonstrabo easdem conditiones esse *sufficientes,* sive quoties aequationes illae numero $\frac{m(m-1)}{2}$ locum habeant, expressionem

$$p_1 dq_1 + p_2 dq_2 + \dots + p_m dq_m$$

esse differentiale completum.

Posito $k = m$, aequatio proposita fit:

$$(1.)\quad 0 = -\left(\frac{\partial p_m}{\partial q_i}\right) + \frac{\partial p_i}{\partial p_{i+1}}\left(\frac{\partial p_m}{\partial q_{i+1}}\right) + \frac{\partial p_i}{\partial p_{i+2}}\left(\frac{\partial p_m}{\partial q_{i+2}}\right) + \dots + \frac{\partial p_i}{\partial p_m}\left(\frac{\partial p_m}{\partial q_m}\right) + \frac{\partial p_i}{\partial q_m}.$$

Uncis rursus utimur, si $p_1, p_2, \dots p_m$ ut solarum $q_1, q_2, \dots q_m$ functiones spectamus, unde pro ipsa $p_m$ perinde scribi potest $\frac{\partial p_m}{\partial q_i}$ sive $\left(\frac{\partial p_m}{\partial q_i}\right)$.

Posito $k = m-1$, fit:

$$0 = -\frac{\partial p_{m-1}}{\partial q_i} + \frac{\partial p_i}{\partial p_{i+1}}\frac{\partial p_{m-1}}{\partial q_{i+1}} + \frac{\partial p_i}{\partial p_{i+2}}\frac{\partial p_{m-1}}{\partial q_{i+2}} + \dots + \frac{\partial p_i}{\partial p_m}\frac{\partial p_{m-1}}{\partial q_m}$$
$$+\frac{\partial p_i}{\partial q_{m-1}} - \frac{\partial p_{m-1}}{\partial p_m}\frac{\partial p_i}{\partial q_m}.$$

Cui aequationi si addimus aequationem (1.) multiplicatam per $\frac{\partial p_{m-1}}{\partial p_m}$, prodit:

$$(2.)\quad 0 = -\left(\frac{\partial p_{m-1}}{\partial q_i}\right) + \frac{\partial p_i}{\partial p_{i+1}}\left(\frac{\partial p_{m-1}}{\partial q_{i+1}}\right) + \frac{\partial p_i}{\partial p_{i+2}}\left(\frac{\partial p_{m-1}}{\partial q_{i+2}}\right) + \dots + \frac{\partial p_i}{\partial p_m}\left(\frac{\partial p_{m-1}}{\partial q_m}\right)$$
$$+\frac{\partial p_i}{\partial q_{m-1}}.$$

Posito $k = m-2$, fit

$$0 = -\frac{\partial p_{m-2}}{\partial q_i} + \frac{\partial p_i}{\partial p_{i+1}}\frac{\partial p_{m-2}}{\partial q_{i+1}} + \frac{\partial p_i}{\partial p_{i+2}}\frac{\partial p_{m-2}}{\partial q_{i+2}} + \dots + \frac{\partial p_i}{\partial p_m}\frac{\partial p_{m-2}}{\partial q_m}$$
$$+\frac{\partial p_i}{\partial q_{m-2}} - \frac{\partial p_{m-2}}{\partial p_{m-1}}\frac{\partial p_i}{\partial q_{m-1}} - \frac{\partial p_{m-2}}{\partial p_m}\frac{\partial p_i}{\partial q_m}.$$

Cui aequationi addo aequationem (1.) per $\frac{\partial p_{m-2}}{\partial p_m}$ et aequationem (2.) per $\frac{\partial p_{m-2}}{\partial p_{m-1}}$ multiplicatam, prodit:

$$(3.)\quad 0 = -\left(\frac{\partial p_{m-2}}{\partial q_i}\right)+\frac{\partial p_i}{\partial p_{i+1}}\left(\frac{\partial p_{m-2}}{\partial q_{i+1}}\right)+\frac{\partial p_i}{\partial p_{i+2}}\left(\frac{\partial p_{m-2}}{\partial q_{i+2}}\right)+\cdots+\frac{\partial p_i}{\partial p_m}\left(\frac{\partial p_{m-2}}{\partial q_m}\right) + \frac{\partial p_i}{\partial q_{m-2}}.$$

Et ita continuando, demonstras aequationem *generalem:*

$$(b.)\quad 0 = -\left(\frac{\partial p_k}{\partial q_i}\right)+\frac{\partial p_i}{\partial p_{i+1}}\left(\frac{\partial p_k}{\partial q_{i+1}}\right)+\frac{\partial p_i}{\partial p_{i+2}}\left(\frac{\partial p_k}{\partial q_{i+2}}\right)+\cdots+\frac{\partial p_i}{\partial p_m}\left(\frac{\partial p_k}{\partial q_m}\right) + \frac{\partial p_i}{\partial q_k},$$

in qua $k$ valores omnes induere potest $m$, $m-1$, $m-2$, ... usque ad $i+1$. Unde, si ipsi $i$ rursus valores 1, 2, 3, ... $m-1$ tribuuntur, numerus aequationum (*b.*) fit $\frac{m(m-1)}{2}$.

*Forma tertia quae est usitata.*

8.

Ex aequationibus (*a.*) theorematis I. deduxi totidem aequationes (*b.*). Iam ex his deducam aequationes

$$(c.)\quad \left(\frac{\partial p_i}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_i}\right)$$

quarum idem est numerus.

Supponam, pro omnibus numeris $i'$ et $k$, qui dato numero $i$ maiores, ipso $m$ non majores sunt, iam probatas esse aequationes:

$$\left(\frac{\partial p_{i'}}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_{i'}}\right).$$

Quarum ope aequatio (*b.*) transformari potest in hanc:

$$0 = -\left(\frac{\partial p_k}{\partial q_i}\right)+\frac{\partial p_i}{\partial p_{i+1}}\left(\frac{\partial p_{i+1}}{\partial q_k}\right)+\frac{\partial p_i}{\partial p_{i+2}}\left(\frac{\partial p_{i+2}}{\partial q_k}\right)+\cdots+\frac{\partial p_i}{\partial p_m}\left(\frac{\partial p_m}{\partial q_k}\right) + \frac{\partial p_i}{\partial q_k},$$

quae eadem est atque haec:

$$0 = -\left(\frac{\partial p_k}{\partial q_i}\right)+\left(\frac{\partial p_i}{\partial q_k}\right).$$

In qua, si placet, etiam $k=i$ ponere licet, quippe quo casu identica fit.

*Valentibus igitur aequationibus (b.), si aequatio*

$$\left(\frac{\partial p_{i'}}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_{i'}}\right)$$

*comprobata est pro omnibus ipsorum $i'$ et $k$ valoribus $i+1$, $i+2$, ... $m$, eadem valebit pro omnibus ipsorum $i'$ et $k$ valoribus $i$, $i+1$, ... $m$.*

Si ponitur $i = m-1$, in aequatione $(b.)$ ipsi $k$ unicus convenit valor $k = m$, unde fit illa:

$$0 = -\left(\frac{\partial p_m}{\partial q_{m-1}}\right) + \frac{\partial p_{m-1}}{\partial p_m}\left(\frac{\partial p_m}{\partial q_m}\right) + \frac{\partial p_{m-1}}{\partial q_m},$$

sive

$$0 = -\left(\frac{\partial p_m}{\partial q_{m-1}}\right) + \left(\frac{\partial p_{m-1}}{\partial q_m}\right).$$

Valent igitur aequationes $(c.)$, in quibus $k > i$ statuatur, si $i = m-1$. Unde ex antecedentibus valebunt etiam, si $i = m-2$; unde ex antecedentibus valebunt etiam, si $i = m-3$, et ita porro; sive valebunt aequationes $(c.)$ pro omnibus ipsius $i$ valoribus $m-1$, $m-2$, $m-3$, ... 2, 1. Q. D. E. Comprobatis aequationibus $(c.)$, sequitur, expressionem

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$$

differentiale completum esse.

Systema conditionum $\frac{m(m-1)}{2}$, quibus satisfieri debet, ut expressio praecedens differentiale completum evadat, sub tribus formis $(a.)$, $(b.)$, $(c.)$ exhibui. Quarum forma $(a.)$ ad solvendum problema propositum sive ad determinandas functiones $p_1$, $p_2$, ... $p_m$ quae expressionem illam differentiale completum efficiant, prae ceteris idonea est.

### *De integrationibus quas e forma prima conditionum integrabilitatis solutio problematis propositi postulet.*

9.

His praeparatis, iam integrationes transigendae accuratius describi possunt. Redit enim problema in determinationem functionum $p_1$, $p_2$, ... $p_m$, quae aequationibus $(a.)$ satisfaciant. Ipsa quidem $p_1$ ut functio ipsarum $p_2$, $p_3$, ... $p_m$, $q_1$, $q_2$, ... $q_m$ per aequationem differentialem partialem propositam data est. Deinde ponendo in $(a.)$ $i = 1$, $k = 2$, determinatur $p_2$ ut functio ipsarum $p_3$, $p_4$, ... $p_m$, $q_1$, $q_2$, ... $q_m$ per aequationem:

$$(1.)\quad \frac{\partial p_1}{\partial q_2} = \frac{\partial p_2}{\partial q_1} - \frac{\partial p_1}{\partial p_2}\frac{\partial p_2}{\partial q_2} - \frac{\partial p_1}{\partial p_3}\frac{\partial p_2}{\partial q_3} - \frac{\partial p_1}{\partial p_4}\frac{\partial p_2}{\partial q_4} - \dots - \frac{\partial p_1}{\partial p_m}\frac{\partial p_2}{\partial q_m}$$
$$+ \frac{\partial p_1}{\partial q_3}\frac{\partial p_2}{\partial p_3} + \frac{\partial p_1}{\partial q_4}\frac{\partial p_2}{\partial p_4} + \dots + \frac{\partial p_1}{\partial q_m}\frac{\partial p_2}{\partial p_m}.$$

Quae est aequatio differentialis partialis *linearis*, cujus nota est integratio. Inventa functione $p_2$, quaecunque aequationi praecedenti satisfacit, ponamus in aequationibus $(a.)$: $i = 1, 2$ atque $k = 3$, prodeunt aequationes:

$$(2.)\quad \begin{cases} \dfrac{\partial p_1}{\partial q_3} = \dfrac{\partial p_3}{\partial q_1} - \dfrac{\partial p_1}{\partial p_2}\dfrac{\partial p_3}{\partial q_2} - \dfrac{\partial p_1}{\partial p_3}\dfrac{\partial p_3}{\partial q_3} - \dots - \dfrac{\partial p_1}{\partial p_m}\dfrac{\partial p_3}{\partial q_m} \\ \qquad + \dfrac{\partial p_1}{\partial q_4}\dfrac{\partial p_3}{\partial p_4} + \dfrac{\partial p_1}{\partial q_5}\dfrac{\partial p_3}{\partial p_5} + \dots + \dfrac{\partial p_1}{\partial q_m}\dfrac{\partial p_3}{\partial p_m}, \\ \dfrac{\partial p_2}{\partial q_3} = \dfrac{\partial p_3}{\partial q_2} - \dfrac{\partial p_2}{\partial p_3}\dfrac{\partial p_3}{\partial q_3} - \dfrac{\partial p_2}{\partial p_4}\dfrac{\partial p_3}{\partial q_4} - \dots - \dfrac{\partial p_2}{\partial p_m}\dfrac{\partial p_3}{\partial q_m} \\ \qquad + \dfrac{\partial p_2}{\partial q_4}\dfrac{\partial p_3}{\partial p_4} + \dfrac{\partial p_2}{\partial q_5}\dfrac{\partial p_3}{\partial p_5} + \dots + \dfrac{\partial p_2}{\partial q_m}\dfrac{\partial p_3}{\partial p_m}. \end{cases}$$

Si functionem $p_1$ non ut ipsarum $p_2, p_3, \dots p_m, q_1, q_2, \dots q_m$, sed substituto ipsius $p_2$ valore per integrationem aequationis (1.) invento, sicuti ipsam $p_2$, ut functionem ipsarum $p_3, p_4, \dots p_m, q_1, q_2, \dots q_m$ considerare placet, multiplicetur aequatio posterior per $\frac{\partial p_1}{\partial p_2}$ et priori addatur. Quo facto obtines, *si $p_1$ et $p_2$ ut functiones quantitatum reliquarum spectantur:*

$$(2.^*)\quad \begin{cases} \dfrac{\partial p_1}{\partial q_3} = \dfrac{\partial p_3}{\partial q_1} - \dfrac{\partial p_1}{\partial p_3}\dfrac{\partial p_3}{\partial q_3} - \dfrac{\partial p_1}{\partial p_4}\dfrac{\partial p_3}{\partial q_4} - \dots - \dfrac{\partial p_1}{\partial p_m}\dfrac{\partial p_3}{\partial q_m} \\ \qquad + \dfrac{\partial p_1}{\partial q_4}\dfrac{\partial p_3}{\partial p_4} + \dfrac{\partial p_1}{\partial q_5}\dfrac{\partial p_3}{\partial p_5} + \dots + \dfrac{\partial p_1}{\partial q_m}\dfrac{\partial p_3}{\partial p_m}, \\ \dfrac{\partial p_2}{\partial q_3} = \dfrac{\partial p_3}{\partial q_2} - \dfrac{\partial p_2}{\partial p_3}\dfrac{\partial p_3}{\partial q_3} - \dfrac{\partial p_2}{\partial p_4}\dfrac{\partial p_3}{\partial q_4} - \dots - \dfrac{\partial p_2}{\partial p_m}\dfrac{\partial p_3}{\partial q_m} \\ \qquad + \dfrac{\partial p_2}{\partial q_4}\dfrac{\partial p_3}{\partial p_4} + \dfrac{\partial p_2}{\partial q_5}\dfrac{\partial p_3}{\partial p_5} + \dots + \dfrac{\partial p_2}{\partial q_m}\dfrac{\partial p_3}{\partial p_m}. \end{cases}$$

Quarum aequationum altera ex altera invenitur indices 1 atque 2 inter se permutando. Binis aequationibus (2.) sive (2.*) ipsa $p_3$ ut functio quantitatum $p_4, p_5, \dots p_m, q_1, q_2, \dots q_m$ determinanda est.

## 10.

Inventa per aequationum praecedentium integrationem etiam functione $p_3$, ponatur in (*a.*) $i=1$, 2, 3, atque $k=4$, prodeunt aequationes tres sequentes:

$$
(3.)\quad\begin{cases}
\dfrac{\partial p_1}{\partial q_4} = \dfrac{\partial p_4}{\partial q_1} - \dfrac{\partial p_1}{\partial p_2}\dfrac{\partial p_4}{\partial q_2} - \dfrac{\partial p_1}{\partial p_3}\dfrac{\partial p_4}{\partial q_3} - \cdots - \dfrac{\partial p_1}{\partial p_m}\dfrac{\partial p_4}{\partial q_m} \\
\qquad + \dfrac{\partial p_1}{\partial q_5}\dfrac{\partial p_4}{\partial p_5} + \dfrac{\partial p_1}{\partial q_6}\dfrac{\partial p_4}{\partial p_6} + \cdots + \dfrac{\partial p_1}{\partial q_m}\dfrac{\partial p_4}{\partial p_m}, \\
\dfrac{\partial p_2}{\partial q_4} = \dfrac{\partial p_4}{\partial q_2} - \dfrac{\partial p_2}{\partial p_3}\dfrac{\partial p_4}{\partial q_3} - \dfrac{\partial p_2}{\partial p_4}\dfrac{\partial p_4}{\partial q_4} - \cdots - \dfrac{\partial p_2}{\partial p_m}\dfrac{\partial p_4}{\partial q_m} \\
\qquad + \dfrac{\partial p_2}{\partial q_5}\dfrac{\partial p_4}{\partial p_5} + \dfrac{\partial p_2}{\partial q_6}\dfrac{\partial p_4}{\partial p_6} + \cdots + \dfrac{\partial p_2}{\partial q_m}\dfrac{\partial p_4}{\partial p_m}, \\
\dfrac{\partial p_3}{\partial q_4} = \dfrac{\partial p_4}{\partial q_3} - \dfrac{\partial p_3}{\partial p_4}\dfrac{\partial p_4}{\partial q_4} - \dfrac{\partial p_3}{\partial p_5}\dfrac{\partial p_4}{\partial q_5} - \cdots - \dfrac{\partial p_3}{\partial p_m}\dfrac{\partial p_4}{\partial q_m} \\
\qquad + \dfrac{\partial p_3}{\partial q_5}\dfrac{\partial p_4}{\partial p_5} + \dfrac{\partial p_3}{\partial q_6}\dfrac{\partial p_4}{\partial p_6} + \cdots + \dfrac{\partial p_3}{\partial q_m}\dfrac{\partial p_4}{\partial p_m}.
\end{cases}
$$

Si substitutis ipsarum $p_2$ et $p_3$ expressionibus per integrationes iam transactas inventis, omnes tres $p_1$, $p_2$, $p_3$ ut solarum $p_4$, $p_5$, ... $p_m$, $q_1$, $q_2$, ... $q_m$ considerare et ad hanc suppositionem differentiationes per partes referre placet, primum aequatio tertia per $\frac{\partial p_2}{\partial p_3}$ multiplicata addatur secundae, prodit:

$$
\frac{\partial p_2}{\partial q_4} = \frac{\partial p_4}{\partial q_2} - \frac{\partial p_2}{\partial p_4}\frac{\partial p_4}{\partial q_4} - \frac{\partial p_2}{\partial p_5}\frac{\partial p_4}{\partial q_5} - \cdots - \frac{\partial p_2}{\partial p_m}\frac{\partial p_4}{\partial q_m}
+ \frac{\partial p_2}{\partial q_5}\frac{\partial p_4}{\partial p_5} + \cdots + \frac{\partial p_2}{\partial q_m}\frac{\partial p_4}{\partial p_m}.
$$

Haec aequatio multiplicata per $\frac{\partial p_1}{\partial p_2}$ et tertia aequationum (3.) multiplicata per $\frac{\partial p_1}{\partial p_3}$ addatur primae, prodit:

$$
\frac{\partial p_1}{\partial q_4} = \frac{\partial p_4}{\partial q_1} - \frac{\partial p_1}{\partial p_4}\frac{\partial p_4}{\partial q_4} - \frac{\partial p_1}{\partial p_5}\frac{\partial p_4}{\partial q_5} - \cdots - \frac{\partial p_1}{\partial p_m}\frac{\partial p_4}{\partial q_m}
+ \frac{\partial p_1}{\partial q_5}\frac{\partial p_4}{\partial p_5} + \cdots + \frac{\partial p_1}{\partial q_m}\frac{\partial p_4}{\partial p_m}.
$$

Determinanda igitur est $p_4$ ut functio ipsarum $p_5$, $p_6$, ... $p_m$, $q_1$, $q_2$, ... $q_m$, quae simul tribus sequentibus aequationibus satisfaciat, *in quibus* $p_1$, $p_2$, $p_3$ *sunt functiones ipsarum* $p_4$, $p_5$, ... $p_m$, $q_1$, $q_2$, ... $q_m$, quales per integrationes praecedentes determinatae sunt:

$$(3.^*)\quad \begin{cases} \dfrac{\partial p_1}{\partial q_4} = \dfrac{\partial p_4}{\partial q_1} - \dfrac{\partial p_1}{\partial p_4}\dfrac{\partial p_4}{\partial q_4} - \dfrac{\partial p_1}{\partial p_5}\dfrac{\partial p_4}{\partial q_5} - \cdots - \dfrac{\partial p_1}{\partial p_m}\dfrac{\partial p_4}{\partial q_m} \\ \qquad\qquad + \dfrac{\partial p_1}{\partial q_5}\dfrac{\partial p_4}{\partial p_5} + \cdots + \dfrac{\partial p_1}{\partial q_m}\dfrac{\partial p_4}{\partial p_m}, \\ \dfrac{\partial p_2}{\partial q_4} = \dfrac{\partial p_4}{\partial q_2} - \dfrac{\partial p_2}{\partial p_4}\dfrac{\partial p_4}{\partial q_4} - \dfrac{\partial p_2}{\partial p_5}\dfrac{\partial p_4}{\partial q_5} - \cdots - \dfrac{\partial p_2}{\partial p_m}\dfrac{\partial p_4}{\partial q_m} \\ \qquad\qquad + \dfrac{\partial p_2}{\partial q_5}\dfrac{\partial p_4}{\partial p_5} + \cdots + \dfrac{\partial p_2}{\partial q_m}\dfrac{\partial p_4}{\partial p_m}, \\ \dfrac{\partial p_3}{\partial q_4} = \dfrac{\partial p_4}{\partial q_3} - \dfrac{\partial p_3}{\partial p_4}\dfrac{\partial p_4}{\partial q_4} - \dfrac{\partial p_3}{\partial p_5}\dfrac{\partial p_4}{\partial q_5} - \cdots - \dfrac{\partial p_3}{\partial p_m}\dfrac{\partial p_4}{\partial q_m} \\ \qquad\qquad + \dfrac{\partial p_3}{\partial q_5}\dfrac{\partial p_4}{\partial p_5} + \cdots + \dfrac{\partial p_3}{\partial q_m}\dfrac{\partial p_4}{\partial p_m}. \end{cases}$$

Quae aequationes tres plane similes sunt et commutando indices 1, 2, 3, aliae ex aliis obtinentur.

*Aequationes differentiales partiales lineares simultaneae, quibus ad singulas quantitates p eruendas satisfieri oportet; quae formam quartam conditionum integrabilitatis constituunt.*

## 11.

Sic pergendo, determinatis $p_1, p_2, \ldots p_i$, ut functionibus ipsarum $p_{i+1}, p_{i+2}, \ldots p_m, q_1, q_2, \ldots q_m$, generaliter determinanda erit $p_{i+1}$ ut functio ipsarum $p_{i+2}, p_{i+3}, \ldots p_m, q_1, q_2, \ldots q_m$ per aequationes sequentes, quae sunt numero $i$:

$$(\alpha.)\quad \begin{cases} \dfrac{\partial p_1}{\partial q_{i+1}} = \dfrac{\partial p_{i+1}}{\partial q_1} - \dfrac{\partial p_1}{\partial p_{i+1}}\dfrac{\partial p_{i+1}}{\partial q_{i+1}} - \dfrac{\partial p_1}{\partial p_{i+2}}\dfrac{\partial p_{i+1}}{\partial q_{i+2}} - \cdots - \dfrac{\partial p_1}{\partial p_m}\dfrac{\partial p_{i+1}}{\partial q_m} \\ \qquad\qquad + \dfrac{\partial p_1}{\partial q_{i+2}}\dfrac{\partial p_{i+1}}{\partial p_{i+2}} + \cdots + \dfrac{\partial p_1}{\partial q_m}\dfrac{\partial p_{i+1}}{\partial p_m}, \\ \dfrac{\partial p_2}{\partial q_{i+1}} = \dfrac{\partial p_{i+1}}{\partial q_2} - \dfrac{\partial p_2}{\partial p_{i+1}}\dfrac{\partial p_{i+1}}{\partial q_{i+1}} - \dfrac{\partial p_2}{\partial p_{i+2}}\dfrac{\partial p_{i+1}}{\partial q_{i+2}} - \cdots - \dfrac{\partial p_2}{\partial p_m}\dfrac{\partial p_{i+1}}{\partial q_m} \\ \qquad\qquad + \dfrac{\partial p_2}{\partial q_{i+2}}\dfrac{\partial p_{i+1}}{\partial p_{i+2}} + \cdots + \dfrac{\partial p_2}{\partial q_m}\dfrac{\partial p_{i+1}}{\partial p_m}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial p_i}{\partial q_{i+1}} = \dfrac{\partial p_{i+1}}{\partial q_i} - \dfrac{\partial p_i}{\partial p_{i+1}}\dfrac{\partial p_{i+1}}{\partial q_{i+1}} - \dfrac{\partial p_i}{\partial p_{i+2}}\dfrac{\partial p_{i+1}}{\partial q_{i+2}} - \cdots - \dfrac{\partial p_i}{\partial p_m}\dfrac{\partial p_{i+1}}{\partial q_m} \\ \qquad\qquad + \dfrac{\partial p_i}{\partial q_{i+2}}\dfrac{\partial p_{i+1}}{\partial p_{i+2}} + \cdots + \dfrac{\partial p_i}{\partial q_m}\dfrac{\partial p_{i+1}}{\partial p_m}. \end{cases}$$

Aequationes ($\alpha$.) constituunt formam *quartam*, qua exhiberi possunt conditiones integrabilitatis expressionis $p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$. E qua forma haec colligis. Data $p_1$ ut functione reliquarum quantitatum per ipsam aequationem differentialem partialem propositam, invenitur $p_2$ per integrationem unius aequationis differentialis partialis linearis inter $2m-1$ variabiles; deinde $p_3$ satisfacere debet simul duabus aequationibus differentialibus partialibus linearibus, quae singulae sunt inter $2m-3$ variabiles; deinde $p_4$ satisfacere debet simul tribus aequationibus differentialibus partialibus linearibus quae singulae sunt inter $2m-5$ variabiles, et ita porro. *Ac generaliter, inventis ipsarum* $p_1$, $p_2$, ... $p_i$ *expressionibus per quantitates* $p_{i+1}$, $p_{i+2}$, ... $p_m$, $q_1$, $q_2$, ... $q_m$ *determinatur* $p_{i+1}$ *per* $i$ *aequationes differentiales partiales lineares, quibus singulis satisfacere debet et quae singulae sunt inter* $2m-2i+1$ *variabiles.* Numerum igitur variabilium in investigatione cujusque insequentis functionis *duabus unitatibus* minui videmus; numerus quidem aequationum, quibus simul satisfacere debet functio quaesita, pro quaque insequente functione unitate crescit, sed hanc integrationem simultaneam, a qua abhorruisse videntur Analystae, non tantis difficultatibus impeditam esse infra patebit. Attamen antequam ipsam aggrediar integrationem istam simultaneam, conditiones integrabilitatis sub aliis adhuc formis exhibebo.

### *Theorema de forma conditionum integrabilitatis maxime generali.*

### 12.

Si loco $i+1$ scribimus $k$ atque per $i$ numerum quemlibet ipso $k$ minorem denotamus, aequationes ($\alpha$.) sic repraesentare licet:

$$(\alpha.)\quad \begin{cases} 0 = \dfrac{\partial p_i}{\partial q_k} + \dfrac{\partial p_i}{\partial p_k}\dfrac{\partial p_k}{\partial q_k} + \dfrac{\partial p_i}{\partial p_{k+1}}\dfrac{\partial p_k}{\partial q_{k+1}} + \dfrac{\partial p_i}{\partial p_{k+2}}\dfrac{\partial p_k}{\partial q_{k+2}} + \cdots + \dfrac{\partial p_i}{\partial p_m}\dfrac{\partial p_k}{\partial q_m} \\ \qquad - \dfrac{\partial p_k}{\partial q_i} \qquad - \dfrac{\partial p_i}{\partial q_{k+1}}\dfrac{\partial p_k}{\partial p_{k+1}} - \dfrac{\partial p_i}{\partial q_{k+2}}\dfrac{\partial p_k}{\partial p_{k+2}} - \cdots - \dfrac{\partial p_i}{\partial q_m}\dfrac{\partial p_k}{\partial p_m}. \end{cases}$$

In hac aequatione est $p_k$ functio ipsarum $p_{k+1}$, $p_{k+2}$, ... $p_m$, $q_1$, $q_2$, ... $q_m$; functio autem $p_i$ praeter has quantitates etiam ipsam $p_k$ continet. Iam vero patet, expressionem

$$\frac{\partial p_i}{\partial p_{k'}}\frac{\partial p_k}{\partial q_{k'}} - \frac{\partial p_i}{\partial q_{k'}}\frac{\partial p_k}{\partial p_{k'}}$$

eandem manere, sive in formandis $\dfrac{\partial p_i}{\partial p_{k'}}$, $\dfrac{\partial p_i}{\partial q_{k'}}$ differentietur etiam quatenus $p_{k'}$, $q_{k'}$ a $p_k$ implicantur, sive tantum, quod in aequatione praecedente suppositum est, quatenus in $p_i$ explicite praeter $p_k$ inveniuntur. Priori casu enim accederent termini se invicem destruentes:

$$\frac{\partial p_i}{\partial p_k}\left\{\frac{\partial p_k}{\partial p_{k'}}\frac{\partial p_k}{\partial q_{k'}}-\frac{\partial p_k}{\partial q_{k'}}\frac{\partial p_k}{\partial p_{k'}}\right\}.$$

Praeterea, si ipsa $p_i$ differentiatur secundum $q_k$ etiam quatenus $q_k$ implicatur ab ipsa $p_k$, quae in expressione ipsius $p_i$ invenitur, scribere licet $\frac{\partial p_i}{\partial q_k}$ loco $\frac{\partial p_i}{\partial q_k}+\frac{\partial p_i}{\partial p_k}\frac{\partial p_k}{\partial q_k}$. *Unde aequationem praecedentem, si et $p_i$ et $p_k$ tamquam solarum $p_{k+1}$, $p_{k+2}$, ... $p_m$, $q_1$, $q_2$, ... $q_m$ functiones consideras, sic exhibere licet:*

$$(\beta.)\quad \frac{\partial p_k}{\partial q_i}-\frac{\partial p_i}{\partial q_k} =$$

$$\frac{\partial p_i}{\partial p_{k+1}}\frac{\partial p_k}{\partial q_{k+1}}+\frac{\partial p_i}{\partial p_{k+2}}\frac{\partial p_k}{\partial q_{k+2}}+\cdots+\frac{\partial p_i}{\partial p_m}\frac{\partial p_k}{\partial q_m}$$

$$-\frac{\partial p_i}{\partial q_{k+1}}\frac{\partial p_k}{\partial p_{k+1}}-\frac{\partial p_i}{\partial q_{k+2}}\frac{\partial p_k}{\partial p_{k+2}}-\cdots-\frac{\partial p_i}{\partial q_m}\frac{\partial p_k}{\partial p_m}.$$

Ordo, in quem variabiles $q$ et quae iis respondent $p$ disposuimus, indicibus subscriptis indicatus, prorsus arbitrarius est. Qua de re in formula praecedente $(\beta.)$ variabiles $q_i$, $q_k$ binae quaelibet esse possunt e numero variabilium $q$, et $q_{k+1}$, $q_{k+2}$, ... $q_m$ aliae quaelibet harum variabilium ab illis duabus diversae et cuiuslibet numeri, qui tamen numerum $m-2$ superare non potest. Statuendae autem sunt a $q_i$, $q_k$ diversae, quum in formula $(\beta.)$ suppositum sit $i<k$ ideoque $i$ inter numeros $k+1$, $k+2$, ... $m$ non inveniatur. Habemus igitur theorema sequens:

### Theorema II.

*Sint $p_1$, $p_2$, ... $p_m$ eiusmodi functiones ipsarum $q_1$, $q_2$, ... $q_m$ ut expressio $p_1dq_1+p_2dq_2+\cdots+p_mdq_m$ sit differentiale completum; si binae quaelibet $p_i$ et $p_k$ exprimuntur praeter $q_1$, $q_2$, ... $q_m$ per alias quasdam e quantitatibus $p$ a $p_i$ et $p_k$ diversas, $p_\lambda$, $p_\mu$ etc. quotcunque placet, id quod infinitis modis licet, atque differentiationes per partes instituendae ad hanc repraesentationem referuntur, erit*

$$\frac{\partial p_k}{\partial q_i}-\frac{\partial p_i}{\partial q_k} = \frac{\partial p_i}{\partial p_\lambda}\frac{\partial p_k}{\partial q_\lambda}+\frac{\partial p_i}{\partial p_\mu}\frac{\partial p_k}{\partial q_\mu}+\cdots$$

$$-\frac{\partial p_i}{\partial q_\lambda}\frac{\partial p_k}{\partial p_\lambda}-\frac{\partial p_i}{\partial q_\mu}\frac{\partial p_k}{\partial p_\mu}-\cdots.$$

Neque necessarium est, ut in theoremate praecedente $p_i$ atque $p_k$ easdem aut eundem numerum quantitatum $p$ contineant; casus enim, quo functio datas quantitates continet, eum amplectitur, quo functio aliquas harum quantitatum vel omnes non involvit.

*Theorematis antecedentis demonstratio directa.*

13.

Theorema praecedens facile etiam directa via deducis ex aequationibus $\left(\frac{\partial p_i}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_i}\right)$. Primum enim probari potest, *in formula proposita expressionem ad dextram immutatam manere, si differentialia secundum $q_\lambda$, $q_\mu$, ... sumta uncis includantur, sive esse*

$$(1.)\quad \left\{\begin{array}{l} \frac{\partial p_i}{\partial p_\lambda}\frac{\partial p_k}{\partial q_\lambda}+\frac{\partial p_i}{\partial p_\mu}\frac{\partial p_k}{\partial q_\mu}+\cdots \\ -\frac{\partial p_k}{\partial p_\lambda}\frac{\partial p_i}{\partial q_\lambda}-\frac{\partial p_k}{\partial p_\mu}\frac{\partial p_i}{\partial q_\mu}-\cdots \end{array}\right\} = \left\{\begin{array}{l} \frac{\partial p_i}{\partial p_\lambda}\left(\frac{\partial p_k}{\partial q_\lambda}\right)+\frac{\partial p_i}{\partial p_\mu}\left(\frac{\partial p_k}{\partial q_\mu}\right)+\cdots \\ -\frac{\partial p_k}{\partial p_\lambda}\left(\frac{\partial p_i}{\partial q_\lambda}\right)-\frac{\partial p_k}{\partial p_\mu}\left(\frac{\partial p_i}{\partial q_\mu}\right)-\cdots \end{array}\right\}$$

Repraesentemus enim aequationis antecedentis dextram partem hoc modo:

$$\sum_\lambda \frac{\partial p_i}{\partial p_\lambda}\left(\frac{\partial p_k}{\partial q_\lambda}\right)-\sum_\lambda \frac{\partial p_k}{\partial p_\lambda}\left(\frac{\partial p_i}{\partial q_\lambda}\right),$$

subscripta $\lambda$ indicando, summam ad omnes valores $\lambda$, $\mu$, ... extendendam esse. Erit porro:

$$\left(\frac{\partial p_k}{\partial q_\lambda}\right) = \frac{\partial p_k}{\partial q_\lambda}+\sum_{\lambda'} \frac{\partial p_k}{\partial p_{\lambda'}}\left(\frac{\partial p_{\lambda'}}{\partial q_\lambda}\right),$$

$$\left(\frac{\partial p_i}{\partial q_\lambda}\right) = \frac{\partial p_i}{\partial q_\lambda}+\sum_{\lambda'} \frac{\partial p_i}{\partial p_{\lambda'}}\left(\frac{\partial p_{\lambda'}}{\partial q_\lambda}\right),$$

subscripta $\lambda'$ similiter summam indicando ad eosdem valores $\lambda$, $\mu$, ... extendi.

Hinc prodit

$$\begin{aligned} &\sum_\lambda \left\{\frac{\partial p_i}{\partial p_\lambda}\left(\frac{\partial p_k}{\partial q_\lambda}\right)-\frac{\partial p_k}{\partial p_\lambda}\left(\frac{\partial p_i}{\partial q_\lambda}\right)\right\}-\sum_\lambda\left\{\frac{\partial p_i}{\partial p_\lambda}\frac{\partial p_k}{\partial q_\lambda}-\frac{\partial p_k}{\partial p_\lambda}\frac{\partial p_i}{\partial q_\lambda}\right\} \\ &= \sum_\lambda\left\{\frac{\partial p_i}{\partial p_\lambda}\sum_{\lambda'}\frac{\partial p_k}{\partial p_{\lambda'}}\left(\frac{\partial p_{\lambda'}}{\partial q_\lambda}\right)-\frac{\partial p_k}{\partial p_\lambda}\sum_{\lambda'}\frac{\partial p_i}{\partial p_{\lambda'}}\left(\frac{\partial p_{\lambda'}}{\partial q_\lambda}\right)\right\} \\ &= \sum_\lambda\sum_{\lambda'}\frac{\partial p_i}{\partial p_\lambda}\frac{\partial p_k}{\partial p_{\lambda'}}\left(\frac{\partial p_{\lambda'}}{\partial q_\lambda}\right)-\sum_\lambda\sum_{\lambda'}\frac{\partial p_k}{\partial p_\lambda}\frac{\partial p_i}{\partial p_{\lambda'}}\left(\frac{\partial p_{\lambda'}}{\partial q_\lambda}\right). \end{aligned}$$

Indicibus $\lambda$ et $\lambda'$ quum omnino iidem valores conveniant, $\lambda$ et $\lambda'$ in duabus summis praecedentibus inter se commutare licet. Quod si in posteriore facimus, expressio antecedens fit:

$$\sum_\lambda \sum_{\lambda'} \frac{\partial p_i}{\partial p_\lambda} \frac{\partial p_k}{\partial p_{\lambda'}} \left\{ \left(\frac{\partial p_{\lambda'}}{\partial q_\lambda}\right) - \left(\frac{\partial p_\lambda}{\partial q_{\lambda'}}\right) \right\},$$

quae est expressio evanescens, quia

$$\left(\frac{\partial p_{\lambda'}}{\partial q_\lambda}\right) = \left(\frac{\partial p_\lambda}{\partial q_{\lambda'}}\right).$$

Unde aequatio (1.) comprobata est.

Iam ex aequatione (1.) sequitur:

$$\begin{aligned}
&\frac{\partial p_i}{\partial p_\lambda}\frac{\partial p_k}{\partial q_\lambda} + \frac{\partial p_i}{\partial p_\mu}\frac{\partial p_k}{\partial q_\mu} + \cdots \\
&- \frac{\partial p_k}{\partial p_\lambda}\frac{\partial p_i}{\partial q_\lambda} - \frac{\partial p_k}{\partial p_\mu}\frac{\partial p_i}{\partial q_\mu} - \cdots \\
&= \frac{\partial p_i}{\partial p_\lambda}\left(\frac{\partial p_\lambda}{\partial q_k}\right) + \frac{\partial p_i}{\partial p_\mu}\left(\frac{\partial p_\mu}{\partial q_k}\right) + \cdots \\
&- \frac{\partial p_k}{\partial p_\lambda}\left(\frac{\partial p_\lambda}{\partial q_i}\right) - \frac{\partial p_k}{\partial p_\mu}\left(\frac{\partial p_\mu}{\partial q_i}\right) - \cdots \\
&= \left(\frac{\partial p_i}{\partial q_k}\right) - \frac{\partial p_i}{\partial q_k} - \left(\frac{\partial p_k}{\partial q_i}\right) + \frac{\partial p_k}{\partial q_i} \\
&= -\left(\frac{\partial p_i}{\partial q_k}\right) + \left(\frac{\partial p_k}{\partial q_i}\right).
\end{aligned}$$

Q. D. E.

Si loco $k$ in formulis ($\beta$.) ponitur $i$, atque $\lambda$ loco $i$, patet e formulis illis sive e theor. II., in formulis ($\alpha$.) esse $p_1, p_2, \ldots p_i$ tales ipsorum $q_1, q_2, \ldots q_m$, $p_{i+1}, p_{i+2}, \ldots p_m$ functiones, ut inter binas earum $p_k$, $p_\lambda$ locum habeat aequatio:

$$\begin{aligned}
&\frac{\partial p_k}{\partial q_\lambda} - \frac{\partial p_\lambda}{\partial q_k} = \\
&\frac{\partial p_k}{\partial q_{i+1}}\frac{\partial p_\lambda}{\partial p_{i+1}} + \frac{\partial p_k}{\partial q_{i+2}}\frac{\partial p_\lambda}{\partial p_{i+2}} + \cdots + \frac{\partial p_k}{\partial q_m}\frac{\partial p_\lambda}{\partial p_m} \\
&- \frac{\partial p_k}{\partial p_{i+1}}\frac{\partial p_\lambda}{\partial q_{i+1}} - \frac{\partial p_k}{\partial p_{i+2}}\frac{\partial p_\lambda}{\partial q_{i+2}} - \cdots - \frac{\partial p_k}{\partial p_m}\frac{\partial p_\lambda}{\partial q_m}.
\end{aligned}$$

Haec est relatio, qua fit, sicuti infra videbimus, ut aequationes ($\alpha$.) simul integrari possint.

*Problema alio modo proponitur. Functiones, quibus constantibus aequiparatis ipsae $p_i$ per $q_1, q_2, \dots q_m$ exprimantur, aequationibus simultaneis $\frac{n(n-1)}{2}$ definiuntur.*

## 14.

Problema de integratione *completa* aequationis differentialis partialis inter $m+1$ variabiles $V, q_1, q_2, \dots q_m$, quae functionem quaesitam $V$ ipsam non continet, sic etiam proponi potest.

Sit $V$ ipsarum $q_1, q_2, \dots q_m$ functio $m$ constantes $h_1, h_2, \dots h_m$ involvens, quarum nulla per additionem tantum ei iuncta sit; sint $p_1, p_2, \dots p_m$ differentialia partialia ipsius $V$ respective secundum $q_1, q_2, \dots q_m$ sumta. Quae differentialia partialia, quum ipsas constantes quoque $h_1, h_2, \dots h_m$ involvant, vice versa aequari possunt $h_1, h_2, \dots h_m$ ipsarum $q_1, q_2, \dots q_m$, $p_1, p_2, \dots p_m$ functionibus. Sint aequationes sic inventae:

$$H_1 = h_1, \quad H_2 = h_2, \quad \dots \quad H_m = h_m,$$

designantibus $H_1, H_2, \dots H_m$ functiones ipsarum $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ a se invicem independentes et nullam constantium $h_1, h_2, \dots h_m$ involventes. *Quaeritur, data una harum aequationum, ex. gr.*

$$H_1 = h_1,$$

*indagare reliquas $m-1$.*

Investigemus aequationes conditionales *identicas*, quibus satisfacere debent functiones $H_1, H_2, \dots H_m$, ut ipsis $p_1, p_2, \dots p_m$ per $q_1, q_2, \dots q_m$ expressis ope aequationum

$$H_1 = h_1, \quad H_2 = h_2, \quad H_3 = h_3, \quad \dots \quad H_m = h_m,$$

expressio differentialis

$$p_1 dq_1 + p_2 dq_2 + \dots + p_m dq_m$$

sit differentiale completum $dV$.

Ponamus in theoremate II. loco indicum $i, k$ indices 1, 2, ac loco indicum $\lambda, \mu, \dots$ omnes reliquos 3, 4, 5, ... $m$; unde eruitur aequatio:

$$(1.)\quad \begin{cases} 0 = \dfrac{\partial p_1}{\partial q_2} - \dfrac{\partial p_2}{\partial q_1} + \dfrac{\partial p_1}{\partial p_3}\dfrac{\partial p_2}{\partial q_3} + \dfrac{\partial p_1}{\partial p_4}\dfrac{\partial p_2}{\partial q_4} + \dots + \dfrac{\partial p_1}{\partial p_m}\dfrac{\partial p_2}{\partial q_m} \\ \qquad - \dfrac{\partial p_2}{\partial p_3}\dfrac{\partial p_1}{\partial q_3} - \dfrac{\partial p_2}{\partial p_4}\dfrac{\partial p_1}{\partial q_4} - \dots - \dfrac{\partial p_2}{\partial p_m}\dfrac{\partial p_1}{\partial q_m}. \end{cases}$$

Sint

$$H_i = h_i, \quad H_k = h_k$$

duae quaelibet ex aequationibus propositis, quarum ope determinentur $p_1$ et $p_2$

ut functiones ipsarum $p_3, p_4, \dots p_m, q_1, q_2, \dots q_m$, quarum quantitatum ipsae $p_1$ et $p_2$ in aequatione praecedente functiones esse supponuntur. Sumtis deinde ipsarum $p_1$ et $p_2$ differentialibus partialibus secundum quantitates illas, substituantur in differentialibus illis, quae etiam constantes $h_i$ et $h_k$ involvunt, loco harum constantium functiones iis aequivalentes $H_i$ et $H_k$, unde emergent eorum valores per quantitates $p_1, p_2, \dots p_m, q_1, q_2, \dots q_m$ expressi absque ulla constanti $h$. Quos valores si in aequatione (1.) substituimus, aequatio illa evadere debet *identica,* cum nulla exstare possit aequatio inter quantitates $p_1, p_2, \dots p_m, q_1, q_2, \dots q_m$ a constantibus $h_1, h_2, \dots h_m$ prorsus libera nisi aequatio identica sit.

Ad eruendos valores differentialium partialium ipsarum $p_1$ et $p_2$, in aequatione (1.) substituendos, aequationes $H_i = h_i$, $H_k = h_k$ secundum $p_3, p_4, \dots p_m$, differentiemus. Sint $r$ et $t$ binae quaelibet harum quantitatum, erit

$$\begin{aligned}
\frac{\partial H_i}{\partial p_1}\frac{\partial p_1}{\partial r}+\frac{\partial H_i}{\partial p_2}\frac{\partial p_2}{\partial r} &= -\frac{\partial H_i}{\partial r},\\
\frac{\partial H_i}{\partial p_1}\frac{\partial p_1}{\partial t}+\frac{\partial H_i}{\partial p_2}\frac{\partial p_2}{\partial t} &= -\frac{\partial H_i}{\partial t},\\
\frac{\partial H_k}{\partial p_1}\frac{\partial p_1}{\partial r}+\frac{\partial H_k}{\partial p_2}\frac{\partial p_2}{\partial r} &= -\frac{\partial H_k}{\partial r},\\
\frac{\partial H_k}{\partial p_1}\frac{\partial p_1}{\partial t}+\frac{\partial H_k}{\partial p_2}\frac{\partial p_2}{\partial t} &= -\frac{\partial H_k}{\partial t}.
\end{aligned}$$

Unde, multiplicatis prima et quarta, secunda et tertia, subductisque productis, nanciscimur:

$$(2.)\quad \left\{\begin{aligned}
&\left(\frac{\partial H_i}{\partial p_1}\frac{\partial H_k}{\partial p_2}-\frac{\partial H_i}{\partial p_2}\frac{\partial H_k}{\partial p_1}\right)\left(\frac{\partial p_1}{\partial r}\frac{\partial p_2}{\partial t}-\frac{\partial p_2}{\partial r}\frac{\partial p_1}{\partial t}\right)\\
&\qquad = \frac{\partial H_i}{\partial r}\frac{\partial H_k}{\partial t}-\frac{\partial H_i}{\partial t}\frac{\partial H_k}{\partial r}.
\end{aligned}\right.$$

Porro ex aequatione prima et tertia sequitur:

$$(3.)\quad \left\{\begin{aligned}
\left(\frac{\partial H_i}{\partial p_1}\frac{\partial H_k}{\partial p_2}-\frac{\partial H_i}{\partial p_2}\frac{\partial H_k}{\partial p_1}\right)\frac{\partial p_1}{\partial r} &= \frac{\partial H_i}{\partial p_2}\frac{\partial H_k}{\partial r}-\frac{\partial H_i}{\partial r}\frac{\partial H_k}{\partial p_2}\\
-\left(\frac{\partial H_i}{\partial p_1}\frac{\partial H_k}{\partial p_2}-\frac{\partial H_i}{\partial p_2}\frac{\partial H_k}{\partial p_1}\right)\frac{\partial p_2}{\partial r} &= \frac{\partial H_i}{\partial p_1}\frac{\partial H_k}{\partial r}-\frac{\partial H_i}{\partial r}\frac{\partial H_k}{\partial p_1}.
\end{aligned}\right.$$

Multiplicemus aequationem (1.) per

$$\frac{\partial H_i}{\partial p_1}\frac{\partial H_k}{\partial p_2}-\frac{\partial H_i}{\partial p_2}\frac{\partial H_k}{\partial p_1}$$

ac ponamus in aequationibus (3.) $q_1$ et $q_2$ loco $r$, in aequatione (2.) $q_3, q_4, \dots q_m$

loco $r$, simulque respective $p_3, p_4, \dots p_m$ loco $t$. Quo facto ex aequatione (1.) prodit:

$$(\gamma.)\quad \begin{cases} \frac{\partial H_i}{\partial p_1}\frac{\partial H_k}{\partial q_1}+\frac{\partial H_i}{\partial p_2}\frac{\partial H_k}{\partial q_2}+\cdots+\frac{\partial H_i}{\partial p_m}\frac{\partial H_k}{\partial q_m} \\ -\frac{\partial H_i}{\partial q_1}\frac{\partial H_k}{\partial p_1}-\frac{\partial H_i}{\partial q_2}\frac{\partial H_k}{\partial p_2}-\cdots-\frac{\partial H_i}{\partial q_m}\frac{\partial H_k}{\partial p_m} = 0. \end{cases}$$

Quae est aequatio identica quaesita, a constantibus $h$ prorsus libera.

## 15.

Si in aequatione ($\gamma$.) indicibus $i$ et $k$ valores omnes tribuuntur, quos induere possunt, nanciscimur aequationes $\frac{m(m-1)}{2}$, quae et ipsae tamquam conditiones spectari possunt, ut expressio

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$$

integrabilis evadat. Habetur enim etiam theorema inversum:

### Theorema III.

*Sint* $H_1, H_2, \dots H_m$ *variabilium* $p_1, p_2, \dots p_m, q_1, q_2, \dots q_m$ *functiones quaecunque a se independentes, quarum binae quaelibet* $H_i$, $H_{i'}$ *satisfaciant aequationi:*

$$0 = \frac{\partial H_i}{\partial p_1}\frac{\partial H_{i'}}{\partial q_1}+\frac{\partial H_i}{\partial p_2}\frac{\partial H_{i'}}{\partial q_2}+\cdots+\frac{\partial H_i}{\partial p_m}\frac{\partial H_{i'}}{\partial q_m}$$
$$-\frac{\partial H_i}{\partial q_1}\frac{\partial H_{i'}}{\partial p_1}-\frac{\partial H_i}{\partial q_2}\frac{\partial H_{i'}}{\partial p_2}-\cdots-\frac{\partial H_i}{\partial q_m}\frac{\partial H_{i'}}{\partial p_m};$$

*si ex aequationibus*

$$H_1 = h_1, \quad H_2 = h_2, \quad \dots \quad H_m = h_m,$$

*in quibus* $h_1, h_2, \dots h_m$ *sunt constantes arbitrariae ipsas functiones* $H_1, H_2, \dots H_m$ *non afficientes, eruuntur ipsarum* $p_1, p_2, \dots p_m$ *valores per* $q_1, q_2, \dots q_m$ *expressi, expressio*

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$$

*differentiale completum fit.*

Quod est theorema gravissimum.

*Theorema antecedens de solutione problematis $\frac{n(n-1)}{2}$ aequationibus simultaneis definienda directa via confirmatur.*

16.

Demonstratio directa praecedentis theorematis haec sese offert. E differentiatione aequationis

$$H_i = h_i$$

secundum $q_{k'}$ facta sequitur, si subscripta $k$ signo summationis indicamus, summam ad valores 1, 2, ... $m$ ipsius $k$ extendi *):

$$\sum_k \frac{\partial H_i}{\partial p_k}\left(\frac{\partial p_k}{\partial q_{k'}}\right)+\frac{\partial H_i}{\partial q_{k'}} = 0.$$

Unde etiam multiplicatione per $\frac{\partial H_{i'}}{\partial p_{k'}}$ facta:

$$\sum_k \frac{\partial H_i}{\partial p_k}\frac{\partial H_{i'}}{\partial p_{k'}}\left(\frac{\partial p_k}{\partial q_{k'}}\right)+\frac{\partial H_{i'}}{\partial p_{k'}}\frac{\partial H_i}{\partial q_{k'}} = 0.$$

In qua aequatione loco $k'$ ponendo omnes ejus valores 1, 2, ... $m$, fit:

$$\sum_k \sum_{k'} \frac{\partial H_i}{\partial p_k}\frac{\partial H_{i'}}{\partial p_{k'}}\left(\frac{\partial p_k}{\partial q_{k'}}\right)+\sum_{k'}\frac{\partial H_{i'}}{\partial p_{k'}}\frac{\partial H_i}{\partial q_{k'}} = 0.$$

Unde etiam, permutando $H_i$ et $H_{i'}$,

$$\sum_k \sum_{k'} \frac{\partial H_i}{\partial p_{k'}}\frac{\partial H_{i'}}{\partial p_k}\left(\frac{\partial p_k}{\partial q_{k'}}\right)+\sum_{k'}\frac{\partial H_{i'}}{\partial q_{k'}}\frac{\partial H_i}{\partial p_{k'}} = 0.$$

Hanc aequationem detrahendo de antecedente, cum sit ex hypothesi:

$$\sum_{k'}\left\{\frac{\partial H_{i'}}{\partial p_{k'}}\frac{\partial H_i}{\partial q_{k'}}-\frac{\partial H_{i'}}{\partial q_{k'}}\frac{\partial H_i}{\partial p_{k'}}\right\} = 0,$$

eruimus:

$$\sum_k \sum_{k'}\left(\frac{\partial H_i}{\partial p_k}\frac{\partial H_{i'}}{\partial p_{k'}}-\frac{\partial H_i}{\partial p_{k'}}\frac{\partial H_{i'}}{\partial p_k}\right)\left(\frac{\partial p_k}{\partial q_{k'}}\right) = 0.$$

Permutando $k$ et $k'$, quippe quibus iidem valores 1, 2, ... $m$ conveniunt, expressionem ad laevam sic quoque scribere licet:

$$-\sum_k \sum_{k'}\left(\frac{\partial H_i}{\partial p_k}\frac{\partial H_{i'}}{\partial p_{k'}}-\frac{\partial H_i}{\partial p_{k'}}\frac{\partial H_{i'}}{\partial p_k}\right)\left(\frac{\partial p_{k'}}{\partial q_k}\right).$$

---

*) Simili notatione saepius in sequentibus utar, quoties sub signo summatorio plures indices inveniuntur, quorum alii constantes, alii, ut ita dicam, summantes sunt; maioris perspicuitatis causa posteriores signo summatorio subscribam.

Unde aequationem antecedentem hoc modo repraesentare possumus:

$$\Sigma\left(\frac{\partial H_i}{\partial p_k}\frac{\partial H_{i'}}{\partial p_{k'}}-\frac{\partial H_i}{\partial p_{k'}}\frac{\partial H_{i'}}{\partial p_k}\right)\left\{\left(\frac{\partial p_k}{\partial q_{k'}}\right)-\left(\frac{\partial p_{k'}}{\partial q_k}\right)\right\} = 0,$$

siquidem extenditur summa ad $\frac{m(m-1)}{2}$ combinationes numerorum 1, 2, ... $m$ pro ipsis $k$ et $k'$ ponendas, sive si ipsi $k$ sub signo summatorio valores 1, 2, ... $m-1$ tribuuntur, et pro singulis $k$ ipsis $k'$ valores $k+1$, $k+2$, ... $m$.

Si in aequatione praecedente pro ipsis $i$ et $i'$ bini quilibet e numeris 1, 2, ... $m$ ponuntur, eruuntur ex ea $\frac{m(m-1)}{2}$ aequationes. In quibus si quantitates

$$\left(\frac{\partial p_k}{\partial q_{k'}}\right)-\left(\frac{\partial p_{k'}}{\partial q_k}\right)$$

ut incognitas consideramus, sunt aequationes illae respectu harum incognitarum *lineares*, numerus incognitarum idem atque aequationum, et partes constantes omnes evanescentes. Unde ipsae quoque incognitae omnes evanescunt, sive pro quolibet ipsorum $k$ et $k'$ valore fit

$$\left(\frac{\partial p_k}{\partial q_{k'}}\right)-\left(\frac{\partial p_{k'}}{\partial q_k}\right) = 0.$$

Q. D. E.

Demonstratione antecedente etiam maxime directa via comprobari potuisset, si

$$\left(\frac{\partial p_k}{\partial q_{k'}}\right)-\left(\frac{\partial p_{k'}}{\partial q_k}\right) = 0,$$

sive si

$$p_1 dq_1+p_2 dq_2+\cdots+p_m dq_m$$

integrabilis sit, fieri

$$\begin{aligned}&\frac{\partial H_i}{\partial p_1}\frac{\partial H_{i'}}{\partial q_1}+\frac{\partial H_i}{\partial p_2}\frac{\partial H_{i'}}{\partial q_2}+\cdots+\frac{\partial H_i}{\partial p_m}\frac{\partial H_{i'}}{\partial q_m}\\ &-\frac{\partial H_i}{\partial q_1}\frac{\partial H_{i'}}{\partial p_1}-\frac{\partial H_i}{\partial q_2}\frac{\partial H_{i'}}{\partial p_2}-\cdots-\frac{\partial H_i}{\partial q_m}\frac{\partial H_{i'}}{\partial p_m} = 0.\end{aligned}$$

Ceterum, quod in demonstratione antecedente theorematis III adhuc desiderari potest, ut comprobetur e $\frac{m(m-1)}{2}$ aequationibus linearibus:

$$\sum_{k,k'}\left(\frac{\partial H_i}{\partial p_k}\frac{\partial H_{i'}}{\partial p_{k'}}-\frac{\partial H_i}{\partial p_{k'}}\frac{\partial H_{i'}}{\partial p_k}\right)x_{k,k'} = y_{i,i'}$$

in quibus quantitates $x_{k,k'}$ incognitas, quantitates $y_{i,i'}$ partes aequationum constantes designant, nullam e reliquis fluere facile variis modis probatur, quum adeo eiusmodi aequationes ex elementis algebraicis sine negotio generaliter resolvantur. Quae resolutio tum demum illusoria fit, si habetur

$$\Sigma \pm \frac{\partial H_1}{\partial p_1} \frac{\partial H_2}{\partial p_2} \cdots \frac{\partial H_m}{\partial p_m} = 0,$$

indicibus $1, 2, \ldots m$ sub signo summatiorio omnimodis permutatis signisque $\pm$ pro ratione nota alternantibus. Haec autem aequatio ipsa est conditio, ut inter quantitates $H_1, H_2, \ldots H_m, q_1, q_2, \ldots q_m$ aequatio locum habeat, ab ipsis $p$ prorsus libera; quod si foret, haberetur etiam inter ipsas $q_1, q_2, \ldots q_m$ et constantes arbitrarias relatio neque ex aequationibus

$$H_1 = h_1, \quad H_2 = h_2, \quad \ldots \quad H_m = h_m,$$

omnes $p_1, p_2, \ldots p_m$, quod supposuimus, ut functiones ipsarum $q_1, q_2, \ldots q_m$ determinari possent.

*Transformatio systematum aequationum quarum solutione simultanea secundum §. 11 singulae $p_i$ obtinentur.*

## §. 17.

Iam ipsam aggressuri integrationem revertamur ad formam aequationum conditionalium ($\alpha$.) §. 11. Difficultatem rei videmus consistere in invenienda functione quae simul numero $i$ aequationum differentialium partialium linearium satisfaciat. Sit $f$ functio ipsarum $p_{i+1}, p_{i+2}, p_{i+3}, \ldots p_m, q_1, q_2, \ldots q_m$, atque

$$f = a$$

aequatio, qua determinetur functio quaesita $p_{i+1}$ per $p_{i+2}, p_{i+3}, \ldots p_m$, $q_1, q_2, \ldots q_m$, designante $a$ constantem arbitrariam, quae ipsam $f$ non afficiat. Designantibus $p_n$ atque $q_n$ quaslibet e quantitatibus $p_{i+2}, p_{i+3}, \ldots p_m$ atque $q_1, q_2, \ldots q_m$ fit

$$\frac{\partial f}{\partial p_{i+1}} \frac{\partial p_{i+1}}{\partial p_n} = -\frac{\partial f}{\partial p_n},$$

$$\frac{\partial f}{\partial p_{i+1}} \frac{\partial p_{i+1}}{\partial q_n} = -\frac{\partial f}{\partial q_n}.$$

Unde aequationes ($\alpha$.) multiplicatae per $\frac{\partial f}{\partial p_{i+1}}$ in has abeunt:

$$
(d.)\quad\left\{\begin{aligned}
0 &= \frac{\partial f}{\partial q_1}+\frac{\partial p_1}{\partial q_{i+1}}\frac{\partial f}{\partial p_{i+1}}+\frac{\partial p_1}{\partial q_{i+2}}\frac{\partial f}{\partial p_{i+2}}+\cdots+\frac{\partial p_1}{\partial q_m}\frac{\partial f}{\partial p_m}\\
&\quad-\frac{\partial p_1}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}}-\frac{\partial p_1}{\partial p_{i+2}}\frac{\partial f}{\partial q_{i+2}}-\cdots-\frac{\partial p_1}{\partial p_m}\frac{\partial f}{\partial q_m},\\
0 &= \frac{\partial f}{\partial q_2}+\frac{\partial p_2}{\partial q_{i+1}}\frac{\partial f}{\partial p_{i+1}}+\frac{\partial p_2}{\partial q_{i+2}}\frac{\partial f}{\partial p_{i+2}}+\cdots+\frac{\partial p_2}{\partial q_m}\frac{\partial f}{\partial p_m}\\
&\quad-\frac{\partial p_2}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}}-\frac{\partial p_2}{\partial p_{i+2}}\frac{\partial f}{\partial q_{i+2}}-\cdots-\frac{\partial p_2}{\partial p_m}\frac{\partial f}{\partial q_m},\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
0 &= \frac{\partial f}{\partial q_i}+\frac{\partial p_i}{\partial q_{i+1}}\frac{\partial f}{\partial p_{i+1}}+\frac{\partial p_i}{\partial q_{i+2}}\frac{\partial f}{\partial p_{i+2}}+\cdots+\frac{\partial p_i}{\partial q_m}\frac{\partial f}{\partial p_m}\\
&\quad-\frac{\partial p_i}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}}-\frac{\partial p_i}{\partial p_{i+2}}\frac{\partial f}{\partial q_{i+2}}-\cdots-\frac{\partial p_i}{\partial p_m}\frac{\partial f}{\partial q_m}.
\end{aligned}\right.
$$

In his aequationibus considerantur $p_1$, $p_2$, ... $p_i$ tamquam functiones datae ipsarum $p_{i+1}$, $p_{i+2}$, ... $p_m$, $q_1$, $q_2$, ... $q_m$, inter quarum binas $p_k$ et $p_\lambda$ locum habet relatio, quam sub finem §. 13 apposui; et quaerenda est earundem quantitatum functio $f$ talis, quae aequationibus praecedentibus simul omnibus identice satisfaciat.

*Theorema affertur circa aequationum quae supra occurrunt integratione simultanea.*

## §. 18.

Non ego hic immorabor quaestioni generali, quando et quomodo duabus compluribusve aequationibus differentialibus partialibus una eademque functione satisfieri possit, sed ad casum propositum particularem investigationem restringam. Quippe quo praeclaris uti licet artificiis ad integrationem expediendam commodis. Maxime autem res absolvitur theoremate sequente:

### Theorema IV.

*Sint $k$, $\lambda$ quilibet diversi e numeris* 1, 2, ... $i$; *sit* $\varphi = f$ *integrale quodcunque unius ex aequationibus* (*d.*):

$$
\begin{aligned}
0 &= \frac{\partial f}{\partial q_k}+\frac{\partial p_k}{\partial q_{i+1}}\frac{\partial f}{\partial p_{i+1}}+\frac{\partial p_k}{\partial q_{i+2}}\frac{\partial f}{\partial p_{i+2}}+\cdots+\frac{\partial p_k}{\partial q_m}\frac{\partial f}{\partial p_m}\\
&\quad-\frac{\partial p_k}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}}-\frac{\partial p_k}{\partial p_{i+2}}\frac{\partial f}{\partial q_{i+2}}-\cdots-\frac{\partial p_k}{\partial p_m}\frac{\partial f}{\partial q_m},
\end{aligned}
$$

*erit expressio*

$$f = \frac{\partial\varphi}{\partial q_\lambda} + \frac{\partial p_\lambda}{\partial q_{i+1}}\frac{\partial\varphi}{\partial p_{i+1}} + \frac{\partial p_\lambda}{\partial q_{i+2}}\frac{\partial\varphi}{\partial p_{i+2}} + \dots + \frac{\partial p_\lambda}{\partial q_m}\frac{\partial\varphi}{\partial p_m}$$
$$- \frac{\partial p_\lambda}{\partial p_{i+1}}\frac{\partial\varphi}{\partial q_{i+1}} - \frac{\partial p_\lambda}{\partial p_{i+2}}\frac{\partial\varphi}{\partial q_{i+2}} - \dots - \frac{\partial p_\lambda}{\partial p_m}\frac{\partial\varphi}{\partial q_m}$$

*alterum ejusdem aequationis integrale.*

In hoc theoremate designant $p_k$, $p_\lambda$ ipsarum $q_k$, $q_\lambda$, $q_{i+1}$, $q_{i+2}$, ... $q_m$, $p_{i+1}$, $p_{i+2}$, ... $p_m$ functiones, quae satisfaciunt aequationi:

$$\frac{\partial p_k}{\partial q_\lambda} - \frac{\partial p_\lambda}{\partial q_k} =$$
$$\frac{\partial p_k}{\partial q_{i+1}}\frac{\partial p_\lambda}{\partial p_{i+1}} + \frac{\partial p_k}{\partial q_{i+2}}\frac{\partial p_\lambda}{\partial p_{i+2}} + \dots + \frac{\partial p_k}{\partial q_m}\frac{\partial p_\lambda}{\partial p_m}$$
$$- \frac{\partial p_k}{\partial p_{i+1}}\frac{\partial p_\lambda}{\partial q_{i+1}} - \frac{\partial p_k}{\partial p_{i+2}}\frac{\partial p_\lambda}{\partial q_{i+2}} - \dots - \frac{\partial p_k}{\partial p_m}\frac{\partial p_\lambda}{\partial q_m}.$$

Quae functiones si etiam alias praeter $q_k$ et $q_\lambda$ e quantitatibus $q_1$, $q_2$, ... $q_i$ involvunt, eae tamquam quantitates constantes considerantur.

*Quomodo ope theorematis antecedentis integratio simultanea succedat, ostenditur.*

## §. 19.

Ope theorematis praecedentis sic absolvitur integratio proposita. Sint $\varphi'_\lambda$, $\varphi''_\lambda$, $\varphi'''_\lambda$, etc. functiones, quae proveniunt ex expressione

$$\frac{\partial f}{\partial q_\lambda} + \frac{\partial p_\lambda}{\partial q_{i+1}}\frac{\partial f}{\partial p_{i+1}} + \frac{\partial p_\lambda}{\partial q_{i+2}}\frac{\partial f}{\partial p_{i+2}} + \dots + \frac{\partial p_\lambda}{\partial q_m}\frac{\partial f}{\partial p_m}$$
$$- \frac{\partial p_\lambda}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}} - \frac{\partial p_\lambda}{\partial p_{i+2}}\frac{\partial f}{\partial q_{i+2}} - \dots - \frac{\partial p_\lambda}{\partial p_m}\frac{\partial f}{\partial q_m},$$

ponendo loco $f$ successive functiones $\varphi$, $\varphi'_\lambda$, $\varphi''_\lambda$, ..., ita ut generaliter habeatur:

$$\varphi_\lambda^{(n)} = \frac{\partial\varphi_\lambda^{(n-1)}}{\partial q_\lambda} + \frac{\partial p_\lambda}{\partial q_{i+1}}\frac{\partial\varphi_\lambda^{(n-1)}}{\partial p_{i+1}} + \frac{\partial p_\lambda}{\partial q_{i+2}}\frac{\partial\varphi_\lambda^{(n-1)}}{\partial p_{i+2}} + \dots + \frac{\partial p_\lambda}{\partial q_m}\frac{\partial\varphi_\lambda^{(n-1)}}{\partial p_m}$$
$$- \frac{\partial p_\lambda}{\partial p_{i+1}}\frac{\partial\varphi_\lambda^{(n-1)}}{\partial q_{i+1}} - \frac{\partial p_\lambda}{\partial p_{i+2}}\frac{\partial\varphi_\lambda^{(n-1)}}{\partial q_{i+2}} - \dots - \frac{\partial p_\lambda}{\partial p_m}\frac{\partial\varphi_\lambda^{(n-1)}}{\partial q_m}.$$

Sit jam $\varphi = f$, integrale quodcunque aequationis

$$(1.)\quad \begin{cases} 0 = f'_1 = \dfrac{\partial f}{\partial q_1} + \dfrac{\partial p_1}{\partial q_{i+1}}\dfrac{\partial f}{\partial p_{i+1}} + \dfrac{\partial p_1}{\partial q_{i+2}}\dfrac{\partial f}{\partial p_{+2}} + \dots + \dfrac{\partial p_1}{\partial q_m}\dfrac{\partial f}{\partial p_m} \\ \qquad - \dfrac{\partial p_1}{\partial p_{i+1}}\dfrac{\partial f}{\partial q_{i+1}} - \dfrac{\partial p_1}{\partial p_{i+2}}\dfrac{\partial f}{\partial q_{i+2}} - \dots - \dfrac{\partial p_1}{\partial p_m}\dfrac{\partial f}{\partial q_m}, \end{cases}$$

erunt e theoremate IV etiam $\varphi_2'$, $\varphi_2''$, $\varphi_2'''$, cet. integralia ejusdem aequationis. Id quod patet, si in theoremate citato in locum ipsius $\varphi$ aliae post alias substituuntur functiones $\varphi_2'$, $\varphi_2''$, cet. Sed exstant tantummodo $2(m-i)$ integralia aequationis praecedentis a se invicem independentia et quorum aliud integrale quodvis functio esse debet, quam functionem praeterea etiam quantitates $q_2, q_3, \dots q_i$ tamquam constantes ingredi possunt. Sit igitur $\varphi_2^{(\mu)}$ *prima* functio, quae per antecedentes $\varphi$, $\varphi_2'$, $\varphi_2''$, ... $\varphi_2^{(\mu-1)}$ et ipsas $q_2$, $q_3$, ... $q_i$ exprimi potest, index $\mu$ numero $2(m-i)$ aut inferior aut certe non major. Statuatur, ipsam $\Pi$ esse functionem ipsarum $\varphi$, $\varphi_2'$, $\varphi_2''$, ... $\varphi_2^{(\mu-1)}$, $q_2$, $q_3$, ... $q_i$, erit etiam

$$f = \Pi$$

integrale aequationis (1.), quippe cujus integralia e theoremate IV. sunt $\varphi$, $\varphi_2'$, $\varphi_2''$, ... $\varphi_2^{(\mu-1)}$, ipsae vero $q_2$, $q_3$, ... $q_i$ in aequatione (1.) pro constantibus habentur. Substituto valore $f = \Pi$ in aequatione:

$$(2.)\quad \begin{cases} 0 = f_2' = \dfrac{\partial f}{\partial q_2} + \dfrac{\partial p_2}{\partial q_{i+1}}\dfrac{\partial f}{\partial p_{i+1}} + \dfrac{\partial p_2}{\partial q_{i+2}}\dfrac{\partial f}{\partial p_{i+2}} + \dots + \dfrac{\partial p_2}{\partial q_m}\dfrac{\partial f}{\partial p_m} \\ \qquad - \dfrac{\partial p_2}{\partial p_{i+1}}\dfrac{\partial f}{\partial q_{i+1}} - \dfrac{\partial p_2}{\partial p_{i+2}}\dfrac{\partial f}{\partial q_{i+2}} - \dots - \dfrac{\partial p_2}{\partial p_m}\dfrac{\partial f}{\partial q_m}, \end{cases}$$

haec aequatio hanc induit formam:

$$(2^a.)\quad 0 = \frac{\partial \Pi}{\partial \varphi}\varphi_2' + \frac{\partial \Pi}{\partial \varphi_2'}\varphi_2'' + \dots + \frac{\partial \Pi}{\partial \varphi_2^{(\mu-1)}}\varphi_2^{(\mu)} + \frac{\partial \Pi}{\partial q_2},$$

in qua variabiles independentes sunt $\varphi$, $\varphi_2'$, $\varphi_2''$, ... $\varphi_2^{(\mu-1)}$, $q_2$. Cujus aequationis integratio jam suppeditat functionem $f = \Pi$, quae satisfaciat simul duabus aequationibus (1.) et (2.).

Evenire potest, ut identice evadat $\varphi_2' = 0$, quo casu sine ulteriore integratione ipsa functio $f = \varphi$, aequationis (1.) integrale, etiam aequationis (2.) integrale habetur. Si generalius est $\varphi_2' = c$, designante $c$ constantem, erit

$$\Pi = \varphi - cq_2 = f$$

utriusque simul aequationis (1.) et (2.) integrale.

## 20.

Postquam antecedentibus monstratum est, quomodo functio $\Pi = f$ inveniatur, quae simul duabus aequationibus (1.) et (2.) satisfacit, id quod ope theorematis IV successit, jam ejusdem theorematis ope ex inventa functione $\Pi$ aliam deducam $\Psi$, quae loco ipsius $f$ posita duabus aequationibus illis simul atque tertiae

$$(3.)\quad \begin{cases} 0=f_3'=\dfrac{\partial f}{\partial q_3}+\dfrac{\partial p_3}{\partial q_{i+1}}\dfrac{\partial f}{\partial p_{i+1}}+\dfrac{\partial p_3}{\partial q_{i+2}}\dfrac{\partial f}{\partial p_{i+2}}+\cdots+\dfrac{\partial p_3}{\partial q_m}\dfrac{\partial f}{\partial p_m} \\ \qquad -\dfrac{\partial p_3}{\partial p_{i+1}}\dfrac{\partial f}{\partial q_{i+1}}-\dfrac{\partial p_3}{\partial p_{i+2}}\dfrac{\partial f}{\partial q_{i+2}}-\cdots-\dfrac{\partial p_3}{\partial p_m}\dfrac{\partial f}{\partial q_m} \end{cases}$$

satisfaciat.

Erunt enim e theoremate IV, siquidem in eo loco $\varphi$ ponimus $\Pi$, atque statuimus $\lambda=3$, ipsi $k$ vero valores 1 et 2 tribuimus, functiones $\Pi_3'$, $\Pi_3''$, ... simul utriusque aequationis (1.) et (2.) integralia. Sit $\Pi_3^{(\mu')}$ proxima functio, quae per praecedentes $\Pi_3$, $\Pi_3'$, ... $\Pi_3^{(\mu'-1)}$ et ipsas $q_3$, $q_4$, ... $q_i$ exprimi potest: numerus $\mu'$ rursus ipsum $2(m-i)$ superare non potest. Posito

$$f = \Psi,$$

designante $\Psi$ functionem ipsarum $\Pi$, $\Pi_3'$, $\Pi_3''$, ... $\Pi_3^{(\mu-1)}$, $q_3$, quam etiam quantitates $q_4$, $q_5$, ... $q_i$ tamquam constantes ingredi possunt, abit (3.) in hanc:

$$(3^a.)\qquad 0 = \frac{\partial\Psi}{\partial\Pi}\Pi_3'+\frac{\partial\Psi}{\partial\Pi_3'}\Pi_3''+\frac{\partial\Psi}{\partial\Pi_3''}\Pi_3'''+\cdots+\frac{\partial\Psi}{\partial\Pi_3^{(\mu'-1)}}\Pi_3^{(\mu')}+\frac{\partial\Psi}{\partial q_3}.$$

Quodcunque integrale hujus aequationis, in qua $\Pi$, $\Pi_3'$, $\Pi_3''$, ... $\Pi_3^{(\mu'-1)}$, $q_3$ sunt variabiles independentes, suppeditat functionem quaesitam $f=\Psi$, quae simul tribus aequationibus (1.), (2.), (3.) satisfacit.

Et sic pergi potest, usque dum habeatur functio $f$ simul omnibus $i$ aequationibus ($d$.) satisfaciens.

## 21.

Ex antecedentibus hic fit integrationum decursus, quibus eruatur functio $i$ aequationibus ($d$.) simul omnibus satisfaciens. Ante omnia quaerenda erat functio $\varphi$ aequationi (1.) satisfaciens. Quam notum est haberi, si

$$\varphi = \text{Constans}$$

est integrale unum quodcunque systematis aequationum differentialium vulgarium sequentis:

$$(a.)\quad \begin{cases} \dfrac{dp_{i+1}}{dq_1}=\dfrac{\partial p_1}{\partial q_{i+1}}, & \dfrac{dq_{i+1}}{dq_1}=-\dfrac{\partial p_1}{\partial p_{i+1}}, \\ \dfrac{dp_{i+2}}{dq_1}=\dfrac{\partial p_1}{\partial q_{i+2}}, & \dfrac{dq_{i+2}}{dq_1}=-\dfrac{\partial p_1}{\partial p_{i+2}}, \\ \cdots & \cdots \\ \dfrac{dp_m}{dq_1}=\dfrac{\partial p_1}{\partial q_m}, & \dfrac{dq_m}{dq_1}=-\dfrac{\partial p_1}{\partial p_m}. \end{cases}$$

Aequatio enim $\varphi =$ Constans integrale dicitur aequationum differentialium vulgarium (*a.*) si per eas aequationi $d\varphi = 0$ identice satisfiat. Id quod fieri non potest nisi aequatio (1.) identice locum habeat.

Inventa functione $\varphi$, ex ea deducantur functiones $\varphi', \varphi'', \dots \varphi^{(\mu-1)}$ — indices subscriptos rejicio — atque exprimatur $\varphi^{(\mu)}$ per $q_2, \varphi, \varphi', \dots \varphi^{(\mu-1)}$, quam expressionem etiam $q_3, q_4, \dots q_i$ afficere possunt tamquam constantes. Quo facto invenitur functio $\Pi$ aequationi (2''.) satisfaciens, si aequatio

$$\Pi = \text{Constans}$$

est unum integrale quodcunque systematis aequationum differentialium vulgarium:

$$\varphi' = \frac{d\varphi}{dq_2}, \quad \varphi'' = \frac{d\varphi'}{dq_2}, \quad \dots \quad \varphi^{(\mu-1)} = \frac{d\varphi^{(\mu-2)}}{dq_2}, \quad \varphi^{(\mu)} = \frac{d\varphi^{(\mu-1)}}{dq_2}.$$

Sit ipsius $\varphi^{(\mu)}$ haec expressio:

$$\varphi^{(\mu)} = \varphi^{(\mu)}(q_2, \varphi, \varphi', \varphi'', \dots \varphi^{(\mu-1)}),$$

sequitur ex antecedentibus, si sit

$$\Pi\left(q_2, \varphi, \frac{d\varphi}{dq_2}, \frac{d^2\varphi}{dq_2^2}, \dots \frac{d^{\mu-1}\varphi}{dq_2^{\mu-1}}\right) = \text{Constans}$$

unum integrale quodcunque aequationis differentialis vulgaris $\mu^{\text{ti}}$ ordinis inter duas variabiles $\varphi$ et $q_2$,

$$(b.) \quad \frac{d^\mu \varphi}{dq_2^\mu} = \varphi^{(\mu)}\left(q_2, \varphi, \frac{d\varphi}{dq_2}, \frac{d^2\varphi}{dq_2^2}, \dots \frac{d^{\mu-1}\varphi}{dq_2^{\mu-1}}\right),$$

fieri

$$\Pi(q_2, \varphi, \varphi', \varphi'', \dots \varphi^{(\mu-1)})$$

functionem $\Pi$ quaesitam, quae simul aequationibus (1.) et (2.) satisfacit.

Tertio loco e functione $\Pi$ deducantur functiones $\Pi', \Pi'', \dots \Pi^{(\mu'-1)}$ atque per has et $q_3$ exprimatur $\Pi^{(\mu')}$; sit expressio inventa

$$\Pi^{(\mu')} = \Pi^{(\mu')}(q_3, \Pi, \Pi', \Pi'', \dots \Pi^{(\mu'-1)});$$

proponatur aequatio differentialis $\mu'^{\text{ti}}$ ordinis inter duas variabiles $\Pi$ et $q_3$:

$$(c.) \quad \frac{d^{\mu'}\Pi}{dq_3^{\mu'}} = \Pi^{(\mu')}\left(q_3, \Pi, \frac{d\Pi}{dq_3}, \frac{d^2\Pi}{dq_3^2}, \dots \frac{d^{\mu'-1}\Pi}{dq_3^{\mu'-1}}\right);$$

cujus integrale unum quodcunque si est

$$\Psi\left(q_3, \Pi, \frac{d\Pi}{dq_3}, \frac{d^2\Pi}{dq_3^2}, \dots \frac{d^{\mu'-1}\Pi}{dq_3^{\mu'-1}}\right) = \text{Constans},$$

est expressio

$$\Psi(q_3, \Pi, \Pi', \Pi'', \dots \Pi^{(\mu'-1)})$$

functio quaesita $\Psi$, quae simul tribus aequationibus (1.), (2.), (3.) satisfacit. Id quod simili demonstratione liquet atque in functione $\Pi$ investiganda dedimus. Functionem $\Psi$ etiam quantitates $q_4$, $q_5$, ... $q_i$ afficere possunt tamquam constantes.

Et sic pergi potest, usque dum habeatur functio $f$ omnibus $i$ aequationibus ($d$.) satisfaciens. Ad quam inveniendam primum quod est principale eruendum est integrale quodcunque systematis aequationum differentialium vulgarium primi ordinis inter $2(m-i)+1$ variabiles, quod locum tenet unius aequationis inter duas variabiles $(2m-2i)^{\text{ti}}$ ordinis. Deinde condendae sunt aliae post alias $i-1$ aequationes differentiales vulgares inter duas variabiles ordinis $\mu^{\text{ti}}$, $\mu'^{\text{ti}}$, $\mu''^{\text{ti}}$, ... $\mu^{(i-2)\,\text{ti}}$, et singularum inveniendum est unum integrale quodcunque, quod formandae aequationi differentiali insequenti inservit. Numeri autem $\mu$, $\mu'$, $\mu''$, ... $\mu^{(i-2)}$ omnes erunt ipso $2(m-i)$ aut minores aut certe non majores. Si postremae aequationis integrale est

$$f = a_i,$$

designante $a_i$ constantem arbitrariam, atque ex hac aequatione petitur ipsius $p_{i+1}$ valor per $p_{i+2}$, $p_{i+3}$, ... $p_m$, $q_1$, $q_2$, ... $q_m$ expressus, erit valor ille talis, qui omnibus $i$ aequationibus ($\alpha$.) §. 11 simul satisfacit. Quo invento etiam $p_1$, $p_2$, ... $p_i$, quae datae supponuntur functiones ipsarum $p_{i+1}$, $p_{i+2}$, ... $p_m$, $q_1$, $q_2$, ... $q_m$, exprimi possunt per $p_{i+2}$, $p_{i+3}$, ... $p_m$, $q_1$, $q_2$, ... $q_m$; et pergi potest ad integrationem simultaneam insequentis systematis $i+1$ aequationum differentialium partialium, quae ex aequationibus ($\alpha$.) proveniunt, si $i+1$, loco $i$ ponitur, et cujus aequationes singulae numerum variabilium *duabus* unitatibus minorem continent.

Aequationes differentiales vulgares inter binas variabiles $\mu^{\text{ti}}$, $\mu'^{\text{ti}}$, cet. ordinis tamquam *auxiliares* spectari possunt; dum systema aequationum differentialium vulgarium ($a$.), quae sunt primi ordinis, sed inter $2(m-i)+1$ variabiles tamquam *principale* considerari potest. Quod systema principale si per eliminationem variabilium omnium praeter duas earumque differentialia ad unam revocas aequationem differentialem inter duas variabiles, ascendet ea ad ordinem $2(m-i)$ neque ad minorem ascendere potest. Ordo autem aequationis cujusvis auxiliaris pendet ab eo, quod inventum est, integrali aequationis auxiliaris praecedentis, et prout hoc vel illud inveneris, ordo major aut inferior fieri potest, qui tamen ordinem $2(m-i)$ aequationis principalis numquam egredi potest. Quin etiam e numeris $\mu$, $\mu'$, ... exsistere possunt qui evanescant, ita ut una aut pluribus aut adeo omnibus integrationibus auxiliaribus omnino supersedeatur.

*Integrationum, quibus totius problematis solutio secundum methodum propositam absolvatur, decursus describitur.*

22.

Si totum negotium inde ab initio prosequimur, hic erit rei processus. Data $p_1$ ut functione ipsarum $p_2$, $p_3$, ... $p_m$, $q_1$, $q_2$, ... $q_m$, quae est aequatio differentialis partialis proposita, reliquae quantitates $p_2$, $p_3$, ... $p_m$ ita determinandae sunt tamquam functiones ipsarum $q_1$, $q_2$, ... $q_m$, ut evadat expressio

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m,$$

ipsa quoque $p_1$ per $q_1$, $q_2$, ... $q_m$ expressa, differentiale completum; quo facto erit

$$V = \int \{p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m\}$$

functio incognita, aequationi differentiali partiali propositae satisfaciens.

Conditur primum systema aequationum differentialium vulgarium sequens:

$$(1.)\quad \begin{cases} \dfrac{dp_2}{dq_1} = \dfrac{\partial p_1}{\partial q_2}, & \dfrac{dq_2}{dq_1} = -\dfrac{\partial p_1}{\partial p_2}, \\ \dfrac{dp_3}{dq_1} = \dfrac{\partial p_1}{\partial q_3}, & \dfrac{dq_3}{dq_1} = -\dfrac{\partial p_1}{\partial p_3}, \\ \cdots & \cdots \\ \dfrac{dp_m}{dq_1} = \dfrac{\partial p_1}{\partial q_m}, & \dfrac{dq_m}{dq_1} = -\dfrac{\partial p_1}{\partial p_m}. \end{cases}$$

Cujus systematis si est integrale quodcunque

$$f_1 = a_1,$$

designante $a_1$ constantem arbitrariam, ex hac aequatione determinatur $p_2$ ut functio quantitatum $p_3$, $p_4$, ... $p_m$, $q_1$, $q_2$, ... $q_m$, unde etiam $p_1$ ut functio earundem quantitatum determinari potest. Quo facto, conditur systema aequationum differentialium vulgarium sequens:

$$(2.)\quad \begin{cases} \dfrac{dp_3}{dq_1} = \dfrac{\partial p_1}{\partial q_3}, & \dfrac{dq_3}{dq_1} = -\dfrac{\partial p_1}{\partial p_3}, \\ \dfrac{dp_4}{dq_1} = \dfrac{\partial p_1}{\partial q_4}, & \dfrac{dq_4}{dq_1} = -\dfrac{\partial p_1}{\partial p_4}, \\ \cdots & \cdots \\ \dfrac{dp_m}{dq_1} = \dfrac{\partial p_1}{\partial q_m}, & \dfrac{dq_m}{dq_1} = -\dfrac{\partial p_1}{\partial p_m}. \end{cases}$$

Cujus systematis si est integrale

$$\varphi = \text{Constans},$$

formantur expressiones

$$\varphi' = \frac{\partial\varphi}{\partial q_2}+\frac{\partial p_2}{\partial p_3}\frac{\partial\varphi}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial\varphi}{\partial q_4}+\cdots+\frac{\partial p_2}{\partial p_m}\frac{\partial\varphi}{\partial q_m}$$
$$-\frac{\partial p_2}{\partial q_3}\frac{\partial\varphi}{\partial p_3}-\frac{\partial p_2}{\partial q_4}\frac{\partial\varphi}{\partial p_4}-\cdots-\frac{\partial p_2}{\partial q_m}\frac{\partial\varphi}{\partial p_m},$$
$$\varphi'' = \frac{\partial\varphi'}{\partial q_2}+\frac{\partial p_2}{\partial p_3}\frac{\partial\varphi'}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial\varphi'}{\partial q_4}+\cdots+\frac{\partial p_2}{\partial p_m}\frac{\partial\varphi'}{\partial q_m}$$
$$-\frac{\partial p_2}{\partial q_3}\frac{\partial\varphi'}{\partial p_3}-\frac{\partial p_2}{\partial q_4}\frac{\partial\varphi'}{\partial p_4}-\cdots-\frac{\partial p_2}{\partial q_m}\frac{\partial\varphi'}{\partial p_m},$$
etc. etc.

usque dum perveniatur ad functionem

$$\varphi^{(\mu)} = \frac{\partial\varphi^{(\mu-1)}}{\partial q_2}+\frac{\partial p_2}{\partial p_3}\frac{\partial\varphi^{(\mu-1)}}{\partial q_3}+\frac{\partial p_2}{\partial p_4}\frac{\partial\varphi^{(\mu-1)}}{\partial q_4}+\cdots+\frac{\partial p_2}{\partial p_m}\frac{\partial\varphi^{(\mu-1)}}{\partial q_m}$$
$$-\frac{\partial p_2}{\partial q_3}\frac{\partial\varphi^{(\mu-1)}}{\partial p_3}-\frac{\partial p_2}{\partial q_4}\frac{\partial\varphi^{(\mu-1)}}{\partial p_4}-\cdots-\frac{\partial p_2}{\partial q_m}\frac{\partial\varphi^{(\mu-1)}}{\partial p_m},$$

quae per antecedentes $\varphi$, $\varphi'$, $\varphi''$, ... $\varphi^{(\mu-1)}$ et ipsam $q_2$ exprimi potest, quod semper evenit pro numero $\mu \leqq 2m-4$. Si expressio ipsius $\varphi^{(\mu)}$ est

$$\varphi^{(\mu)}(q_2,\ \varphi,\ \varphi',\ \varphi'',\ \dots\ \varphi^{(\mu-1)}),$$

formatur aequatio differentialis $\mu^{ti}$ ordinis:

$$(2^a.)\qquad \frac{d^\mu\varphi}{dq_2^\mu} = \varphi^{(\mu)}\Big(q_2,\ \varphi,\ \frac{d\varphi}{dq_2},\ \frac{d^2\varphi}{dq_2^2},\ \dots\ \frac{d^{\mu-1}\varphi}{dq_2^{\mu-1}}\Big).$$

Cuius integrale quodcunque si est

$$f_2\Big(q_2,\ \varphi,\ \frac{d\varphi}{dq_2},\ \frac{d^2\varphi}{dq_2^2},\ \dots\ \frac{d^{\mu-1}\varphi}{dq_2^{\mu-1}}\Big) = a_2,$$

designante $a_2$ constantem arbitrariam, formatur aequatio:

$$f_2 = f_2(q_2,\ \varphi,\ \varphi',\ \varphi'',\ \dots\ \varphi^{(\mu-1)}) = a_2,$$

atque ope aequationum

$$f_1 = a_1,\quad f_2 = a_2$$

exprimuntur $p_1$, $p_2$, $p_3$ per $p_4$, $p_5$, ... $p_m$ $q_1$, $q_2$, ... $q_m$. Quo facto conditur systema aequationum differentialium vulgarium sequens:

$$(3.)\qquad \begin{cases} \dfrac{dp_4}{dq_1} = \dfrac{\partial p_1}{\partial q_4}, & \dfrac{dq_4}{dq_1} = -\dfrac{\partial p_1}{\partial p_4}, \\ \dfrac{dp_5}{dq_1} = \dfrac{\partial p_1}{\partial q_5}, & \dfrac{dq_5}{dq_1} = -\dfrac{\partial p_1}{\partial p_5}, \\ \dots & \dots \\ \dfrac{dp_m}{dq_1} = \dfrac{\partial p_1}{\partial q_m}, & \dfrac{dq_m}{dq_1} = -\dfrac{\partial p_1}{\partial p_m}. \end{cases}$$

Cuius systematis integrale si est

$$\Pi = \text{Constans},$$

formantur functiones:

$$\begin{aligned}
\Pi' &= \frac{\partial \Pi}{\partial q_2} + \frac{\partial p_2}{\partial p_4}\frac{\partial \Pi}{\partial q_4} + \frac{\partial p_2}{\partial p_5}\frac{\partial \Pi}{\partial q_5} + \cdots + \frac{\partial p_2}{\partial p_m}\frac{\partial \Pi}{\partial q_m} \\
&\quad - \frac{\partial p_2}{\partial q_4}\frac{\partial \Pi}{\partial p_4} - \frac{\partial p_2}{\partial q_5}\frac{\partial \Pi}{\partial p_5} - \cdots - \frac{\partial p_2}{\partial q_m}\frac{\partial \Pi}{\partial p_m}, \\
\Pi'' &= \frac{\partial \Pi'}{\partial q_2} + \frac{\partial p_2}{\partial p_4}\frac{\partial \Pi'}{\partial q_4} + \frac{\partial p_2}{\partial p_5}\frac{\partial \Pi'}{\partial q_5} + \cdots + \frac{\partial p_2}{\partial p_m}\frac{\partial \Pi'}{\partial q_m} \\
&\quad - \frac{\partial p_2}{\partial q_4}\frac{\partial \Pi'}{\partial p_4} - \frac{\partial p_2}{\partial q_5}\frac{\partial \Pi'}{\partial p_5} - \cdots - \frac{\partial p_2}{\partial q_m}\frac{\partial \Pi'}{\partial p_m}
\end{aligned}$$

etc. etc.

usque dum perveniatur ad functionem

$$\begin{aligned}
\Pi^{(\nu)} &= \frac{\partial \Pi^{(\nu-1)}}{\partial q_2} + \frac{\partial p_2}{\partial p_4}\frac{\partial \Pi^{(\nu-1)}}{\partial q_4} + \cdots + \frac{\partial p_2}{\partial p_m}\frac{\partial \Pi^{(\nu-1)}}{\partial q_m} \\
&\quad - \frac{\partial p_2}{\partial q_4}\frac{\partial \Pi^{(\nu-1)}}{\partial p_4} - \cdots - \frac{\partial p_2}{\partial q_m}\frac{\partial \Pi^{(\nu-1)}}{\partial p_m},
\end{aligned}$$

quae per antecedentes $\Pi, \Pi', \Pi'', \ldots \Pi^{(\nu-1)}$ et ipsam $q_2$ exprimi potest existente $\nu \leqq 2m-6$. Quam expressionem etiam $q_3$ tamquam constans afficere potest. Scribendo igitur loco $\Pi^{(\nu)}$ hanc expressionem:

$$\Pi^{(\nu)}(q_2, \Pi, \Pi', \ldots \Pi^{(\nu-1)}),$$

conditur aequatio differentialis $\nu^{\text{ti}}$ ordinis:

$$(3^a.) \qquad \frac{d^\nu \Pi}{dq_2^\nu} = \Pi^{(\nu)}\Big(q_2, \Pi, \frac{d\Pi}{dq_2}, \ldots \frac{d^{\nu-1}\Pi}{dq_2^{\nu-1}}\Big).$$

Cuius aequationis integrale aliquod quodcunque si est

$$\Pi_1 = \text{Constans},$$

formantur functiones

$$\begin{aligned}
\Pi_1' &= \frac{\partial \Pi_1}{\partial q_3} + \frac{\partial p_3}{\partial p_4}\frac{\partial \Pi_1}{\partial q_4} + \cdots + \frac{\partial p_3}{\partial p_m}\frac{\partial \Pi_1}{\partial q_m} \\
&\quad - \frac{\partial p_3}{\partial q_4}\frac{\partial \Pi_1}{\partial p_4} - \cdots - \frac{\partial p_3}{\partial q_m}\frac{\partial \Pi}{\partial p_m}, \\
\Pi_1'' &= \frac{\partial \Pi_1'}{\partial q_3} + \frac{\partial p_3}{\partial p_4}\frac{\partial \Pi_1'}{\partial q_4} + \cdots + \frac{\partial p_3}{\partial p_m}\frac{\partial \Pi_1'}{\partial q_m} \\
&\quad - \frac{\partial p_3}{\partial q_4}\frac{\partial \Pi_1'}{\partial p_4} - \cdots - \frac{\partial p_3}{\partial q_m}\frac{\partial \Pi_1'}{\partial p_m},
\end{aligned}$$

etc. etc.

usque dum perveniatur ad functionem $\Pi^{(\nu')}$, quae per praecedentes $\Pi_1$, $\Pi_1'$, $\Pi_1''$, ... $\Pi_1^{(\nu'-1)}$, et ipsam $q_3$ exprimi potest, rursus existente $\nu' \leqq 2m-6$. Quae expressio si est

$$\Pi_1^{(\nu')}(q_3, \Pi_1, \Pi_1', \Pi_1'', \ldots \Pi_1^{(\nu'-1)})$$

conditur aequatio differentialis $\nu'^{\text{ti}}$ ordinis:

$$(3^b.) \qquad \frac{d^{\nu'}\Pi_1}{dq_3^{\nu'}} = \Pi_1^{(\nu')}\left(q_3, \Pi_1, \frac{d\Pi_1}{dq_3}, \frac{d^2\Pi_1}{dq_3^2}, \ldots \frac{d^{\nu'-1}\Pi_1}{dq_3^{\nu'-1}}\right),$$

cuius unum quaeritur integrale quodcunque

$$f_3\left(q_3, \Pi_1, \frac{d\Pi_1}{dq_3}, \frac{d^2\Pi_1}{dq_3^2}, \ldots \frac{d^{\nu'-1}\Pi_1}{dq_3^{\nu'-1}}\right) = a_3,$$

designante $a_3$ constantem arbitrariam. Quo invento formatur aequatio

$$f_3 = f_3(q_3, \Pi_1, \Pi_1', \Pi_1'', \ldots \Pi_1^{(\nu'-1)}) = a_3,$$

et ope trium aequationum,

$$f_1 = a_1, \quad f_2 = a_2, \quad f_3 = a_3$$

exprimuntur $p_1$, $p_2$, $p_3$, $p_4$ ut functiones ipsarum $p_5$, ... $p_m$, $q_1$, $q_2$, ... $q_m$. Et ita porro. Totum negotium in has integrationes desinit. Scilicet inventis per methodum assignatam aequationibus

$$f_1 = a_1, \quad f_2 = a_2, \quad \ldots \quad f_{m-2} = a_{m-2},$$

in quibus $a_1$, $a_2$, ... $a_{m-2}$ sunt constantes arbitrariae, quarum unaquaeque $a_i$ functiones $f_{i+1}$, $f_{i+2}$, ... $f_{m-2}$ neque vero ullam eas praecedentem $f_1$, $f_2$, ... $f_i$ afficit, exprimantur ope harum aequationum et aequationis differentialis partialis propositae $p_1$, $p_2$, ... $p_{m-1}$ ut functiones ipsarum $p_m$, $q_1$, $q_2$, ... $q_m$, et proponantur aequationes

$$\frac{dp_m}{dq_1} = \frac{\partial p_1}{\partial q_m}, \quad \frac{dq_m}{dq_1} = -\frac{\partial p_1}{\partial p_m};$$

quae quum duo integralia habeant, sit alterum

$$\psi = \text{Constans},$$

et formentur functiones

$$\psi' = \frac{\partial \psi}{\partial q_2} + \frac{\partial p_2}{\partial p_m}\frac{\partial \psi}{\partial q_m} - \frac{\partial p_2}{\partial q_m}\frac{\partial \psi}{\partial p_m},$$

$$\psi'' = \frac{\partial \psi'}{\partial q_2} + \frac{\partial p_2}{\partial p_m}\frac{\partial \psi'}{\partial q_m} - \frac{\partial p_2}{\partial q_m}\frac{\partial \psi'}{\partial p_m};$$

si $\psi'$ est functio ipsius $\psi$ ipsarumque $q_2$, $q_3$, ... $q_m$,

$$\psi' = \psi'(q_2, \psi)$$

integretur aequatio primi ordinis:

$$\frac{d\psi}{dq_2} = \psi'(q_2, \psi);$$

si vero $\psi'$ non est functio ipsius $\psi$ ipsarumque $q_2, q_3, \ldots q_m$, certe erit $\psi''$ functio ipsarum $\psi$, $\psi'$ et quantitatum $q_2, q_3, \ldots q_m$,

$$\psi'' = \psi''(q_2, \psi, \psi'),$$

quo casu quaeratur alterum integrale aequationis differentialis secundi ordinis:

$$\frac{d^2\psi}{dq_2^2} = \psi''\left(q_2, \psi, \frac{d\psi}{dq_2}\right);$$

quibus in aequationibus considerantur $q_3, q_4, \ldots q_m$ ut constantes; sit integrale huius vel illius aequationis

$$\psi_1 = \text{Constans},$$

designante $\psi_1$ priore casu ipsarum $q_2$, $\psi$, posteriore ipsarum $q_2$, $\psi$, $\frac{d\psi}{dq_2}$ functionem, ac restituatur posteriore casu $\psi'$ loco $\frac{d\psi}{dq_2}$ in functione $\psi_1$, quo facto formentur rursus functiones

$$\psi_1' = \frac{\partial\psi_1}{\partial q_3} + \frac{\partial p_3}{\partial p_m}\frac{\partial\psi_1}{\partial q_m} - \frac{\partial p_3}{\partial q_m}\frac{\partial\psi_1}{\partial p_m},$$

$$\psi_1'' = \frac{\partial\psi_1'}{\partial q_3} + \frac{\partial p_3}{\partial p_m}\frac{\partial\psi_1'}{\partial q_m} - \frac{\partial p_3}{\partial q_m}\frac{\partial\psi_1'}{\partial p_m};$$

erit aut $\psi_1'$ ipsarum $\psi_1, q_3, q_4, \ldots q_m$ aut, si hoc locum non habet, certe $\psi_1''$ ipsarum $\psi_1, \psi_1', q_3, q_4, \ldots q_m$ functio; quaeratur integrale priore casu aequationis

$$\frac{d\psi_1}{dq_3} = \psi_1',$$

posteriore casu aequationis

$$\frac{d^2\psi_1}{dq_3^2} = \psi_1'',$$

siquidem in $\psi_1''$ loco $\psi_1'$ ponitur $\frac{d\psi_1}{dq_3}$, ipsis $q_4, q_5, \ldots q_m$ in hac vel illa aequatione consideratis ut constantibus; si integrale quaesitum est

$$\psi_2 = \text{Constans},$$

ac posteriore casu in $\psi_2$ loco $\frac{d\psi_1}{dq_3}$ restituitur $\psi_1'$, iam simili modo e $\psi_3$ deducatur functio $\psi_4$, ex hac $\psi_5$ et ita porro; postremo ex inventa functione

$\psi_{m-3}$ formetur functio

$$\psi'_{m-3} = \frac{\partial \psi_{m-3}}{\partial q_{m-1}} + \frac{\partial p_{m-1}}{\partial p_m}\frac{\partial \psi_{m-3}}{\partial q_m} - \frac{\partial p_{m-1}}{\partial q_m}\frac{\partial \psi_{m-3}}{\partial p_m},$$

quae si est ipsarum $\psi_{m-3}$, $q_{m-1}$, $q_m$ functio, quaeratur integrale aequationis

$$\frac{d\psi_{m-3}}{dq_{m-1}} = \psi'_{m-3};$$

sin minus, formetur adhuc functio

$$\psi''_{m-3} = \frac{\partial \psi'_{m-3}}{\partial q_{m-1}} + \frac{\partial p_{m-1}}{\partial p_m}\frac{\partial \psi'_{m-3}}{\partial q_m} - \frac{\partial p_{m-1}}{\partial q_m}\frac{\partial \psi'_{m-3}}{\partial p_m},$$

erit $\psi''_{m-3}$ ipsarum $\psi_{m-3}$, $\psi'_{m-3}$, $q_{m-1}$, $q_m$ functio; in qua si loco $\psi'_{m-3}$ ponitur $\frac{d\psi_{m-3}}{dq_{m-1}}$, quaeratur integrale aequationis differentialis secundi ordinis $\frac{d^2\psi_{m-3}}{dq^2_{m-1}} = \psi''_{m-3}$, in hac et illa aequatione considerata $q_m$ ut constante; sit integrale quaesitum

$$f_{m-1} = a_{m-1},$$

in quo posteriore casu restituendum est $\psi'_{m-3}$ loco $\frac{d\psi}{dq_{m-3}}$, designante $a_{m-1}$ constantem arbitrariam; erit inventa functione $f_{m-1}$ totum negotium finitum. Scilicet erutis ex aequationibus

$$f_1 = a_1, \quad f_2 = a_2, \quad \ldots \quad f_{m-1} = a_{m-1}$$

et ex aequatione differentiali partiali proposita ipsarum $p_1$, $p_2$, ... $p_m$ valoribus per $q_1$, $q_2$, ... $q_m$ expressis, fit

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$$

differentiale exactum atque

$$V = \int \{p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m\}$$

integrale aequationis differentialis partialis propositae, praeter constantem arbitrariam additione addendam alias $m-1$ constantes arbitrarias involvens $a_1$, $a_2$, ... $a_{m-1}$.

Systemate aequationum differentialium vulgarium revocato ad unam aequationem differentialem inter duas variabiles, ordo systematis secundum huius aequationis differentialis ordinem aestimetur sive secundum numerum constantium arbitrariarum, quas integratio eius completa secum fert. Iam si aequationum differentialium auxiliarium systemata omnia ad summum ordinem ascendunt, ad quem ascendere possunt, per methodum antecedentibus propositam quaerendum est unum integrale quodcunque $\frac{m(m-1)}{2}$ systematum aequationum differentialium inter duas variabiles; et quidem

| | | |
|---|---|---|
| unius | $(2m-2)^{\text{ti}}$ | ordinis |
| duarum | $(2m-4)^{\text{ti}}$ | - |
| trium | $(2m-6)^{\text{ti}}$ | - |
| . . . | . . . | . . . |
| $m-1$ | $2^{\text{ti}}$ | - |

Sed systematum aequationum auxiliarium ordo plerumque multo inferior evadit; qua de re accuratius dicetur, $m-1$ systematum, quae alia post alia conduntur atque respective $(2m-2)^{\text{ti}}$, $(2m-4)^{\text{ti}}$, ... $2^{\text{ti}}$ ordinis sunt, singulorum unum integrale quaerendum esse; atque insuper pro singulis systematis $(2m-2i)^{\text{ti}}$ ordinis formanda esse $i-1$ systemata auxiliaria alia post alia, quae ordinem $(2m-2i)^{\text{tum}}$ non excedunt, plerumque multo inferioris ordinis sunt, et quorum singulorum unum integrale investigandum est. Methodi hactenus notae poscebant systematis $(2m-2)^{\text{ti}}$ ordinis integrationem completam, quod post unum integrale inventum ad integrationem completam aequationis differentialis vulgaris inter duas variabiles $(2m-3)^{\text{ti}}$ ordinis reducitur. Dicere solebant Analystae, se aequationem differentialem integrasse, quam ad integrationes aequationum differentialium inferiorum ordinum reduxerint. Hac mente aequatio illa $(2m-3)^{\text{ti}}$ ordinis per methodos a me antecedentibus propositas generaliter integrata est, quippe ad aequationes ordinis $(2m-4)^{\text{ti}}$ et inferiorum ordinum reducta.

*Agitur de demonstratione theorematis IV., §. 18, quo antecedentia nituntur.*
*De inversione operationum differentialium.*

## 23.

Demonstrandum restat theorema IV., quo analysis antecedens tota innititur. Quam demonstrationem paullo altius repetam.

Sit $f$ functio $n$ variabilium $x_1, x_2, \dots x_n$, ac proponantur duae expressiones:

$$\mathrm{A}[f] = A_1\frac{\partial f}{\partial x_1}+A_2\frac{\partial f}{\partial x_2}+\cdots+A_n\frac{\partial f}{\partial x_n},$$

$$\mathrm{B}[f] = B_1\frac{\partial f}{\partial x_1}+B_2\frac{\partial f}{\partial x_2}+\cdots+B_n\frac{\partial f}{\partial x_n},$$

in quibus $A_1, A_2, \dots$ atque $B_1, B_2, \dots$ sunt datae ipsarum $x_1, x_2, \dots x_n$ functiones quaecunque. Ipsae

$$\mathrm{A}[f],\quad \mathrm{B}[f]$$

sunt notationes mere symbolicae, expressiones notantes, quae post certas operationes circa functionem $f$ transactas prodeunt; quas operationes *primam* et *secundam* dicam. Subiiciamus expressionem $\mathrm{B}[f]$ operationi primae, expressionem $\mathrm{A}[f]$ operationi secundae et expressiones inde prodeuntes alteram de altera deducamus, dico, *expressionem*

$$\mathrm{A}[\mathrm{B}[f]]-\mathrm{B}[\mathrm{A}[f]]$$

*differentialia partialia secunda functionis $f$ non continere, sed in ipsam formam redire:*

$$\mathrm{C}[f] = C_1\frac{\partial f}{\partial x_1}+C_2\frac{\partial f}{\partial x_2}+\cdots+C_n\frac{\partial f}{\partial x_n}.$$

Nam in altera expressione

$$\mathrm{A}[\mathrm{B}[f]]$$

evoluta multiplicatur $\frac{\partial^2 f}{\partial x_i \partial x_i}$ per $A_i B_i$ atque $\frac{\partial^2 f}{\partial x_i \partial x_k}$, si $i$ et $k$ inter se diversi sunt, per $A_i B_k + A_k B_i$; altera vero expressio quum de altera prodeat, $A$ et $B$ inter se permutando quo coefficientes illi non mutantur, ex utriusque expressionis differentia terminos illos prorsus abire patet. Eruitur porro in aequatione inventa:

$$\mathrm{A}[\mathrm{B}[f]]-\mathrm{B}[\mathrm{A}[f]] = \mathrm{C}[f]$$

terminus generalis

$$\begin{aligned} C_i = {} & A_1\frac{\partial B_i}{\partial x_1}+A_2\frac{\partial B_i}{\partial x_2}+\cdots+A_n\frac{\partial B_i}{\partial x_n} \\ & -B_1\frac{\partial A_i}{\partial x_1}-B_2\frac{\partial A_i}{\partial x_2}-\cdots-B_n\frac{\partial A_i}{\partial x_n}. \end{aligned}$$

Statuatur generaliter

$$\mathrm{A}^i[f] = \mathrm{A}[\mathrm{A}^{i-1}[f]],$$

ita ut sit:

$$\mathrm{A}^2[f] = \mathrm{A}[\mathrm{A}\ [f]],$$
$$\mathrm{A}^3[f] = \mathrm{A}[\mathrm{A}^2[f]],$$
$$\cdots\cdots$$

ac simili modo sit generaliter:

$$\mathrm{B}^i[f] = \mathrm{B}[\mathrm{B}^{i-1}[f]],$$

porro

$$\mathrm{B}^k\mathrm{A}^i[f] = \mathrm{B}^k[\mathrm{A}^i[f]],$$
$$\mathrm{A}^l\mathrm{B}^k\mathrm{A}^i[f] = \mathrm{A}^l[\mathrm{B}^k\mathrm{A}^i[f]],$$
$$\cdots\cdots\cdots$$

ita ut obtineatur ex. gr. expressio

$$\mathrm{B}^m \mathrm{A}^l \mathrm{B}^k \mathrm{A}^i [f],$$

si functio $f$ subjicitur $i$ vicibus iteratis operationi primae, expressio proveniens $k$ vicibus iteratis operationi secundae, expressio proveniens $l$ vicibus iteratis rursus operationi primae, expressio proveniens $m$ vicibus iteratis rursus operationi secundae. His positis supponamus, expressionem

$$\begin{aligned} C_i = {} & A_1 \frac{\partial B_i}{\partial x_1} + A_2 \frac{\partial B_i}{\partial x_2} + \cdots + A_n \frac{\partial B_i}{\partial x_n} \\ & - B_1 \frac{\partial A_i}{\partial x_1} - B_2 \frac{\partial A_i}{\partial x_2} - \cdots - B_n \frac{\partial A_i}{\partial x_n} \end{aligned}$$

identice evanescere pro quolibet ipsius $i$ valore, erit identice, quaecunque sit $f$ functio,

$$\mathrm{A}\mathrm{B}[f] = \mathrm{B}\mathrm{A}[f],$$

sive duarum operationum ordo interverti potest. Unde deduci potest theorema generale *expressionem*

$$\mathrm{B}^m \mathrm{A}^l \mathrm{B}^k \mathrm{A}^i [f]$$

*eandem evasuram, quicunque sit operationum ordo.*

Ad demonstrandam propositionem praecedentem generalem, observo, fieri

$$\begin{aligned} \mathrm{B}^k \mathrm{A}[f] &= \mathrm{B}^{k-1} \mathrm{B}\mathrm{A}[f] \quad = \mathrm{B}^{k-1} \mathrm{A}\mathrm{B}[f] \\ &= \mathrm{B}^{k-2} \mathrm{B}\mathrm{A}\mathrm{B}[f] = \mathrm{B}^{k-2} \mathrm{A}\mathrm{B}^2[f] \\ &= \mathrm{B}^{k-3} \mathrm{B}\mathrm{A}\mathrm{B}^2[f] = \mathrm{B}^{k-3} \mathrm{A}\mathrm{B}^3[f] \\ &= \mathrm{B}^{k-4} \mathrm{B}\mathrm{A}\mathrm{B}^3[f] \ldots = \mathrm{B}\mathrm{A}\mathrm{B}^{k-1}[f] \\ &= \mathrm{A}\mathrm{B}^k[f]. \end{aligned}$$

Unde

$$\begin{aligned} \mathrm{B}^k \mathrm{A}^i[f] &= \mathrm{B}^k \mathrm{A}\mathrm{A}^{i-1}[f] \quad = \mathrm{A}\mathrm{B}^k \mathrm{A}^{i-1}[f] = \mathrm{A}\mathrm{B}^k \mathrm{A}\mathrm{A}^{i-2}[f] \\ &= \mathrm{A}\mathrm{A}\mathrm{B}^k \mathrm{A}^{i-2}[f] = \mathrm{A}^2 \mathrm{B}^k \mathrm{A}\mathrm{A}^{i-3}[f] \\ &= \mathrm{A}^2 \mathrm{A}\mathrm{B}^k \mathrm{A}^{i-3}[f] = \mathrm{A}^3 \mathrm{B}^k \mathrm{A}\mathrm{A}^{i-4}[f] \ldots \\ &= \mathrm{A}^{i-1} \mathrm{B}^k \mathrm{A}[f] \quad = \mathrm{A}^{i-1} \mathrm{A}\mathrm{B}^k[f] = \mathrm{A}^i \mathrm{B}^k[f]. \end{aligned}$$

Hinc etiam eruitur

$$\begin{aligned} \mathrm{A}^l \mathrm{B}^k \mathrm{A}^i[f] \quad &= \mathrm{A}^l \mathrm{A}^i \mathrm{B}^k[f] \quad = \mathrm{A}^{i+l} \mathrm{B}^k[f] \quad = \mathrm{B}^k \mathrm{A}^{i+l}[f] \\ \mathrm{B}^m \mathrm{A}^l \mathrm{B}^k \mathrm{A}^i[f] &= \mathrm{B}^m \mathrm{B}^k \mathrm{A}^{i+l}[f] = \mathrm{B}^{m+k} \mathrm{A}^{i+l}[f] \\ &= \mathrm{A}^{i+l} \mathrm{B}^{m+k}[f]. \end{aligned}$$

. . . . . . . . . . . . . . . . . . . . . . .

Unde propositio demonstranda patet.

*Formula §. antecedente inventa alia via confirmatur.*

### 24.

Propositio inventa, si $C_i = 0$, pro quolibet ipsius $i$ valore, fieri

$$\mathrm{AB}[f] = \mathrm{BA}[f]$$

his considerationibus confirmatur. Sint $x_1, x_2, \dots x_n$ functiones duarum variabilium $t$ et $u$, quae in ipsa $f$ neque in ipsis $A_i$, $B_i$ praeterea non inveniantur explicite. Quas functiones supponamus determinari per aequationes:

$$(1.)\quad \begin{cases} \dfrac{\partial x_1}{\partial t} = A_1, & \dfrac{\partial x_2}{\partial t} = A_2, \quad \dots \quad \dfrac{\partial x_n}{\partial t} = A_n, \\ \dfrac{\partial x_1}{\partial u} = B_1, & \dfrac{\partial x_2}{\partial u} = B_2, \quad \dots \quad \dfrac{\partial x_n}{\partial u} = B_n. \end{cases}$$

Quae aequationes, ut locum habere possint, fieri debet pro quolibet ipsius $i$ valore:

$$(2.)\quad \frac{\partial B_i}{\partial t} - \frac{\partial A_i}{\partial u} = C_i = 0.$$

Sequitur autem e (1.):

$$\frac{\partial f}{\partial t} = \mathrm{A}[f], \quad \frac{\partial f}{\partial u} = \mathrm{B}[f],$$

unde etiam

$$\frac{\partial \frac{\partial f}{\partial t}}{\partial u} = \mathrm{BA}[f], \quad \frac{\partial \frac{\partial f}{\partial u}}{\partial t} = \mathrm{AB}[f].$$

Quae expressiones, quum differentiationum secundum $t$ et $u$ institutarum ordo inverti possit, inter se aequales existunt. Quod est theorema propositum.

*De usu formulae inventae in integratione aequationum differentialium partialium linearium.*

### 25.

Antecedentibus erat $f$ functio quaecunque. Iam supponamus, esse $f$ integrale aequationis

$$(1.)\quad 0 = A_1\frac{\partial \varphi}{\partial x_1} + A_2\frac{\partial \varphi}{\partial x_2} + \dots + A_n\frac{\partial \varphi}{\partial x_n},$$

sive esse $f$ functionem talem, ut identice habeatur

$$\mathrm{A}[f] = 0.$$

Iam, *si rursus* $B_1, B_2, \dots B_n$ *sunt functiones ipsarum* $x_1, x_2, \dots x_n$ *tales* *ut pro quolibet ipsius i valore, identice sit*

$$0 = C_i = A_1\frac{\partial B_i}{\partial x_1}+A_2\frac{\partial B_i}{\partial x_2}+\cdots+A_n\frac{\partial B_i}{\partial x_n}$$
$$-B_1\frac{\partial A_i}{\partial x_1}-B_2\frac{\partial A_i}{\partial x_2}-\cdots-B_n\frac{\partial A_i}{\partial x_n},$$

*sequitur e propositione demonstrata, etiam functionem*

$$\mathrm{B}[f] = B_1\frac{\partial f}{\partial x_1}+B_2\frac{\partial f}{\partial x_2}+\cdots+B_n\frac{\partial f}{\partial x_n}$$

*esse aequationis* (1.) *integrale, sive generalius functionem* $\mathrm{B}^m[f]$. Quippe quod ut fiat, identice esse debet

$$\mathrm{AB}^m[f] = 0.$$

Sed quum sint quantitates $C_i = 0$, fit identice

$$\mathrm{AB}^m[f] = \mathrm{B}^m\mathrm{A}[f],$$

quae expressio identice evanescit, quum ex hypothesi expressio $\mathrm{A}[f]$ identice evanescat.

Fieri potest, ut expressio $\mathrm{B}[f]$ et ipsa identice evanescat sive constanti aequalis evadat. Quod vero si locum non habet, cognitis ipsis $B_1, B_2, \ldots B_n$ e quovis integrali aequationis (1.) $\varphi = f$ alterum $\varphi = \mathrm{B}[f]$ deduci potest, ex hoc, posito novo integrali in locum prioris, tertium $\varphi = \mathrm{B}^2[f]$ et ita porro. Sed quum constet aequationem (1.) plura quam $n-1$ integralia non habere a se independentia, habemus propositionem,

*si pro quovis ipsius i valore sit*

$$0 = A_1\frac{\partial B_i}{\partial x_1}+A_2\frac{\partial B_i}{\partial x_2}+\cdots+A_n\frac{\partial B_i}{\partial x_n}$$
$$-B_1\frac{\partial A_i}{\partial x_1}-B_2\frac{\partial A_i}{\partial x_2}-\cdots-B_n\frac{\partial A_i}{\partial x_n},$$

*atque*

$$0 = A_1\frac{\partial f}{\partial x_1}+A_2\frac{\partial f}{\partial x_2}+\cdots+A_n\frac{\partial f}{\partial x_n} = \mathrm{A}[f],$$

*inter functiones* $f$, $\mathrm{B}[f]$, $\mathrm{B}^2[f]$, ... $\mathrm{B}^{n-1}[f]$ *unam vel plures aequationes dari, quas ipsae* $x_1, x_2, \ldots x_n$ *non ingrediuntur.*

*Antecedentium in aequationes problematis propositi applicatio. Theorema generale expressiones* $[\varphi, \psi]$ *concernens.*

## 26.

Iam ipsis $A_1, A_2, \ldots A_n, B_1, B_2, \ldots B_n$ valores quosdam tribuamus particulares, quibus fit, ut expressiones $C_i$ omnes ejusdem functionis evadant differentialia partialia. Quam deinde expressionem si evanescere statuimus, etiam ipsae $C_i$ pro omnibus ipsius $i$ valoribus evanescunt, quae est conditio requisita. Initio autem generaliorem propositionem condam. In finem propositum pono

$$n = 2m,$$

atque in loco variabilium independentium $x_1, x_2, \dots x_n$ introduco systema duplex variabilium

$$\begin{array}{llll} q_1, & q_2, & \dots & q_m, \\ p_1, & p_2, & \dots & p_m, \end{array}$$

atque statuo:

$$\begin{aligned} \mathrm{A}[f] &= A_1^0\frac{\partial f}{\partial q_1}+A_2^0\frac{\partial f}{\partial q_2}+\dots+A_m^0\frac{\partial f}{\partial q_m} \\ &\quad +A_1^1\frac{\partial f}{\partial p_1}+A_2^1\frac{\partial f}{\partial p_2}+\dots+A_m^1\frac{\partial f}{\partial p_m}, \\ \mathrm{B}[f] &= B_1^0\frac{\partial f}{\partial q_1}+B_2^0\frac{\partial f}{\partial q_2}+\dots+B_m^0\frac{\partial f}{\partial q_m} \\ &\quad +B_1^1\frac{\partial f}{\partial p_1}+B_2^1\frac{\partial f}{\partial p_2}+\dots+B_m^1\frac{\partial f}{\partial p_m}; \end{aligned}$$

tandem sit:

$$\begin{gathered} \mathrm{AB}[f]-\mathrm{BA}[f] = \mathrm{C}[f] = \\ C_1^0\frac{\partial f}{\partial q_1}+C_2^0\frac{\partial f}{\partial q_2}+\dots+C_m^0\frac{\partial f}{\partial q_m} \\ +C_1^1\frac{\partial f}{\partial p_1}+C_2^1\frac{\partial f}{\partial p_2}+\dots+C_m^1\frac{\partial f}{\partial p_m}. \end{gathered}$$

His positis, fit

$$\begin{aligned} C_i^0 &= \sum_k\left\{A_k^0\frac{\partial B_i^0}{\partial q_k}+A_k^1\frac{\partial B_i^0}{\partial p_k}-B_k^0\frac{\partial A_i^0}{\partial q_k}-B_k^1\frac{\partial A_i^0}{\partial p_k}\right\}, \\ C_i^1 &= \sum_k\left\{A_k^0\frac{\partial B_i^1}{\partial q_k}+A_k^1\frac{\partial B_i^1}{\partial p_k}-B_k^0\frac{\partial A_i^1}{\partial q_k}-B_k^1\frac{\partial A_i^1}{\partial p_k}\right\}, \end{aligned}$$

siquidem ipsi $k$ sub signo $\Sigma$ valores $1, 2, \dots m$ tribuuntur. Iam ut expressiones sub signo $\Sigma$ evadant differentialia partialia ejusdem expressionis, statuo

$$\begin{aligned} A_k^0 &= \frac{\partial\varphi}{\partial p_k}, \quad A_k^1 = -\frac{\partial\varphi}{\partial q_k}, \\ B_k^0 &= \frac{\partial\psi}{\partial p_k}, \quad B_k^1 = -\frac{\partial\psi}{\partial q_k}, \end{aligned}$$

unde

$$A_k^0\frac{\partial B_i^0}{\partial q_k}-B_k^1\frac{\partial A_i^0}{\partial p_k} = \frac{\partial\varphi}{\partial p_k}\,\frac{\partial^2\psi}{\partial p_i\partial q_k}+\frac{\partial\psi}{\partial q_k}\,\frac{\partial^2\varphi}{\partial p_i\partial p_k} = \frac{\partial\,\dfrac{\partial\varphi}{\partial p_k}\,\dfrac{\partial\psi}{\partial q_k}}{\partial p_i}.$$

Unde etiam permutando $A$ et $B$, $\varphi$ et $\psi$ fit:

$$B_k^0 \frac{\partial A_i^0}{\partial q_k} - A_k^1 \frac{\partial B_i^0}{\partial p_k} = \frac{\partial \frac{\partial \varphi}{\partial q_k} \frac{\partial \psi}{\partial p_k}}{\partial p_i},$$

ideoque

$$C_i^0 = -\frac{\partial \Sigma_k \left\{ \frac{\partial \varphi}{\partial q_k} \frac{\partial \psi}{\partial p_k} - \frac{\partial \varphi}{\partial p_k} \frac{\partial \psi}{\partial q_k} \right\}}{\partial p_i}.$$

Permutando $p$ et $q$, unde simul permutari debent $A$ et $A^1$, $B$ et $B^1$, mutatur etiam $C_i^0$ in $C_i^1$. Unde formula praecedens suppeditat permutando $p$ et $q$:

$$C_i^1 = \frac{\partial \Sigma_k \left\{ \frac{\partial \varphi}{\partial q_k} \frac{\partial \psi}{\partial p_k} - \frac{\partial \varphi}{\partial p_k} \frac{\partial \psi}{\partial q_k} \right\}}{\partial q_i}.$$

*Designabo sequentibus per* $[f, \varphi]$ *expressionem sequentem:*

$$[f, \varphi] = \frac{\partial f}{\partial q_1} \frac{\partial \varphi}{\partial p_1} + \frac{\partial f}{\partial q_2} \frac{\partial \varphi}{\partial p_2} + \cdots + \frac{\partial f}{\partial q_m} \frac{\partial \varphi}{\partial p_m}$$
$$- \frac{\partial f}{\partial p_1} \frac{\partial \varphi}{\partial q_1} - \frac{\partial f}{\partial p_2} \frac{\partial \varphi}{\partial q_2} - \cdots - \frac{\partial f}{\partial p_m} \frac{\partial \varphi}{\partial q_m},$$

unde erit

$$[f, f] = 0, \quad [f, \varphi] = -[\varphi, f].$$

Qua introducta notatione iam erit pro iis, quos ipsis $A_i$, $B_i$, $A_i^1$, $B_i^1$ valores tribuimus:

$$\begin{aligned} \mathrm{A}[f] &= [f, \varphi], \\ \mathrm{B}[f] &= [f, \psi], \\ \mathrm{AB}[f] &= [[f, \psi], \varphi], \\ \mathrm{BA}[f] &= [[f, \varphi], \psi]. \end{aligned}$$

Porro

$$C_i^0 = -\frac{\partial [\varphi, \psi]}{\partial p_i}, \quad C_i^1 = \frac{\partial [\varphi, \psi]}{\partial q_i}.$$

Quibus substitutis valoribus fit

$$\mathrm{C}[f] = [[\varphi, \psi], f].$$

Unde tandem formula supra inventa

$$\mathrm{AB}[f] - \mathrm{BA}[f] = \mathrm{C}[f]$$

in hanc abit:

$$[[f, \psi], \varphi] - [[f, \varphi], \psi] = [[\varphi, \psi], f],$$

quae concinnius sic exhibetur:

$$[[f, \varphi], \psi] + [[\varphi, \psi], f] + [[\psi, f], \varphi] = 0,$$

sive habetur:

### Theorema V.

*Generaliter designetur, quaecunque sint $R$ et $S$ ipsarum $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ functiones, per signum $[R, S]$ haec expressio:*

$$[R,S] = \frac{\partial R}{\partial q_1}\frac{\partial S}{\partial p_1}+\frac{\partial R}{\partial q_2}\frac{\partial S}{\partial p_2}+\dots+\frac{\partial R}{\partial q_m}\frac{\partial S}{\partial p_m}$$
$$-\frac{\partial R}{\partial p_1}\frac{\partial S}{\partial q_1}-\frac{\partial R}{\partial p_2}\frac{\partial S}{\partial q_2}-\dots-\frac{\partial R}{\partial p_m}\frac{\partial S}{\partial q_m},$$

*ponatur*

$$[\varphi,\psi]=F,\quad [\psi,f]=\Phi,\quad [f,\varphi]=\Psi,$$

*erit identice:*

$$[F,f]+[\Phi,\varphi]+[\Psi,\psi] = 0.$$

Quod est gravissimum theorema.

*De systemate aequationum differentialium vulgarium, quod aequationi $[f,\varphi]=0$ respondeat, de eiusque tertio integrali e binis quibuslibet inveniendo.*

27.

Sit $f$ data functio, erit aequatio

$$[f,\varphi] = 0$$

aequatio differentialis partialis, cui functio $\varphi$ satisfacere debet. Atque notum est, obtineri omnes functiones $\varphi$ aequationi

$$(1.)\quad \begin{cases} 0=[f,\varphi]=\dfrac{\partial f}{\partial q_1}\dfrac{\partial \varphi}{\partial p_1}+\dfrac{\partial f}{\partial q_2}\dfrac{\partial \varphi}{\partial p_2}+\dots+\dfrac{\partial f}{\partial q_m}\dfrac{\partial \varphi}{\partial p_m} \\ \qquad -\dfrac{\partial f}{\partial p_1}\dfrac{\partial \varphi}{\partial q_1}-\dfrac{\partial f}{\partial p_2}\dfrac{\partial \varphi}{\partial q_2}-\dots-\dfrac{\partial f}{\partial p_m}\dfrac{\partial \varphi}{\partial q_m} \end{cases}$$

satisfacientes, si quaeruntur integralia systematis aequationum differentialium vulgarium sequentis:

$$(2.)\quad \begin{cases} dp_1 : dp_2 : \dots : dp_m : \quad dq_1 : \quad dq_2 : \dots : \quad dq_m = \\ \dfrac{\partial f}{\partial q_1}:\dfrac{\partial f}{\partial q_2}:\dots:\dfrac{\partial f}{\partial q_m}:-\dfrac{\partial f}{\partial p_1}:-\dfrac{\partial f}{\partial p_2}:\dots:-\dfrac{\partial f}{\partial p_m}. \end{cases}$$

Quoties enim huius systematis aequationum differentialium vulgarium integrale quodcunque est

$$\varphi = \text{Const.},$$

erit $\varphi$ functio aequationi (1.) satisfaciens. Iam sit

$$\psi = \text{Const.}$$

alterum integrale quodcunque aequationum (2.), erit identice:

$$[f,\varphi]=0,\quad [f,\psi]=0,$$

sive si notationem §. antecedentis adhibitam rursus adhibemus:

$$\Psi = 0, \qquad \Phi = 0.$$

Hoc autem casu aequatio identica theoremate V. proposita in hanc abit:

$$[f, F] = 0,$$

Unde sequitur, aequationum (2.) integrale quoque esse

$$F = [\varphi, \psi] = \text{Const.};$$

sive habetur

### Theorema VI.

*Sint*

$$\varphi = \text{Const.}, \qquad \psi = \text{Const.}$$

*duo integralia quaecunque aequationum:*

$$dp_1 : dp_2 : \cdots : dp_m : dq_1 : dq_2 : \cdots : dq_m = \frac{\partial f}{\partial q_1} : \frac{\partial f}{\partial q_2} : \cdots : \frac{\partial f}{\partial q_m} : -\frac{\partial f}{\partial p_1} : -\frac{\partial f}{\partial p_2} : \cdots : -\frac{\partial f}{\partial p_m},$$

*erit aequatio:*

$$\text{Const.} = [\varphi, \psi] = \frac{\partial \varphi}{\partial q_1}\frac{\partial \psi}{\partial p_1} + \frac{\partial \varphi}{\partial q_2}\frac{\partial \psi}{\partial p_2} + \cdots + \frac{\partial \varphi}{\partial q_m}\frac{\partial \psi}{\partial p_m} - \frac{\partial \varphi}{\partial p_1}\frac{\partial \psi}{\partial q_1} - \frac{\partial \varphi}{\partial p_2}\frac{\partial \psi}{\partial q_2} - \cdots - \frac{\partial \varphi}{\partial p_m}\frac{\partial \psi}{\partial q_m}$$

*tertium integrale eiusdem aequationum differentialium vulgarium systematis.*

*Dilucidationes circa theorema §. antecedente propositum.*

### 28.

Antecedentibus evenire potest, ut functio $[\varphi, \psi]$ in quantitatem constantem sive generalius in ipsarum $\varphi$ et $\psi$ functionem abeat, quo casu e duobus integralibus inventis tertium ratione quam theoremate praecedente indicavi non derivatur. Sed observo, hos casus tantum ut exceptionales considerandos esse. Generaliter dicere debemus *e duobus integralibus aequationum:*

$$dq_1 : dq_2 : \ldots : dq_m : dp_1 : dp_2 : \ldots : dp_m = \frac{\partial U}{\partial p_1} : \frac{\partial U}{\partial p_2} : \ldots : \frac{\partial U}{\partial p_m} : -\frac{\partial U}{\partial q_1} : -\frac{\partial U}{\partial q_2} : \ldots : -\frac{\partial U}{\partial q_m}$$

*deduci posse per solas differentiationes tertium, ex hoc combinato cum duobus propositis quartum et quintum, etc. etc., ita ut e datis duobus integralibus per solas operationes differentiationis per partes cuncta deducantur propositi aequationum differentialium vulgarium systematis integralia.* Scilicet, si aequationum propositarum integralia, quorum numerus $2m-1$, haec sunt:

$$u_1 = a_1, \quad u_2 = a_2, \quad \ldots \quad u_{2m-1} = a_{2m-1},$$

designantibus $a_1$, $a_2$, ... $a_{2m-1}$ constantes arbitrarias, quae ipsas functiones $u_1$, $u_2$, ... $u_{2m-1}$ non ingrediuntur, erit expressio *generalis* duorum integralium:

$$\Theta\,(u_1, u_2, \ldots u_{2m-1}) = \text{Const.},$$

$$\Theta_1(u_1, u_2, \ldots u_{2m-1}) = \text{Const.}$$

Ac, nisi functionibus $\Theta$, $\Theta_1$ formae quaedam particulares concilientur, semper eveniet, ut ex his duobus integralibus

$$\Theta = \text{Const.}, \quad \Theta_1 = \text{Const.}$$

per methodum theoremate praecedente propositam identidem repetitam cuncta integralia proveniant. Ac reapse semper infinitis modis bina ejusmodi integralia, $\Theta = \text{Const.}$, $\Theta_1 = \text{Const.}$, assignare licet, e quibus per operationes propositas cuncta reliqua derivari possunt. Id quod eo majoris momenti est, quum systema aequationum differentialium vulgarium propositum idem est, cuius integratio motum suppeditat numeri cuiuslibet punctorum materialium, quae viribus quibuscunque attractionum seu repulsionum sollicitantur, ac praeterea quibuscunque conditionibus subiecta sunt. Ad theoremata antecedentia V. et VI. perveni necessitate quadam coactus, dum inquirerem, quinam sit aequationum ($\alpha$.) §. 11 habitus et quaenam compositio, quibus eveniat, ut omnibus simul una eademque functione $p_{i+1}$ satisfacere liceat. Nam hoc fieri posse aliunde constabat, quum satis notum esset, extare functionem $V$ aequationi differentiali partiali propositae satisfacientem, quae $m-1$ constantes arbitrarias involvat, unde etiam patebat, inveniri posse praeter aequationem illam propositam alias $m-1$ aequationes inter quantitates $q_1$, $q_2$, ... $q_m$, $p_1$, $p_2$, ... $p_m$, totidem constantes arbitrarias involventes. Hinc rursus concludis, semper dari functionem $p_{i+1}$ aequationibus ($\alpha$.) omnibus simul satisfacientem eamque continere posse constantem arbitrariam. Inquirens autem in conditiones possibilitatis eiusmodi integrationis simultaneae quum ad theorema fundamentale VI. delapsus essem, ingenue fateor, theorema illud me per aliquod tempus pro invento plane novo habuisse. Quid enim magis mirum fingi potest ac paene fidem superans, quam quod inde sequitur et mox videbimus, *in omnibus problematibus mechanicis, in quibus virium vivarum conservatio locum habet, generaliter e duobus integralibus praeter principium illud inventis reliqua omnia absque ulla ulteriore* *integratione inveniri posse?* Hoc theorema quomodo notum crederes, quum in nullo Tractatu Mechanico, in nullo Tractatu Analytico, in quo de integratione aequationum differentialium agitur, reperiatur, quum tamen ubique tamquam summum Calculi Integralis inventum circumferri deberet. Attamen inventum illud — ipso nescio auctore dicam? — inde ab annis novem et viginti *) factum est ab ill. *Poisson,* quippe quod prorsus idem est atque propositio illa, in formulis eius perturbàtoriis, quibus differentialia elementorum perturbatorum lineariter exprimuntur per differentialia partialia functionis perturbatricis respectu elementorum sumta, coefficientes per quos differentialia illa partialia functionis perturbatricis multiplicantur, et quorum formatio eadem atque expressionum $[\varphi, \psi]$ a viro ill. inventa est, a tempore $t$ liberos esse sive solorum elementorum esse functiones. Quae propositio vix pro nova et memorabili habebatur; nam quum formulae perturbatoriae *Lagrangianae* et *Poissonianae* aliae aliarum inversae sint et quum de suis formulis ill. *Lagrange* iam coefficientium a tempore independentiam demonstrasset, res sponte de *Poissonianis* formulis patebat seu certe mathematicis nihil habere videbatur, quod admirationem movere possit. Scilicet sola formatio differentialium elementorum perturbatorum curabatur, et quum formulae *Lagrangianae* eum in finem commodiores censerentur, formulae *Poissonianae* et propositio illa stupenda nonnisi ut propter demonstrationis difficultatem memorabiles obiter citabantur. Nemo, quantum scio, suscepit, propositionem illam per se examinare, nullo ad theoriam perturbationum respectu habito, quod si quis fecisset, fugere eum non potuisset, quantum sit eius in tractando *imperturbato* problemate momentum, eamque esse totius Mechanicae Analyticae gravissimam propositionem cuius analoga per totum Calculum Integralem non extat. Ill. *Lagrange* dum in Mech. Anal. (Vol. II, sect. VIII, art. 6) memorat, in formulis perturbatoriis *Poissonianis* coefficientes differentialium partialium functionis perturbatricis a tempore independentes esse, „sed demonstratio directa“ addit, „huius proprietatis singularis fit perdifficilis, uti videre licet in pulchra commentatione Cl. *Poisson* inserta tomo VIII Diarii Scholae Polytechnicae, ac nemo unquam fortasse eam quaesitum ivisset, nisi antea constasset de veritate huius theorematis“. Videmus ipsum summum magistrum

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*) Commentatio citata lucem vidit mense Decembri anni 1809; unde iam commentationem, quam legis, sub finem anni 1838 scriptam esse conjicis. Quod etiam cum eo consentaneum est, quod formularum quarundam in hac commentatione traditarum in nota mentio fit sub die $21^{\text{mo}}$ mensis Novembris anni 1838 cum Academia scientiarum Berolinensi communicata. Cf. finem §. 70. *C.*

ne suspicatum quidem esse, quid sit id, quod re vera theorema singulare reddat. Habemus hic praeclarum exemplum, nisi animo praeformata sint problemata, fieri posse, ut vel ante oculos posita gravissima inventa non videamus. Formaverat ill. *Poisson* e binis integralibus per differentiationes partiales coefficientes formularum, quibus elementorum perturbatorum differentialia exprimuntur, eosque a tempore liberos esse docuit. Sed quum animi mathematicorum toti in formulas perturbatorias intenti essent, huius inventi id tantum ut memorabile notabatur, coefficientes formularum perturbatoriarum a tempore non pendere, non id multo magis admirabile, e binis integralibus per differentiationes partiales formari posse expressionem tertiam a tempore non pendentem. Cuiusmodi tamen expressio generaliter est tertium integrale. Putabatur ea propositio nihil novi suppeditare ultra *Lagrangiana* inventa, quum propositionis *Lagrangianae,* quae tamquam aequivalens considerabatur, in imperturbato problemate omnino nullus usus sit, nisi quod, uti ipse autor eiusmodi usum circumspiciens innuit, eius propositionis ope examinare liceat, an inventae expressiones coordinatarum per elementa et tempus iustae sint. At propositio, ad quam differentialia elementorum perturbatorum directe quaerens pervenit ill. *Poisson,* summi momenti est in indagandis ipsis problematis imperturbati integralibus, eique tamquam fundamento superstruere contingit theoriam plane novam integrationis problematum mechanicorum, in quibus principium conservationis virium vivarum valet, ac generalius omnium problematum, quae ad integrationem aequationis differentialis partialis primi ordinis revocari possunt, ad quae demonstrari potest etiam problemata *isoperimetrica* maxime generalia pertinere. Et quamvis fere totum hoc opusculum illo innitatur fundamento ac maxime versetur in enucleandis proprietatibus functionum $[\varphi, \psi]$, quae tertium integrale formatum e duobus propositis sive coefficientes formularum perturbatoriarum ab ill. *Poisson* traditarum suppeditant: tamen longe abesse credo, ut omnia exhauriat, quae ex hoc fonte in integrationem aequationum differentialium *dynamicarum* redundare possint, immo plurima gravissima curas posteriores exspectant.

Quum omnibus casibus et utile sit nec elegantia careat, propositiones omnes ad meras identitates revocare, theorema VI. tamquam Corollarium deduxi de aequatione identica nova et simplicissima, quam theoremate V. proposui et quae ad alias quoque quaestiones usui esse potest. Revertimur ad propositum.

*Demonstratio theorematis* IV.

29.

Ex theoremate VI. deducamus theorema IV., quod demonstratu propositum est, et quo nova methodus nostra aequationes differentiales partiales primi ordinis inter numerum quemcunque variabilium integrandi innitebatur.

Docet theorema VI., si identice sit:

$$[f, \varphi] = 0, \qquad [f, \psi] = 0,$$

fore etiam identice

$$[f, [\varphi, \psi]] = 0.$$

Unde etiam permutando $\varphi$ et $f$ sequitur, *si identice sit*

$$[\varphi, f] = 0, \qquad [\varphi, \psi] = 0,$$

*fore etiam*

$$[\varphi, [f, \psi]] = 0.$$

Designantibus $\varkappa$ et $\lambda$ binos quoscunque e numeris 1, 2, 3, ... $i$ inter se diversos, statuamus, functionem $f$ e variabilibus $q_1, q_2, \ldots q_i, p_1, p_2, \ldots p_i$ tantum continere duas $q_\varkappa$, $q_\lambda$, ac praeterea functioni $\varphi$ adhuc terminum $-p_\varkappa$, functioni $\psi$ terminum $-p_\lambda$ additione iunctum esse, ita ut sit:

$$\frac{\partial f}{\partial p_\varkappa} = \frac{\partial f}{\partial p_\lambda} = \frac{\partial \varphi}{\partial p_\lambda} = \frac{\partial \psi}{\partial p_\varkappa} = 0,$$

$$\frac{\partial \varphi}{\partial p_\varkappa} = \frac{\partial \psi}{\partial p_\lambda} = -1.$$

Hinc erit

$$(1.)\quad \begin{cases} [\varphi, f] = \dfrac{\partial \varphi}{\partial q_{i+1}}\dfrac{\partial f}{\partial p_{i+1}} + \dfrac{\partial \varphi}{\partial q_{i+2}}\dfrac{\partial f}{\partial p_{i+2}} + \cdots + \dfrac{\partial \varphi}{\partial q_m}\dfrac{\partial f}{\partial p_m} \\ \qquad + \dfrac{\partial f}{\partial q_\varkappa} - \dfrac{\partial \varphi}{\partial p_{i+1}}\dfrac{\partial f}{\partial q_{i+1}} - \dfrac{\partial \varphi}{\partial p_{i+2}}\dfrac{\partial f}{\partial q_{i+2}} - \cdots - \dfrac{\partial \varphi}{\partial p_m}\dfrac{\partial f}{\partial q_m}, \end{cases}$$

$$(2.)\quad \begin{cases} [\psi, f] = \dfrac{\partial \psi}{\partial q_{i+1}}\dfrac{\partial f}{\partial p_{i+1}} + \dfrac{\partial \psi}{\partial q_{i+2}}\dfrac{\partial f}{\partial p_{i+2}} + \cdots + \dfrac{\partial \psi}{\partial q_m}\dfrac{\partial f}{\partial p_m} \\ \qquad + \dfrac{\partial f}{\partial q_\lambda} - \dfrac{\partial \psi}{\partial p_{i+1}}\dfrac{\partial f}{\partial q_{i+1}} - \dfrac{\partial \psi}{\partial p_{i+2}}\dfrac{\partial f}{\partial q_{i+2}} - \cdots - \dfrac{\partial \psi}{\partial p_m}\dfrac{\partial f}{\partial q_m}, \end{cases}$$

porro

$$(3.)\quad \begin{cases} [\varphi, \psi] = -\dfrac{\partial \varphi}{\partial q_\lambda} + \dfrac{\partial \varphi}{\partial q_{i+1}}\dfrac{\partial \psi}{\partial p_{i+1}} + \dfrac{\partial \varphi}{\partial q_{i+2}}\dfrac{\partial \psi}{\partial p_{i+2}} + \cdots + \dfrac{\partial \varphi}{\partial q_m}\dfrac{\partial \psi}{\partial p_m} \\ \qquad + \dfrac{\partial \psi}{\partial q_\varkappa} - \dfrac{\partial \varphi}{\partial p_{i+1}}\dfrac{\partial \psi}{\partial q_{i+1}} - \dfrac{\partial \varphi}{\partial p_{i+2}}\dfrac{\partial \psi}{\partial q_{i+2}} - \cdots - \dfrac{\partial \varphi}{\partial p_m}\dfrac{\partial \psi}{\partial q_m}. \end{cases}$$

Quibus ipsarum

$$[\varphi, f], \quad [\psi, f], \quad [\varphi, \psi]$$

valoribus substitutis, docet propositio praecedens, *designantibus* $\varphi$ *et* $\psi$ *tales ipsarum* $q_\varkappa$, $q_\lambda$, $q_{i+1}$, $q_{i+2}$, ... $q_m$, $p_{i+1}$, $p_{i+2}$, ... $p_m$ *functiones, quae satisfaciant aequationi:*

$$0 = -\frac{\partial\varphi}{\partial q_\lambda} + \frac{\partial\varphi}{\partial q_{i+1}}\frac{\partial\psi}{\partial p_{i+1}} + \frac{\partial\varphi}{\partial q_{i+2}}\frac{\partial\psi}{\partial p_{i+2}} + \dots + \frac{\partial\varphi}{\partial q_m}\frac{\partial\psi}{\partial p_m}$$
$$\frac{\partial\psi}{\partial q_\varkappa} - \frac{\partial\varphi}{\partial p_{i+1}}\frac{\partial\psi}{\partial q_{i+1}} - \frac{\partial\varphi}{\partial p_{i+2}}\frac{\partial\psi}{\partial q_{i+2}} - \dots - \frac{\partial\varphi}{\partial p_m}\frac{\partial\psi}{\partial q_m},$$

*ubi sit*

$$f = F$$

*integrale aequationis:*

$$0 = \frac{\partial\varphi}{\partial q_{i+1}}\frac{\partial F}{\partial p_{i+1}} + \frac{\partial\varphi}{\partial q_{i+2}}\frac{\partial F}{\partial p_{i+2}} + \dots + \frac{\partial\varphi}{\partial q_m}\frac{\partial F}{\partial p_m}$$
$$+\frac{\partial F}{\partial q_\varkappa} - \frac{\partial\varphi}{\partial p_{i+1}}\frac{\partial F}{\partial q_{i+1}} - \frac{\partial\varphi}{\partial p_{i+2}}\frac{\partial F}{\partial q_{i+2}} - \dots - \frac{\partial\varphi}{\partial p_m}\frac{\partial F}{\partial q_m},$$

*fore expressionem*

$$F = \frac{\partial\psi}{\partial q_{i+1}}\frac{\partial f}{\partial p_{i+1}} + \frac{\partial\psi}{\partial q_{i+2}}\frac{\partial f}{\partial p_{i+2}} + \dots + \frac{\partial\psi}{\partial q_m}\frac{\partial f}{\partial p_m}$$
$$+\frac{\partial f}{\partial q_\lambda} - \frac{\partial\psi}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}} - \frac{\partial\psi}{\partial p_{i+2}}\frac{\partial f}{\partial q_{i+2}} - \dots - \frac{\partial\psi}{\partial p_m}\frac{\partial f}{\partial q_m},$$

*eiusdem aequationis alterum integrale.* Quod est theorema IV., siquidem loco $\varphi$ et $\psi$ scribitur $p_\varkappa$ et $p_\lambda$ atque $\varphi$ loco $F$. Unde iam, quae demonstranda restabant, demonstrata sunt.

*Quum antecedentia forma quarta conditionum integrabilitatis innitantur, iam ut problema variis rationibus condatur, reditur ad formam primam.*

## 30.

Methodo integrationis propositae dilucidationes addam.

Sit

$$f = a,$$

designante $a$ constantem, aequatio differentialis partialis proposita; inventae sunt per methodum propositam aequationes

$$f_1 = a_1, \quad f_2 = a_2, \quad \dots \quad f_{m-1} = a_{m-1},$$

e quibus propositae iunctis, determinandae erant $p_1$, $p_2$, ... $p_m$ ut ipsarum $q_1$, $q_2$, ... $q_m$ functiones. Eratque

$f$ functio ipsarum $p_1,\ p_2,\ p_3,\ p_4,\ \dots\dots\ p_m,\ q_1,\ q_2,\ \dots\ q_m,$
$f_1$ functio ipsarum $a,\ p_2,\ p_3,\ p_4,\ \dots\dots\ p_m,\ q_1,\ q_2,\ \dots\ q_m,$
$f_2$ functio ipsarum $a,\ a_1,\ p_3,\ p_4,\ \dots\dots\ p_m,\ q_1,\ q_2,\ \dots\ q_m,$
$f_3$ functio ipsarum $a,\ a_1,\ a_2,\ p_4,\ \dots\dots\ p_m,\ q_1,\ q_2,\ \dots\ q_m,$
. . . . . . . . . . . . . . . . . . . .
$f_{m-2}$ functio ipsarum $a,\ a_1,\ \dots\ a_{m-3},\ p_{m-1},\ p_m,\ q_1,\ q_2,\ \dots\ q_m,$
$f_{m-1}$ functio ipsarum $a,\ a_1,\ \dots\ a_{m-3},\ a_{m-2},\ p_m,\ q_1,\ q_2,\ \dots\ q_m.$

Quantitates $a_1,\ a_2,\ \dots\ a_{m-1}$ sunt constantes arbitrariae, $a$ est constans data, quam nullitati aequiparare licet, quam tamen uniformitatis gratia conservo. Determinatis ex aequationibus

$$f = a,\quad f_1 = a_1,\quad f_2 = a_2,\ \dots\ f_{i-1} = a_{i-1}$$

ipsis $p_1,\ p_2,\ \dots\ p_i$ per $p_{i+1},\ p_{i+2},\ \dots\ p_m,\ q_1,\ q_2,\ \dots\ q_m$, functio $f_i$ numero $i$ aequationum ($d$.) §. 17 identice satisfaciebat, unde etiam functio $p_{i+1}$, ex aequatione $f_i = a_i$ expressa per $p_{i+2},\ p_{i+3},\ \dots\ p_m,\ q_1,\ q_2,\ \dots\ q_m$, satisfaciebat $i$ aequationibus ($\alpha$.) §. 11. At si ope aequationum

$$f = a,\quad f_1 = a_1,\quad \dots\quad f_{i-1} = a_{i-1}$$

non $p_1,\ p_2,\ \dots\ p_i$ per reliquas quantitates $p_{i+1},\ p_{i+2},\ \dots\ p_m,\ q_1,\ q_2,\ \dots\ q_m$, sed, quemadmodum in theoremate I. §. 6 factum est, e primis $i$ quantitatum

$$p_1,\ p_2,\ p_3,\ \dots\ p_m,\ q_1,\ q_2,\ \dots\ q_m$$

unaquaeque per insequentes exhibetur, ita ut ex aequatione $f = a$ exprimatur $p_1$ per $p_2,\ p_3$ etc., e $f_1 = a_1$ exprimatur $p_2$ per $p_3,\ p_4$ etc., e $f_2 = a_2$ exprimatur $p_3$ per $p_4,\ p_5$ etc.: tum vidimus initio huius commentationis numerum $i$ aequationum ($\alpha$.) cum totidem convenire sequentibus:

$$(\alpha.)\quad \begin{cases} 0 = -\dfrac{\partial p_{i+1}}{\partial q_1} + \dfrac{\partial p_1}{\partial p_2}\dfrac{\partial p_{i+1}}{\partial q_2} + \dfrac{\partial p_1}{\partial p_3}\dfrac{\partial p_{i+1}}{\partial q_3} + \dots + \dfrac{\partial p_1}{\partial p_m}\dfrac{\partial p_{i+1}}{\partial q_m} \\ \qquad + \dfrac{\partial p_1}{\partial q_{i+1}} - \dfrac{\partial p_1}{\partial q_{i+2}}\dfrac{\partial p_{i+1}}{\partial p_{i+2}} - \dfrac{\partial p_1}{\partial q_{i+3}}\dfrac{\partial p_{i+1}}{\partial p_{i+3}} - \dots - \dfrac{\partial p_1}{\partial q_m}\dfrac{\partial p_{i+1}}{\partial p_m}, \\ 0 = -\dfrac{\partial p_{i+1}}{\partial q_2} + \dfrac{\partial p_2}{\partial p_3}\dfrac{\partial p_{i+1}}{\partial q_3} + \dfrac{\partial p_2}{\partial p_4}\dfrac{\partial p_{i+1}}{\partial q_4} + \dots + \dfrac{\partial p_2}{\partial p_m}\dfrac{\partial p_{i+1}}{\partial q_m} \\ \qquad + \dfrac{\partial p_2}{\partial q_{i+1}} - \dfrac{\partial p_2}{\partial q_{i+2}}\dfrac{\partial p_{i+1}}{\partial p_{i+2}} - \dfrac{\partial p_2}{\partial q_{i+3}}\dfrac{\partial p_{i+1}}{\partial p_{i+3}} - \dots - \dfrac{\partial p_2}{\partial q_m}\dfrac{\partial p_{i+1}}{\partial p_m}, \\ \dots\dots\dots\dots\dots\dots \\ 0 = -\dfrac{\partial p_{i+1}}{\partial q_i} + \dfrac{\partial p_i}{\partial p_{i+1}}\dfrac{\partial p_{i+1}}{\partial q_{i+1}} + \dfrac{\partial p_i}{\partial p_{i+2}}\dfrac{\partial p_{i+1}}{\partial q_{i+2}} + \dots + \dfrac{\partial p_i}{\partial p_m}\dfrac{\partial p_{i+1}}{\partial q_m} \\ \qquad + \dfrac{\partial p_i}{\partial q_{i+1}} - \dfrac{\partial p_i}{\partial q_{i+2}}\dfrac{\partial p_{i+1}}{\partial p_{i+2}} - \dfrac{\partial p_i}{\partial q_{i+3}}\dfrac{\partial p_{i+1}}{\partial p_{i+3}} - \dots - \dfrac{\partial p_i}{\partial q_m}\dfrac{\partial p_{i+1}}{\partial p_m}. \end{cases}$$

Quae sunt ipsae aequationes ($a.$) in theoremate I. propositae, siquidem in theoremate illo statuitur $k=i+1$ atque loco $i$ successive ponuntur numeri $1, 2, 3, \dots i$. E quibus ipsis aequationibus ($a.$) supra aequationes ($\alpha.$) deductae sunt.

*Introductis functionibus $f$, quae in solutione problematis singulae constantibus aequantur, aequationibus supra adhibitis forma communis $[f_i, f_k]=0$ conciliatur.*

31.

Multiplicemus $i$ aequationes praecedentes per

$$\frac{\partial f_i}{\partial p_{i+1}}\frac{\partial f}{\partial p_1},\quad \frac{\partial f_i}{\partial p_{i+1}}\frac{\partial f_1}{\partial p_2},\quad \dots\quad \frac{\partial f_i}{\partial p_{i+1}}\frac{\partial f_{i-1}}{\partial p_i},$$

atque sola in eas differentialia partialia functionum $f, f_1, f_2, \dots f_i$ introducamus, quod per aequationes $f=a,\ f_1=a_1,\ f_2=a_2,\ \dots\ f_i=a_i$ licet. Quo facto induent aequationes praecedentes hanc formam:

$$(a'.)\quad \begin{cases} 0 = \dfrac{\partial f}{\partial p_1}\dfrac{\partial f_i}{\partial q_1}+\dfrac{\partial f}{\partial p_2}\dfrac{\partial f_i}{\partial q_2}+\dfrac{\partial f}{\partial p_3}\dfrac{\partial f_i}{\partial q_3}+\dots+\dfrac{\partial f}{\partial p_m}\dfrac{\partial f_i}{\partial q_m} \\ \qquad -\dfrac{\partial f}{\partial q_{i+1}}\dfrac{\partial f_i}{\partial p_{i+1}}-\dfrac{\partial f}{\partial q_{i+2}}\dfrac{\partial f_i}{\partial p_{i+2}}-\dots-\dfrac{\partial f}{\partial q_m}\dfrac{\partial f_i}{\partial p_m}, \\ 0 = \dfrac{\partial f_1}{\partial p_2}\dfrac{\partial f_i}{\partial q_2}+\dfrac{\partial f_1}{\partial p_3}\dfrac{\partial f_i}{\partial q_3}+\dfrac{\partial f_1}{\partial p_4}\dfrac{\partial f_i}{\partial q_4}+\dots+\dfrac{\partial f_1}{\partial p_m}\dfrac{\partial f_i}{\partial q_m} \\ \qquad -\dfrac{\partial f_1}{\partial q_{i+1}}\dfrac{\partial f_i}{\partial p_{i+1}}-\dfrac{\partial f_1}{\partial q_{i+2}}\dfrac{\partial f_i}{\partial p_{i+2}}-\dots-\dfrac{\partial f_1}{\partial q_m}\dfrac{\partial f_i}{\partial p_m}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ 0 = \dfrac{\partial f_{i-1}}{\partial p_i}\dfrac{\partial f_i}{\partial q_i}+\dfrac{\partial f_{i-1}}{\partial p_{i+1}}\dfrac{\partial f_i}{\partial q_{i+1}}+\dfrac{\partial f_{i-1}}{\partial p_{i+2}}\dfrac{\partial f_i}{\partial q_{i+2}}+\dots+\dfrac{\partial f_{i-1}}{\partial p_m}\dfrac{\partial f_i}{\partial q_m} \\ \qquad -\dfrac{\partial f_{i-1}}{\partial q_{i+1}}\dfrac{\partial f_i}{\partial p_{i+1}}-\dfrac{\partial f_{i-1}}{\partial q_{i+2}}\dfrac{\partial f_i}{\partial p_{i+2}}-\dots-\dfrac{\partial f_{i-1}}{\partial q_m}\dfrac{\partial f_i}{\partial p_m}. \end{cases}$$

Quum functiones $f_k$ quantitates $p_1, p_2, \dots p_k$ non involvant, has aequationes omnes in eandem formam redigere licet sequentem:

$$(a''.)\quad \begin{cases} 0 = \dfrac{\partial f}{\partial p_1}\dfrac{\partial f_i}{\partial q_1} + \dfrac{\partial f}{\partial p_2}\dfrac{\partial f_i}{\partial q_2} + \cdots + \dfrac{\partial f}{\partial p_m}\dfrac{\partial f_i}{\partial q_m} \\ \qquad - \dfrac{\partial f}{\partial q_1}\dfrac{\partial f_i}{\partial p_1} - \dfrac{\partial f}{\partial q_2}\dfrac{\partial f_i}{\partial p_2} - \cdots - \dfrac{\partial f}{\partial q_m}\dfrac{\partial f_i}{\partial p_m}, \\ 0 = \dfrac{\partial f_1}{\partial p_1}\dfrac{\partial f_i}{d q_1} + \dfrac{\partial f_1}{\partial p_2}\dfrac{\partial f_i}{\partial q_2} + \cdots + \dfrac{\partial f_1}{\partial p_m}\dfrac{\partial f_i}{\partial q_m} \\ \qquad - \dfrac{\partial f_1}{\partial q_1}\dfrac{\partial f_i}{\partial p_1} - \dfrac{\partial f_1}{\partial q_2}\dfrac{\partial f_i}{\partial p_2} - \cdots - \dfrac{\partial f_1}{\partial q_m}\dfrac{\partial f_i}{\partial p_m}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \dfrac{\partial f_{i-1}}{\partial p_1}\dfrac{\partial f_i}{\partial q_1} + \dfrac{\partial f_{i-1}}{\partial p_2}\dfrac{\partial f_i}{\partial q_2} + \cdots + \dfrac{\partial f_{i-1}}{\partial p_m}\dfrac{\partial f_i}{\partial q_m} \\ \qquad - \dfrac{\partial f_{i-1}}{\partial q_1}\dfrac{\partial f_i}{\partial p_1} - \dfrac{\partial f_{i-1}}{\partial q_2}\dfrac{\partial f_i}{\partial p_2} - \cdots - \dfrac{\partial f_{i-1}}{\partial q_m}\dfrac{\partial f_i}{\partial p_m}. \end{cases}$$

Termini enim qui, ut eadem forma aequationum omnium sit, addendi erant, sua sponte evanescunt. E notatione antecedentibus proposita aequationes praecedentes sic exhibentur:

$$(a'''.)\quad 0 = [f_i, f],\quad 0 = [f_i, f_1],\quad 0 = [f_i, f_2],\quad \ldots\quad 0 = [f_i, f_{i-1}].$$

Consideremus unam aequationum praecedentium:

$$0 = [f_i, f_k],$$

in qua $k$ quemcunque e numeris $0, 1, 2, \ldots i-1$ designat. Designante $n$ unum e numeris $0, 1, \ldots i-1$, continebit sive altera sive utraque functio $f_i$, $f_k$ constantem $a_n$. Cuius in locum si ponimus functionem $f_n$ constanti illi aequivalentem, abit expressio $[f_i, f_k]$ in hanc:

$$[f_i, f_k] + \frac{\partial f_i}{\partial a_n}[f_n, f_k] + \frac{\partial f_k}{\partial a_n}[f_i, f_n],$$

quum expressio accedens

$$\frac{\partial f_i}{\partial a_n}\frac{\partial f_k}{\partial a_n}[f_n, f_n]$$

sponte evanescat. Sed ex aequationibus $(a'''.)$ et e systemate aequationum, quae ipsum systema aequationum $(a'''.)$ antecedunt, sive ad minores ipsius $i$ valores pertinent, sequitur:

$$[f_n, f_k] = 0,\qquad [f_i, f_n] = 0.$$

Unde videmus, aequationem

$$[f_i, f_k] = 0$$

eandem formam retinere, si in formandis differentialibus partialibus functionum $f_i$, $f_k$ in locum constantis alicuius $a_n$, quam functiones illae involvunt, functio aequivalens substituitur $f_n$. Si $n$ unus e numeris $k$, $k+1$, ... $i-1$, altera functio $f_k$ constantem $a_n$ non continet, quo igitur casu in demonstratione praecedente ponendum est $\frac{\partial f_k}{\partial a_n}=0$, sive termini in $\frac{\partial f_k}{\partial a_n}$ multiplicati rejiciendi.

Prorsus eadem ratione demonstratur, aequationem

$$[f_i, f_n] = 0$$

immutatam manere, si in altera functionum $f_i$, $f_k$ retineatur $a_n$, in altera in locum eius functio $f_n$ substituatur.

Si eodem modo cum reliquis constantibus arbitrariis agis, quas functiones $f_i$, $f_k$ involvunt, deducis propositionem generalem: *aequationes*

$$[f_i, f_k] = 0$$

*adhuc valere, si in altera aut in utraque functione $f_i$, $f_k$ ante differentiationes partiales instituendas in locum unius vel plurium vel omnium constantium arbitrariarum, quas continent, functiones aequivalentes substituantur, seu generalius, quascunque mutationes functiones $f_i$, $f_k$ ante differentiationes partiales factas auxilio aequationum $f=a$, $f_1=a_1$, ... $f_{m-1}=a_{m-1}$ subeant.* Quae propositio etiam e theoremate II. §. 12 derivari potuisset.

*E forma allata aequationes §. 14 $[H_i, H_k]=0$ denuo obtinentur. Systema aequationum differentialium vulgarium, cuius aequationes $H_i=$ Const. sunt integralia.*

### 32.

Si in aequationibus

$$f=a, \quad f_1=a_1, \quad f_2=a_2, \quad \ldots \quad f_{m-1}=a_{m-1}$$

e quaque functione $f_i$ ope aequationum

$$f=a, \quad f_1=a_1, \quad f_2=a_2, \quad \ldots \quad f_{i-1}=a_{i-1}$$

constantes $a$, $a_1$, $a_2$, ... $a_{i-1}$, quas $f_i$ continet, eliminantur atque functio inde proveniens vocatur

$$H_i = f_i,$$

obtinemus aequationes

$$H=a, \quad H_1=a_1, \quad H_2=a_2, \quad \ldots \quad H_{m-1}=a_{m-1},$$

in quibus $H_i$ sunt functiones ipsarum $p_1$, $p_2$, ... $p_m$, $q_1$, $q_2$, ... $q_m$ absque ullis constantibus arbitrariis. Pro illis autem functionibus aequationes

$$[H_i, H_k] = 0$$

*identicae* evadere debent, quum expressio ad laevam nullam constantem arbitrariam contineat. Quas aequationes supra §. 14. iam dedi, ubi $H_i$, $h_i$ loco $H_{i-1}$, $a_{i-1}$ scriptum erat. Adhibitis functionibus $H$ propositio antecedens sic enunciari potest:

*valere aequationem*

$$[H_i, H_k] = 0,$$

*quaecunque variabilium $p_1$, $p_2$, etc. e functionibus $H_i$, $H_k$ ope aequationum $H = a$, $H_1 = a_1$, ... $H_{m-1} = a_{m-1}$ ante differentiationes partiales instituendas eliminatae sint, sive quascunque mutationes ope harum aequationum functiones $H_i$, $H_k$ subierint.*

Ex aequationibus

$$[H, H_1] = 0, \quad [H, H_2] = 0, \quad \dots \quad [H, H_{m-1}] = 0$$

sequitur, aequationes

$$H = a, \quad H_1 = a_1, \quad H_2 = a_2, \quad \dots \quad H_{m-1} = a_{m-1}$$

esse $m$ integralia systematis aequationum differentialium vulgarium:

$$dq_1 : dq_2 : \dots : dq_m : dp_1 : dp_2 : \dots : dp_m =$$
$$\frac{\partial H}{\partial p_1} : \frac{\partial H}{\partial p_2} : \dots : \frac{\partial H}{\partial p_m} : -\frac{\partial H}{\partial q_1} : -\frac{\partial H}{\partial q_2} : \dots : -\frac{\partial H}{\partial q_m},$$

in quibus $H$ eadem est functio, quam supra $f$ vocavi. Unde etiam aequationes

$$f = a, \quad f_1 = a_1, \quad f_2 = a_2, \quad \dots \quad f_{m-1} = a_{m-1},$$

quae cum aequationibus illis conveniunt, tamquam systema $m$ aequationum integralium systematis aequationum differentialium vulgarium praecedentis considerari possunt. Sed quum systema hoc habeat $2m-1$ integralia, restat ut reliqua $m-1$ indagentur. Eum in finem observo sequentia.

*Systematis aequationum differentialium vulgarium propositi reliqua integralia investigantur.*

### 33.

Aequationes $f = a$, $f_1 = a_1$, ... $f_{m-1} = a_{m-1}$ sive aequationes $H = a$, $H_1 = a_1$, $H_2 = a_2$, ... $H_{m-1} = a_{m-1}$ ita formatae sunt, ut, expressis earum beneficio ipsis $p_1$, $p_2$, ... $p_m$ per $q_1$, $q_2$, ... $q_m$, expressio

$$p_1 dq_1 + p_2 dq_2 + \dots + p_m dq_m$$

evadat differentiale completum. Valores illi ipsarum $p_1$, $p_2$, ... $p_m$ praeter variabiles $q_1$, $q_2$, ... $q_m$ adhuc involvunt constantem $a$ et constantes arbitrarias $a_1$, $a_2$, ... $a_{m-1}$. Secundum quarum unam $a_i$, si expressionem

$$p_1 dq_1 + p_2 dq_2 + \dots + p_m dq_m$$

differentiamus, prodibit expressio

$$\frac{\partial p_1}{\partial a_i} dq_1 + \frac{\partial p_2}{\partial a_i} dq_2 + \dots + \frac{\partial p_m}{\partial a_i} dq_m,$$

quae et ipsa differentiale completum esse debet. Iam vero ex aequatione

$$f = a,$$

quae est aequatio differentialis partialis proposita, sequitur differentiando secundum $a_i$:

$$\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial a_i}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial a_i}+\cdots+\frac{\partial f}{\partial p_m}\frac{\partial p_m}{\partial a_i} = 0.$$

Unde ex aequationibus differentialibus vulgaribus propositis

$$dq_1 : dq_2 : \ldots : dq_m = \frac{\partial f}{\partial p_1} : \frac{\partial f}{\partial p_2} : \ldots : \frac{\partial f}{\partial p_m}$$

deducere possumus aequationem:

$$\frac{\partial p_1}{\partial a_i}dq_1+\frac{\partial p_2}{\partial a_i}dq_2+\cdots+\frac{\partial p_m}{\partial a_i}dq_m = 0,$$

in qua est expressio ad laevam differentiale completum. Quo integrato positisque in locum ipsius $a_i$ valoribus ejus $a_1$, $a_2$, ... $a_{m-1}$, prodeunt $m-1$ integralia nova quaesita:

$$\int\left\{\frac{\partial p_1}{\partial a_1}dq_1+\frac{\partial p_2}{\partial a_1}dq_2+\cdots+\frac{\partial p_m}{\partial a_1}dq_m\right\} = b_1,$$

$$\int\left\{\frac{\partial p_1}{\partial a_2}dq_1+\frac{\partial p_2}{\partial a_2}dq_2+\cdots+\frac{\partial p_m}{\partial a_2}dq_m\right\} = b_2,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\int\left\{\frac{\partial p_1}{\partial a_{m-1}}dq_1+\frac{\partial p_2}{\partial a_{m-1}}dq_2+\cdots+\frac{\partial p_m}{\partial a_{m-1}}dq_m\right\} = b_{m-1},$$

in quibus sunt $b_1$, $b_2$, ... $b_{m-1}$ constantes novae arbitrariae.

Systema aequationum differentialium vulgarium ita propositum est, ut differentialia variabilium datis quantitatibus existant proportionalia. Fingatur differentiale auxiliare $dt$, cuius ope quantitates proportionales evadant inter se aequales, unde systema propositum fit:

$$\frac{dq_1}{dt}=\frac{\partial f}{\partial p_1},\quad \frac{dq_2}{dt}=\frac{\partial f}{\partial p_2},\quad \ldots \quad \frac{dq_m}{dt}=\frac{\partial f}{\partial p_m},$$

$$\frac{dp_1}{dt}=-\frac{\partial f}{\partial q_1},\quad \frac{dp_2}{dt}=-\frac{\partial f}{\partial q_2},\quad \ldots \quad \frac{dp_m}{dt}=-\frac{\partial f}{\partial q_m}.$$

Hinc fit

$$\frac{\partial p_1}{\partial a}dq_1+\frac{\partial p_2}{\partial a}dq_2+\cdots+\frac{\partial p_2}{\partial a}dq_m$$

$$= \left\{\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial a}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial a}+\cdots+\frac{\partial f}{\partial p_m}\frac{\partial p_m}{\partial a}\right\}dt.$$

At differentiando aequationem propositam $f=a$ secundum $a$ prodit:

$$\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial a}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial a}+\cdots+\frac{\partial f}{\partial p_m}\frac{\partial p_m}{\partial a} = 1,$$

unde aequatio antecedens abit in sequentem:

$$\frac{\partial p_1}{\partial a}dq_1+\frac{\partial p_2}{\partial a}dq_2+\cdots+\frac{\partial p_2}{\partial a}dq_m = dt,$$

cuius pars ad laevam est differentiale exactum. Hinc videmus, ut quantitas auxiliaris $t$ obtineatur per solas quadraturas, non esse necessarium, ut omnes quantitates $p_1, p_2, \dots p_m, q_1, q_2, \dots q_m$ per unam ex earum numero exprimantur, atque tum ex una aequationum differentialium propositarum, ex. gr. ex aequatione

$$dt = \frac{dq_1}{\frac{\partial f}{\partial p_1}}$$

valor ipsius $t$ per quadraturam eruatur, sed expressis $p_1, p_2, \dots p_m$ ope aequationum $f=a,\ f_1=a_1,\ \dots\ f_{m-1}=a_{m-1}$ per $q_1, q_2, \dots q_m$, haberi $t$ per aequationem

$$t+b = \int\left\{\frac{\partial p_1}{\partial a}dq_1+\frac{\partial p_2}{\partial a}dq_2+\cdots+\frac{\partial p_m}{\partial a}dq_m\right\},$$

in qua $b$ est nova constans arbitraria.

*De antecedentibus theorema conditur. Designatis illis quae desiderabantur integralibus per $f'_i=b_i$, vel $H'_i=b_i$, expressionum $[H_i, H'_k]$, $[H'_i, H'_k]$ valores indagantur.*

## 34.

*Si $V$ est integrale aequationis differentialis partialis propositae,*

$$f(q_1, q_2, \dots q_m, p_1, p_2, \dots p_m) = a,$$

*quale invenitur per aequationem*

$$V = \int\{p_1dq_1+p_2dq_2+\cdots+p_mdq_m\},$$

*in qua $p_1, p_2, \dots p_m$ ope aequationum $f=a,\ f_1=a_1,\ f_2=a_2,\ \dots\ f_{m-1}=a_{m-1}$ per $q_1, q_2, \dots q_m$ expressae sunt, licet integralia aequationum differentialium vulgarium*

$$\frac{dq_1}{dt}=\frac{\partial f}{\partial p_1},\quad \frac{dq_2}{dt}=\frac{\partial f}{\partial p_2},\quad \dots\quad \frac{dq_m}{dt}=\frac{\partial f}{\partial p_m},$$

$$\frac{dp_1}{dt}=-\frac{\partial f}{\partial q_1},\quad \frac{dp_2}{dt}=-\frac{\partial f}{\partial q_2},\quad \dots\quad \frac{dp_m}{dt}=-\frac{\partial f}{\partial q_m},$$

*hoc modo repraesentare,*

$$\frac{\partial V}{\partial q_1}=p_1,\quad \frac{\partial V}{\partial q_2}=p_2,\quad \dots\quad \frac{\partial V}{\partial q_m}=p_m,$$

$$\frac{\partial V}{\partial a_1}=b_1,\quad \frac{\partial V}{\partial a_2}=b_2,\quad \dots\quad \frac{\partial V}{\partial a_{m-1}}=b_{m-1},\quad \frac{\partial V}{\partial a}=t+b,$$

*in quibus* $a, a_1, a_2, \dots a_{m-1}, b, b_1, b_2, \dots b_{m-1}$ *sunt* $2m$ *constantes arbitrariae.* Unde integratio completa est.

Theorema praecedens gravissimum, iam olim a me demonstratum, est amplificatio alius theorematis ab ill. *Hamilton* inventi, quo primus aequationes differentiales vulgares dynamicas ad aequationes differentiales partiales revocavit. Sed ille binas simul adhibuit aequationes differentiales partiales, quo praeter necessitatem problema intricabatur. Eratque eo tempore integratio aequationis differentialis partialis $f = a$ problema multo difficilius et quod multo plures postulabat integrationes quam integratio systematis aequationum differentialium vulgarium, quae simul sunt aequationes differentiales dynamicae

$$\frac{dq_i}{dt} = \frac{\partial f}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial f}{\partial q_i}.$$

Qua de re tum temporis vir ill. multo magis aequationum differentialium partialium integrationem quam dynamicam promovisse existimandus erat. Neque vero viri illustris merito derogatum esse volo. Summum enim videtur quum in omni scientia tum in analysi mathematica nexus novus patefactus inter ea, quae nullo vinculo videbantur coniuncta.

Statuamus, designante $i$ unum quemlibet e numeris $0, 1, 2, \dots m-1$:

$$\frac{\partial V}{\partial a_i} = \int \left\{ \frac{\partial p_1}{\partial a_i} dq_1 + \frac{\partial p_2}{\partial a_i} dq_2 + \dots + \frac{\partial p_m}{\partial a_i} dq_m \right\} = f'_i,$$

et quemadmodum supra (§. 32.) suppositum est, e functione $f_{i-1}$ prodire functionem $H_{i-1}$, si in functione illa loco constantium $a, a_1, a_2, \dots a_{i-2}$, quas continet, ponantur functiones $H, H_1, H_2, \dots H_{i-2}$, ita iam supponamus, e functionibus $f'_{i-1}$ prodire functiones $H'_{i-1}$, si loco constantium $a, a_1, a_2, \dots a_{m-1}$, quas continent, ponantur respective functiones $H, H_1, H_2, \dots H_{m-1}$, ita ut etiam $m$ functiones $H'_1, H'_2, \dots H'_{m-1}$ sint ipsarum $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ functiones, constantes $a, a_1, \dots a_{m-1}$ non continentes. Designantibus $i, k$ binos quoslibet e numeris $0, 1, 2, \dots m-1$, erat *identice*

$$[H_i, H_k] = 0;$$

iam valorem expressionum

$$[H_i, H'_k]$$

investigemus.

Ac primum observo, in expressione illa loco $H'_k$ poni posse functionem $f'_k$, e qua $H'_k$ obtinetur ponendo loco $a, a_1, a_2, \dots a_{m-1}$ functiones $f, f_1, f_2, \dots f_{m-1}$. Etenim, si eandem substitutionem facimus, postquam expressiones $[H_i, f'_k]$, $\frac{\partial f'_k}{\partial a}$, $\frac{\partial f'_k}{\partial a_1}$, ... $\frac{\partial f'_k}{\partial a_{m-1}}$ formatae sunt, *identice* fit, sicuti e formatione expressionis $[H_i, H'_k]$ facile sequitur:

$$[H_i, H'_k] = [H_i, f'_k] + \frac{\partial f'_k}{\partial a}[H_i, H] + \frac{\partial f'_k}{\partial a_1}[H_i, H_1] + \cdots + \frac{\partial f'_k}{\partial a_{m-1}}[H_i, H_{m-1}].$$

Unde, expressionibus

$$[H_i, H], \quad [H_i, H_1], \quad [H_i, H_2], \quad \ldots \quad [H_i, H_{m-1}]$$

identice evanescentibus, quum insuper functio $f'_k$ solas $q_1, q_2, \ldots q_m, a, a_1, \ldots a_{m-1}$ neque quantitates $p_1, p_2, \ldots p_m$ contineat:

$$[H_i, H'_k] = [H_i, f'_k] = -\left\{\frac{\partial H_i}{\partial p_1}\frac{\partial f'_k}{\partial q_1} + \frac{\partial H_i}{\partial p_2}\frac{\partial f'_k}{\partial q_2} + \cdots + \frac{\partial H_i}{\partial p_m}\frac{\partial f'_k}{\partial q_m}\right\}$$

$$= -\left\{\frac{\partial H_i}{\partial p_1}\frac{\partial p_1}{\partial a_k} + \frac{\partial H_i}{\partial p_2}\frac{\partial p_2}{\partial a_k} + \cdots + \frac{\partial H_i}{\partial p_m}\frac{\partial p_m}{\partial a_k}\right\}.$$

Unde habetur

$$[H_i, H'_k] = [H_i, f'_k] = -\frac{\partial H_i}{\partial a_k},$$

siquidem in functione $H_i$ in locum variabilium $p_1, p_2, \ldots p_m$ ad formandum differentiale partiale $\frac{\partial H_i}{\partial a_k}$ substituuntur eorum valores, qui obtinentur ex aequationibus:

$$H = a, \quad H_1 = a_1, \quad \ldots \quad H_{m-1} = a_{m-1},$$

*sive erit identice* $[H_i, H'_k] = 0$, *quoties* $i$ *et* $k$ *diversi sunt, atque* $= -1$, *si* $i = k$.

Quod attinet ad valores expressionum

$$[H'_i, H'_k],$$

primum observo, haberi per easdem considerationes, quibus antecedentibus usi sumus,

$$[H'_i, H'_k] = [H'_i, f'_k],$$

eamque aequationem identicam fieri, si post formatam expressionem $[H'_i, f'_k]$ in ea loco ipsarum $a, a_1, a_2, \ldots a_{m-1}$ restituamus functiones $H, H_1, \ldots H_{m-1}$. Quod si in fine operationum signis nostris indicatarum efficimus, fit identice:

$$[H'_i, f'_k] = [f'_i, f'_k] + \frac{\partial f'_i}{\partial a}[H, f'_k] + \frac{\partial f'_i}{\partial a_1}[H_1, f'_k] + \cdots + \frac{\partial f'_i}{\partial a_{m-1}}[H_{m-1}, f'_k];$$

porro antecedentibus vidimus, si $i$ et $k$ diversi sint, fieri:

$$[H_i, f'_k] = 0;$$

expressio autem $[f_i', f_k']$ in nihilum abit, quum neque $f_i'$ neque $f_k'$ quantitates $p_1, p_2, \dots p_m$ contineant; unde, si $i$ et $k$ inter se diversi sunt, fit

$$[H_i', H_k'] = [H_i', f_k'] = 0.$$

Aequatio autem $[H_i', H_k'] = 0$, si $i = k$, sponte patet. *Jam igitur aequationum differentialium*

$$\frac{dq_1}{dt} = \frac{\partial f}{\partial p_1}, \quad \frac{dq_2}{dt} = \frac{\partial f}{\partial p_2}, \quad \dots \quad \frac{dq_m}{dt} = \frac{\partial f}{\partial p_m},$$

$$\frac{dp_1}{dt} = -\frac{\partial f}{\partial q_1}, \quad \frac{dp_2}{dt} = -\frac{\partial f}{\partial q_2}, \quad \dots \quad \frac{dp_m}{dt} = -\frac{\partial f}{\partial q_m}$$

*2m integralia inventa*

$$f = H = a, \quad H_1 = a_1, \quad H_2 = a_2, \quad \dots \quad H_{m-1} = a_{m-1},$$

$$H' = b + t, \quad H_1' = b_1, \quad H_2' = b_2, \quad \dots \quad H_{m-1}' = b_{m-1}$$

*ita comparata sunt, ut tribuendo ipsis i et k valores* 1, 2, 3, ... *m, identice sit:*

$$[H_i, H_k] = 0, \qquad [H_i', H_k'] = 0,$$

*ac si i et k inter se diversi sunt,*

$$[H_i, H_k'] = 0,$$

*denique*

$$[H_i, H_i'] = -1.$$

Quae sunt propositiones in theoria nostra *fundamentales.*

### *De modificatione formularum praecedentium, qua opus est, si functio f ipsam continet variabilem t, quae supra tanquam auxiliaris spectabatur.*

### 35.

Supposuimus, in systemate aequationum propositarum:

$$(1.) \qquad \frac{dq_i}{dt} = \frac{\partial f}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial f}{\partial q_i},$$

functionem $f$ ipsas tantum $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ neque quantitatem $t$ continere. Sed facile iste casus, quo $f$ etiam $t$ continet, ad praecedentem revocatur. Statuamus enim in praecedentibus variabilibus $q_1, q_2, \dots q_m$ accedere variabilem $t$, unde posito

$$\frac{\partial V}{\partial t} = u$$

etiam statuendum erit:

$$(2.) \qquad dV = u\,dt + p_1 dq_1 + p_2 dq_2 + \dots + p_m dq_m.$$

Insuper in aequatione differentiali partiali proposita loco $f$ scribatur $u+f$, ita ut evadat illa:

$$(3.)\qquad u+f=a \quad \text{sive} \quad \frac{\partial V}{\partial t}=-f+a\ ^{*)},$$

functione $f$ involvente ipsam $t$ neque vero ipsam $u$. His statutis mutationibus formulae propositae sponte ad casum patent, quo $f$ ipsam $t$ involvit. Nam quum sit

$$\frac{\partial(u+f)}{\partial u}=1, \qquad \frac{\partial(u+f)}{\partial t}=\frac{\partial f}{\partial t},$$
$$\frac{\partial(u+f)}{\partial q_i}=\frac{\partial f}{\partial q_i}, \qquad \frac{\partial(u+f)}{\partial p_i}=\frac{\partial f}{\partial p_i},$$

aequationes differentiales vulgares, quarum integrationem vidimus §§. 32, 33 pendere ab integratione aequationis differentialis partialis propositae, hae evadunt:

$$dt:dq_1:dq_2:\ldots:dq_m:\ du\ :\ dp_1\ :\ dp_2:\ldots:\ dp_m\ =$$
$$1:\frac{\partial f}{\partial p_1}:\frac{\partial f}{\partial p_2}:\ldots:\frac{\partial f}{\partial p_m}:-\frac{\partial f}{\partial t}:-\frac{\partial f}{\partial q_1}:-\frac{\partial f}{\partial q_2}:\ldots:-\frac{\partial f}{\partial q_m},$$

quae eaedem sunt atque aequationes (1.), accedente, si placet, aequatione

$$(4.)\qquad \frac{du}{dt}=-\frac{\partial f}{\partial t}.$$

Praeter aequationem propositam $u+f=a$ sint

$$(5.)\qquad f_1=a_1,\quad f_2=a_2,\quad \ldots\quad f_m=a_m$$

integralia aequationum (1.) per methodum a me supra propositam indaganda. E quibus ipsae $u$, $p_1$, $p_2$, ... $p_m$ per $t$, $q_1$, $q_2$, ... $q_m$ ita determinantur, ut fiat

$$u\,dt+p_1dq_1+p_2dq_2+\cdots+p_mdq_m$$

expressio integrabilis. Numerus aequationum (5.) unitate maior est atque in quaestionibus praecedentibus, quum variabilibus independentibus $q_1$, $q_2$, ... $q_m$ accesserit nova variabilis $t$. Ac per regulam praescriptam erit $f_1=a_1$ integrale quodcunque aequationum (1.); in quibus quum neque $u$ neque constans $a$ obveniat, etiam $f_1$ neque $u$ neque $a$ continebit, idemque valebit de functionibus $f_2$, $f_3$, ... $f_m$. Quarum $f_i$ erit functio ipsarum $p_i$, $p_{i+1}$, ... $p_m$, $t$, $q_1$, $q_2$, ... $q_m$, $a_1$, $a_2$, ... $a_{i-1}$. Si in $f_2$ loco $a_1$ restituimus functionem $f_1$, prodeat $f_2=H_2$; si in $f_3$ loco $a_1$, $a_2$ restituimus functiones $f_1$, $H_2$, prodeat $f_3=H_3$, et ita porro. Unde generaliter fit $H_i$ functio ipsarum $p_1$, $p_2$, ... $p_m$, $t$, $q_1$, $q_2$, ... $q_m$ a constantibus arbitrariis vacua, quae ope aequationum $f_1=a_1$, $f_2=a_2$, ... $f_{i-1}=a_{i-1}$

*) Constantem $a$ per totam disquisitionem sequentem etiam $=0$ ponere licet.

ipsi $f_i$ aequalis evadit. In locum igitur aequationum (5.) hae quoque adhiberi possunt:

$$(6.) \qquad H_1 = a_1, \quad H_2 = a_2, \quad \ldots \quad H_m = a_m,$$

quae et ipsa erunt aequationum (1.) integralia. Inventis aequationibus (5.) earumque ope expressis $f, p_1, p_2, \ldots p_m$ per quantitates $t, q_1, q_2, \ldots q_m, a_1, a_2, \ldots a_m$, habentur e §. 34 reliqua integralia aequationum (1.):

$$(7.) \quad \begin{cases} \dfrac{\partial V}{\partial a_1} = \displaystyle\int \left\{ -\frac{\partial f}{\partial a_1} dt + \frac{\partial p_1}{\partial a_1} dq_1 + \frac{\partial p_2}{\partial a_1} dq_2 + \cdots + \frac{\partial p_m}{\partial a_1} dq_m \right\} = b_1, \\ \dfrac{\partial V}{\partial a_2} = \displaystyle\int \left\{ -\frac{\partial f}{\partial a_2} dt + \frac{\partial p_1}{\partial a_2} dq_1 + \frac{\partial p_2}{\partial a_2} dq_2 + \cdots + \frac{\partial p_m}{\partial a_2} dq_m \right\} = b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial V}{\partial a_m} = \displaystyle\int \left\{ -\frac{\partial f}{\partial a_m} dt + \frac{\partial p_1}{\partial a_m} dq_1 + \frac{\partial p_2}{\partial a_m} dq_2 + \cdots + \frac{\partial p_m}{\partial a_m} dq_m \right\} = b_m, \end{cases}$$

designantibus $b_1, b_2, \ldots b_m$ novas constantes arbitrarias.

Aequatio, quae e §. 34 addi potest,

$$\frac{\partial V}{\partial a} = t + b,$$

hic mere identica evadit; nam quum expressiones ipsarum $f, p_1, p_2, \ldots p_m$ constantem $a$ non contineant, atque sit $u = a - f$, eruimus differentiando (2.) secundum $a$ et integrando:

$$\frac{\partial V}{\partial a} = \int dt,$$

quod aequationem praecedentem sponte suppeditat.

Si in functionibus $\dfrac{\partial V}{\partial a_1}, \dfrac{\partial V}{\partial a_2}, \ldots \dfrac{\partial V}{\partial a_m}$ loco ipsarum $a_1, a_2, \ldots a_m$ ponimus functiones $H_1, H_2, \ldots H_m$, functiones inde prodeuntes vocemus rursus $H'_1, H'_2, \ldots H'_m$. Tum, pro his functionibus sicuti supra, valebunt aequationes

$$[H_i, H_k] = 0, \quad [H_i, H'_k] = 0, \quad [H'_i, H'_k] = 0,$$
$$[H_i, H'_i] = -1,$$

si quidem notatione

$$[\varphi, \psi]$$

semper designamus expressionem

$$[\varphi, \psi] = \frac{\partial \varphi}{\partial q_1}\frac{\partial \psi}{\partial p_1} + \frac{\partial \varphi}{\partial q_2}\frac{\partial \psi}{\partial p_2} + \cdots + \frac{\partial \varphi}{\partial q_m}\frac{\partial \psi}{\partial p_m}$$
$$- \frac{\partial \varphi}{\partial p_1}\frac{\partial \psi}{\partial q_1} - \frac{\partial \varphi}{\partial p_2}\frac{\partial \psi}{\partial q_2} - \cdots - \frac{\partial \varphi}{\partial p_m}\frac{\partial \psi}{\partial q_m}.$$

Quamquam enim casu, quem hic consideramus, variabilibus

$$q_1, \quad q_2, \quad \ldots \quad q_m, \quad p_1, \quad p_2, \quad \ldots \quad p_m$$

accedunt variabiles

$$t, \quad u,$$

unde videri posset, aequationibus praecedentibus accedere debere terminos e differentiatione secundum has variabiles provenientes: nullo tamen modo in formandis expressionibus

$$[H_i, H_k], \quad [H_i, H'_k], \quad [H'_i, H'_k]$$

opus est, ut habeamus respectum ad variabilem $t$ functiones $H_i$, $H'_i$ afficientem atque etiam respectu huius differentiationes partiales instituamus. Nam quum functiones $H_i$, $H'_i$ non contineant ipsam $u$, evanescunt termini, qui addendi forent,

$$\frac{\partial H_i}{\partial t}\frac{\partial H_k}{\partial u} - \frac{\partial H_k}{\partial t}\frac{\partial H_i}{\partial u},$$

$$\frac{\partial H_i}{\partial t}\frac{\partial H'_k}{\partial u} - \frac{\partial H'_k}{\partial t}\frac{\partial H_i}{\partial u},$$

$$\frac{\partial H'_i}{\partial t}\frac{\partial H'_k}{\partial u} - \frac{\partial H'_k}{\partial t}\frac{\partial H'_i}{\partial u}.$$

Si ponitur

$$u + f = H,$$

solis expressionibus

$$[H, H_i], \quad [H, H'_i]$$

termini accedunt provenientes e differentiatione respectu ipsarum $t$, $u$ instituta. Habetur enim, quum $f$ ipsam $u$ non involvat,

$$\frac{\partial H}{\partial u} = 1,$$

unde termini addendi valores obtinent sequentes:

$$\frac{\partial H}{\partial t}\frac{\partial H_i}{\partial u} - \frac{\partial H_i}{\partial t}\frac{\partial H}{\partial u} = -\frac{\partial H_i}{\partial t},$$

$$\frac{\partial H}{\partial t}\frac{\partial H'_i}{\partial u} - \frac{\partial H'_i}{\partial t}\frac{\partial H}{\partial u} = -\frac{\partial H'_i}{\partial t}.$$

Unde sequitur:

$$[H, H_i] - \frac{\partial H_i}{\partial t} = [f, H_i] - \frac{\partial H_i}{\partial t} = 0,$$

$$[H, H'_i] - \frac{\partial H'_i}{\partial t} = [f, H'_i] - \frac{\partial H'_i}{\partial t} = 0.$$

Quae formulae etiam inde deducuntur, quod sint aequationes

$$H_i = \text{Const.}, \qquad H'_i = \text{Const.}$$

integralia aequationum differentialium propositarum,

$$\frac{dq_i}{dt} = \frac{\partial f}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial f}{\partial q_i},$$

ita ut, substitutis his aequationibus, identice fiat

$$\frac{dH_i}{dt} = 0, \quad \frac{dH_i'}{dt} = 0,$$

quod formulas praecedentes suppeditat. Si ponimus analogiam notationis adhibitae servantes

$$t = \frac{\partial V}{\partial a} = H',$$

manent aequationes

$$[H', H_i] = 0, \quad [H', H_i'] = 0,$$

quum functiones $H'$, $H_i$, $H_i'$ ipsam $u$ non involvant, ideoque termini addendi evanescant. Quod attinet ad expressionem

$$[H, H'],$$

observo eam evanescere, quia $H'$ nullam contineat variabilium

$$q_1, \quad q_2, \quad \ldots \quad q_m, \quad p_1, \quad p_2, \quad \ldots \quad p_m$$

sed unicam $t$.

*Applicatio in aequationes dynamicas. Quae sub Lagrangiana forma proponuntur.*

36.

In formam aequationum differentialium vulgarium propositarum

$$\frac{dq_i}{dt} = \frac{\partial f}{\partial p_i}, \quad \frac{\partial p_i}{\partial t} = -\frac{\partial f}{\partial q_i}$$

aequationes differentiales dynamicas revocari posse omnibus casibus, quibus principium minimae actionis sive principium conservationis virium vivarum locum habeat, primus, quantum scio, ill. *Hamilton* docuit. Adstruam primum formulas dynamicas generales *Lagrangianas* ex iisque deinde formulas propositas deducam.

Proponantur $n$ puncta materialia, quorum massae $m_1, m_2, \ldots m_n$; sint $x_i$, $y_i$, $z_i$ coordinatae orthogonales puncti, cuius massa $m_i$; ac sollicitetur punctum illud secundum directiones axium coordinatarum viribus $X_i$, $Y_i$, $Z_i$, erunt problemata mechanica, quae hic consideramus et pro quibus dicta principia valent, ea, in quibus expressio

$$\Sigma m_i \{X_i dx_i + Y_i dy_i + Z_i dz_i\},$$

extensa ad omnia $n$ corpora, est differentiale completum. Cuius integrale si vocatur $U$, erunt aequationes differentiales dynamicae contentae hac aequatione symbolica

$$\Sigma m_i \left\{ \frac{d^2 x_i}{dt^2} \delta x_i + \frac{d^2 y_i}{dt^2} \delta y_i + \frac{d^2 z_i}{dt^2} \delta z_i \right\} = \delta U,$$

cui satisfieri debet per variationes omnes virtuales $\delta x_i$, $\delta y_i$, $\delta z_i$, sive per variationes conditiones non turbantes, quibus $n$ puncta materialia subiecta sunt. Id quod docuit olim ill. *Lagrange* combinando principium *d'Alemberti* cum principio velocitatum virtualium. Posito autem

$$\frac{dx_i}{dt} = x'_i, \qquad \frac{dy_i}{dt} = y'_i, \qquad \frac{dz_i}{dt} = z'_i,$$

fit aequatio illa symbolica:

$$\frac{d\Sigma m_i \{ x'_i \delta x_i + y'_i \delta y_i + z'_i \delta z_i \}}{dt} - \Sigma m_i \{ x'_i \delta x'_i + y'_i \delta y'_i + z'_i \delta z'_i \} = \delta U,$$

quae, posita semissi virium vivarum

$$\tfrac{1}{2} \Sigma m_i \{ x'_i x'_i + y'_i y'_i + z'_i z'_i \} = T,$$

sic etiam repraesentari potest:

$$\frac{d\Sigma m_i \{ x'_i \delta x_i + y'_i \delta y_i + z'_i \delta z_i \}}{dt} = \delta (U+T).$$

Statuendo $U+T=R$, hanc aequationem ita exhibeamus:

$$\delta R = \frac{d\Sigma_i \left\{ \frac{\partial R}{\partial x'_i} \delta x_i + \frac{\partial R}{\partial y'_i} \delta y_i + \frac{\partial R}{\partial z'_i} \delta z_i \right\}}{dt},$$

id quod licet, quum $U$ quantitates $x'_i$, $y'_i$, $z'_i$ omnino non contineat. Exprimamus $3n$ quantitates $x_i$, $y_i$, $z_i$ per $m$ alias quantitates $q_1, q_2, \ldots q_m$ sitque rursus

$$q'_i = \frac{dq_i}{dt},$$

facile probatur expressa etiam $R$ per $q_1, q_2, \ldots q_m, q'_1, q'_2, \ldots q'_m$ fieri:

$$(1.) \quad \begin{cases} \Sigma_i \left\{ \frac{\partial R}{\partial x'_i} \delta x_i + \frac{\partial R}{\partial y'_i} \delta y_i + \frac{\partial R}{\partial z'_i} \delta z_i \right\} = \\ \frac{\partial R}{\partial q'_1} \delta q_1 + \frac{\partial R}{\partial q'_2} \delta q_2 + \cdots + \frac{\partial R}{\partial q'_m} \delta q_m . \end{cases}$$

Habetur enim

$$\Sigma_i\left\{\frac{\partial R}{\partial x_i'}\,\delta x_i+\frac{\partial R}{\partial y_i'}\,\delta y_i+\frac{\partial R}{\partial z_i'}\,\delta z_i\right\}=$$

$$\Sigma_i\left\{\frac{\partial R}{\partial x_i'}\frac{\partial x_i}{\partial q_1}+\frac{\partial R}{\partial y_i'}\frac{\partial y_i}{\partial q_1}+\frac{\partial R}{\partial z_i'}\frac{\partial z_i}{\partial q_1}\right\}\delta q_1+$$

$$\Sigma_i\left\{\frac{\partial R}{\partial x_i'}\frac{\partial x_i}{\partial q_2}+\frac{\partial R}{\partial y_i}\frac{\partial y_i}{\partial q_2}+\frac{\partial R}{\partial z_i'}\frac{\partial z_i}{\partial q_2}\right\}\delta q_2+$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\Sigma_i\left\{\frac{\partial R}{\partial x_i'}\frac{\partial x_i}{\partial q_m}+\frac{\partial R}{\partial y_i'}\frac{\partial y_i}{\partial q_m}+\frac{\partial R}{\partial z_i'}\frac{\partial z_i}{\partial q_m}\right\}\delta q_m.$$

At quum sit

$$x_i'=\frac{\partial x_i}{\partial q_1}q_1'+\frac{\partial x_i}{\partial q_2}q_2'+\cdots+\frac{\partial x_i}{\partial q_m}q_m',$$

erit

$$\frac{\partial x_i'}{\partial q_k'}=\frac{\partial x_i}{\partial q_k},\ \text{ac similiter}\ \frac{\partial y_i'}{\partial q_k'}=\frac{\partial y_i}{\partial q_k},\ \frac{\partial z_i'}{\partial q_k'}=\frac{\partial z_i}{\partial q_k},$$

unde

$$\Sigma_i\left\{\frac{\partial R}{\partial x_i'}\frac{\partial x_i}{\partial q_k}+\frac{\partial R}{\partial y_i'}\frac{\partial y_i}{\partial q_k}+\frac{\partial R}{\partial z_i'}\frac{\partial z_i}{\partial q_k}\right\}=$$

$$\Sigma_i\left\{\frac{\partial R}{\partial x_i'}\frac{\partial x_i'}{\partial q_k'}+\frac{\partial R}{\partial y_i'}\frac{\partial y_i'}{\partial q_k'}+\frac{\partial R}{\partial z_i'}\frac{\partial z_i'}{\partial q_k'}\right\}=\frac{\partial R}{\partial q_k'},$$

ideoque

$$\Sigma_i\left\{\frac{\partial R}{\partial x_i'}\delta x_i+\frac{\partial R}{\partial y_i'}\delta y_i+\frac{\partial R}{\partial z_i'}\delta z_i\right\}=\Sigma_k\frac{\partial R}{\partial q_k'}\delta q_k,$$

quod probandum erat. Habemus igitur, expressa $R$ per novas quantitates $q_k$, $q_k'$, aequationem

$$\delta R=\frac{d\left\{\frac{\partial R}{\partial q_1'}\delta q_1+\frac{\partial R}{\partial q_2'}\delta q_2+\cdots+\frac{\partial R}{\partial q_m'}\delta q_m\right\}}{dt}.$$

Si $\pi$ est numerus aequationum conditionalium, quibus $3n$ coordinatae satisfacere debent, fieri debet $m$ aut $=3n-\pi$ aut $>3n-\pi$. Si $m=3n-\pi+\nu$, designante $\nu$ numerum positivum, habetur inter quantitates $q_1, q_2, \ldots q_m$ numerus $\nu$ aequationum conditionalium, unde totidem emergunt inter variationes $\delta q_1$, $\delta q_2$, ... $\delta q_m$ aequationes conditionales. Ac primum quidem, existente $m=3n-\pi$, quantitates $q_1, q_2, \ldots q_m$ a se invicem independentes sunt ideoque variationes $\delta q_1, \delta q_2, \ldots \delta q_m$ prorsus arbitrariae. Hoc igitur casu ex aequatione praecedenti symbolica hoc fluit aequationum differentialium systema:

$$\frac{\partial R}{\partial q_1}=\frac{d\frac{\partial R}{\partial q_1'}}{dt},\quad \frac{\partial R}{\partial q_2}=\frac{d\frac{\partial R}{\partial q_2'}}{dt},\quad \ldots\quad \frac{\partial R}{\partial q_m}=\frac{d\frac{\partial R}{\partial q_m'}}{dt}.$$

Hac forma aequationes differentiales dynamicae iam in editione prima *Mechanicae Lagrangianae* propositae inveniuntur.

*Forma Hamiltoniana aequationum dynamicarum derivatur; quae cum systemate supra considerato congruit. Theorema VI. de tertio integrali e binis quibuslibet inveniendo applicatur in systema dynamicum.*

37.

At ill. *Poisson* in laudatissima commentatione de *Variatione Constantium* (Journal de l'Ecole Polyt. Cah. XV) loco quantitatum $q'_1, q'_2, \dots q'_m$ alias in formulas dynamicas introduxit,

$$p_1 = \frac{\partial R}{\partial q'_1}, \quad p_2 = \frac{\partial R}{\partial q'_2}, \quad \dots \quad p_m = \frac{\partial R}{\partial q'_m}.$$

Quibus ipsis variabilibus adhibitis in locum functionis $R$ ill. *Hamilton* novam introduxit functionem:

$$\begin{aligned} H &= \frac{\partial R}{\partial q'_1} q'_1 + \frac{\partial R}{\partial q'_2} q'_2 + \dots + \frac{\partial R}{\partial q'_m} q'_m - R \\ &= p_1 q'_1 + p_2 q'_2 + \dots + p_m q'_m - R. \end{aligned}$$

Qua functione per $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ expressa, ubi simul has omnes quantitates variamus, obtinemus:

$$\begin{aligned} \delta H = &\; q'_1 \delta p_1 + q'_2 \delta p_2 + \dots + q'_m \delta p_m \\ &- \frac{\partial R}{\partial q_1} \delta q_1 - \frac{\partial R}{\partial q_2} \delta q_2 - \dots - \frac{\partial R}{\partial q_m} \delta q_m. \end{aligned}$$

Evanescit enim expressio

$$\begin{aligned} &p_1 \delta q'_1 + p_2 \delta q'_2 + \dots + p_m \delta q'_m \\ &- \left\{ \frac{\partial R}{\partial q'_1} \delta q'_1 + \frac{\partial R}{\partial q'_2} \delta q'_2 + \dots + \frac{\partial R}{\partial q'_m} \delta q'_m \right\}. \end{aligned}$$

Expressio ipsius $\delta H$ inventa iam suppeditat differentialia partialia functionis $H$ secundum novas variabiles sumta sequentia:

$$\begin{aligned} &\frac{\partial H}{\partial p_1} = q'_1, \quad \frac{\partial H}{\partial p_2} = q'_2, \quad \dots \quad \frac{\partial H}{\partial p_m} = q'_m, \\ &\frac{\partial H}{\partial q_1} = -\frac{\partial R}{\partial q_1}, \quad \frac{\partial H}{\partial q_2} = -\frac{\partial R}{\partial q_2}, \quad \dots \quad \frac{\partial H}{\partial q_m} = -\frac{\partial R}{\partial q_m}, \end{aligned}$$

Quibus valoribus substitutis, vice versa $R$ e functione $H$ obtinetur per formulam:

$$\begin{aligned} R &= p_1 q'_1 + p_2 q'_2 + \dots + p_m q'_m - H \\ &= p_1 \frac{\partial H}{\partial p_1} + p_2 \frac{\partial H}{\partial p_2} + \dots + p_m \frac{\partial H}{\partial p_m} - H. \end{aligned}$$

Introductis igitur quantitatibus $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ ut variabilibus, aequatio inventa symbolica iam haec evadit:

$$\delta\left\{p_1\frac{\partial H}{\partial p_1}+p_2\frac{\partial H}{\partial p_2}+\dots+p_m\frac{\partial H}{\partial p_m}-H\right\} = \frac{d\{p_1\delta q_1+p_2\delta q_2+\dots+p_m\delta q_m\}}{dt}.$$

Facta variatione et differentiatione et substitutis valoribus

$$q'_k = \frac{\partial H}{\partial p_k}$$

venit in aequationis praecedentis utraque parte expressio

$$p_1\delta\cdot\frac{\partial H}{\partial p_1}+p_2\delta\cdot\frac{\partial H}{\partial p_2}+\dots+p_m\delta\cdot\frac{\partial H}{\partial p_m},$$

qua reiecta, hanc nanciscimur formulam:

$$(1.)\quad \begin{cases} -\left\{\frac{\partial H}{\partial q_1}\delta q_1+\frac{\partial H}{\partial q_2}\delta q_2+\dots+\frac{\partial H}{\partial q_m}\delta q_m\right\} = \\ \frac{dp_1}{dt}\delta q_1+\frac{dp_2}{dt}\delta q_2+\dots+\frac{dp_m}{dt}\delta q_m. \end{cases}$$

Si $m = 3n-\pi$, designante $\pi$ numerum aequationum conditionalium, quibus coordinatae punctorum materialium satisfacere debent, quantitates $q_1, q_2, \dots q_m$ a se invicem prorsus independentes sunt, earumque variationes $\delta q_1, \delta q_2, \dots \delta q_m$ omnes arbitrariae. Quo casu ex (1.) fluunt aequationes differentiales *dynamicae* in variabilibus $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ exhibitae:

$$(2.)\quad \begin{cases} \frac{\partial H}{\partial p_1}=\frac{dq_1}{dt}, & \frac{\partial H}{\partial p_2}=\frac{dq_2}{dt}, & \dots & \frac{\partial H}{\partial p_m}=\frac{dq_m}{dt}, \\ -\frac{\partial H}{\partial q_1}=\frac{dp_1}{dt}, & -\frac{\partial H}{\partial q_2}=\frac{dp_2}{dt}, & \dots & -\frac{\partial H}{\partial q_m}=\frac{dp_m}{dt}. \end{cases}$$

Qua forma primus ill. *Hamilton* aequationes *dynamicas* exhibuit, neque parum inde commodi in Mechanicam Analyticam redundasse existimo. Observaverat iam l. c. ill. *Poisson*, valores ipsarum $\frac{dq_i}{dt} = q'_i$ per quantitates $q_i, p_i$ expressos ita comparatos esse, ut habeatur

$$\frac{\partial q'_i}{\partial p_k} = \frac{\partial q'_k}{\partial p_i},$$

(Journal de l'Ecole Polyt. Cah. XV, pag. 275), quae ad finem sibi propositum sufficiebant formulae. E formulis (2.) statim etiam sequentes fluunt:

$$\frac{\partial q'_i}{\partial q_k} = -\frac{\partial p'_k}{\partial p_i}, \quad \frac{\partial p'_i}{\partial q_k} = \frac{\partial p'_k}{\partial q_i},$$

siquidem rursus $p'_i = \frac{dp_i}{dt}$ ponimus.

Forma *Hamiltoniana* aequationum differentialium dynamicarum eadem est atque systematis aequationum differentialium vulgarium, cuius integrationem supra docui §§. 33, 34.

Si in aequatione supra (§. 36 (1.)) probata

$$\Sigma_i\left\{\frac{\partial R}{\partial x_i'}\delta x_i+\frac{\partial R}{\partial y_i'}\delta y_i+\frac{\partial R}{\partial z_i'}\delta z_i\right\} = \Sigma_k\frac{\partial R}{\partial q_k'}\delta q_k,$$

in locum ipsorum $\delta x_i$, $\delta y_i$, $\delta z_i$, $\delta q_k$ simul ponimus $x_i'$, $y_i'$, $z_i'$, $q_k'$, quod licet, quum aequationes conditionales supponantur ipsam $t$ non involvere, eruimus:

$$\Sigma\left\{\frac{\partial R}{\partial x_i'}x_i'+\frac{\partial R}{\partial y_i'}y_i'+\frac{\partial R}{\partial z_i'}z_i'\right\} = \Sigma\frac{\partial R}{\partial q_k'}q_k'.$$

Unde substituto valore $R = T+U$, quum sit

$$\frac{\partial R}{\partial x_i'} = \frac{\partial T}{\partial x_i'} = m_i x_i', \qquad \frac{\partial R}{\partial y_i'} = \frac{\partial T}{\partial y_i'} = m_i y_i', \qquad \frac{\partial R}{\partial z_i'} = \frac{\partial T}{\partial z_i'} = m_i z_i',$$

prodit:

$$2T = \Sigma\frac{\partial R}{\partial q_k'}q_k' = H+R = H+T+U.$$

In applicationibus igitur ad *dynamicam* est functio $H$ ipsi $T-U$ aequalis, unde aequatio

$$H = h,$$

in qua $h$ est constans arbitraria, est ipsa aequatio *conservationem virium vivarum* concernens.

Docet theorema VI. supra probatum, si habentur aequationum (2.) duo alia integralia quaecunque

$$\varphi = a, \qquad \psi = b,$$

in quibus $a$ et $b$ sunt constantes arbitrariae ipsas $\varphi$ et $\psi$ non afficientes, inde generaliter deduci integrale novum:

$$\begin{aligned}\text{Const.} = [\varphi, \psi] = & \frac{\partial\varphi}{\partial q_1}\frac{\partial\psi}{\partial p_1}+\frac{\partial\varphi}{\partial q_2}\frac{\partial\psi}{\partial p_2}+\cdots+\frac{\partial\varphi}{\partial q_m}\frac{\partial\psi}{\partial p_m} \\ & -\frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial q_1}-\frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_2}-\cdots-\frac{\partial\varphi}{\partial p_m}\frac{\partial\psi}{\partial q_m}.\end{aligned}$$

*Exponitur problema de expressione $[\varphi, \psi]$ per maiorem variabilium numerum exhibenda, inter quas aequationes conditionales datae sunt.*

## 38.

Quum propter rei utilitatem tum propter egregiam eius difficultatem tum quia accurate examinare iuvat, quaecunque spectant ad expressionem $[\varphi, \psi]$ tantis proprietatibus gaudentem, investigabo hic expressionem, quam induit $[\varphi, \psi]$, si in ea loco quantitatum $q_1, q_2, \dots q_m$ a se independentium restituuntur $3n$ coordinatae $x_i, y_i, z_i$, quae datis conditionibus quibuscunque satisfacere debent, sive generalius introducitur maior numerus variabilium $\xi_1, \xi_2, \dots \xi_\mu$, inter quas numerus $\mu - m$ relationum locum habent. In hoc problemate supponitur, ipsas $\varphi$ et $\psi$ ut functiones ipsarum $\xi_1, \xi_2, \dots \xi_\mu$ et quantitatum

$$\xi_1' = \frac{d\xi_1}{dt}, \quad \xi_2' = \frac{d\xi_2}{dt}, \quad \dots \quad \xi_\mu' = \frac{d\xi_\mu}{dt}$$

datas esse, atque ipsius $[\varphi, \psi]$ expressio investiganda talis esse debet, ut nullae in ea inveniantur quantitates, ad quarum formationem efficiendam datae esse debent relationes, quibus quantitates $\xi_1, \xi_2, \dots \xi_\mu$ et $q_1, q_2, \dots q_m$ aliae per alias determinantur, ita ut in formula investiganda nulla variabilium $q_1, q_2, \dots q_m$ vestigia remaneant. Iam problema accuratius exponam.

Problema propositum hoc est:

Sint inter $\mu$ quantitates $\xi_1, \xi_2, \dots \xi_\mu$ datae aequationes conditionales numero $\mu - m$,

$$F = 0, \quad \Phi = 0, \quad \text{etc.},$$

unde etiam inter ipsas $\xi_1', \xi_2', \dots \xi_\mu'$ habentur aequationes

$$\frac{dF}{dt} = \frac{\partial F}{\partial \xi_1}\xi_1' + \frac{\partial F}{\partial \xi_2}\xi_2' + \dots + \frac{\partial F}{\partial \xi_\mu}\xi_\mu' = 0,$$

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial \xi_1}\xi_1' + \frac{\partial \Phi}{\partial \xi_2}\xi_2' + \dots + \frac{\partial \Phi}{\partial \xi_\mu}\xi_\mu' = 0,$$

$$\dots\dots\dots\dots\dots$$

Quum inter $\mu$ quantitates $\xi_1, \xi_2, \dots \xi_\mu$ datae sint $\mu - m$ aequationes conditionales, exprimi possunt quantitates illae per $m$ quantitates $q_1, q_2, \dots q_m$ a se invicem independentes. Unde posito

$$q_i' = \frac{dq_i}{dt},$$

exprimi etiam possunt quantitates $\xi_1', \xi_2', \dots \xi_\mu'$ per $q_1, q_2, \dots q_m, q_1', q_2', \dots q_m'$ ope formularum

$$\xi_i' = \frac{\partial \xi_i}{\partial q_1}q_1' + \frac{\partial \xi_i}{\partial q_2}q_2' + \dots + \frac{\partial \xi_i}{\partial q_m}q_m'.$$

Sit etiam $R$ functio quaecunque ipsarum $\xi_1, \xi_2, \dots \xi_\mu, \xi_1', \xi_2', \dots \xi_\mu'$, atque

$$H = \xi_1'\frac{\partial R}{\partial \xi_1'} + \xi_2'\frac{\partial R}{\partial \xi_2'} + \dots + \xi_\mu'\frac{\partial R}{\partial \xi_\mu'} - R,$$

positoque

$$\frac{\partial R}{\partial \xi_1'} = v_1, \quad \frac{\partial R}{\partial \xi_2'} = v_2, \quad \dots \quad \frac{\partial R}{\partial \xi_\mu'} = v_\mu$$

exprimatur $H$ per $\xi_1, \xi_2, \dots \xi_\mu, v_1, v_2, \dots v_\mu$, qua expressione differentiata respective secundum $v_1, v_2, \dots v_\mu$ sequitur ex analysi supra tradita vice versa obtineri quantitates ipsis $\xi'_1, \xi'_2, \dots \xi'_\mu$ aequales, sive fieri

$$\xi'_1 = \frac{\partial H}{\partial v_1}, \quad \xi'_2 = \frac{\partial H}{\partial v_2}, \quad \dots \quad \xi'_\mu = \frac{\partial H}{\partial v_\mu}.$$

Unde cognita expressione illa ipsius $H$, habetur expressio ipsius $R$ per $\xi_1, \xi_2, \dots \xi_\mu, v_1, v_2, \dots v_\mu$ ope formulae

$$R = v_1 \frac{\partial H}{\partial v_1} + v_2 \frac{\partial H}{\partial v_2} + \dots + v_\mu \frac{\partial H}{\partial v_\mu} - H.$$

Substitutis iam expressionibus ipsarum $\xi_1, \xi_2, \dots \xi_\mu$ per $q_1, q_2, \dots q_m$ atque expressionibus ipsarum $\xi'_1, \xi'_2, \dots \xi'_\mu$ per $q_1, q_2, \dots q_m, q'_1, q'_2, \dots q'_m$, exhibeatur $R$ per has $2m$ quantitates; quo facto demonstratur prorsus eadem ratione atque formula (1.) §. 36 demonstrata est, haec aequatio:

$$\xi'_1 \frac{\partial R}{\partial \xi'_1} + \xi'_2 \frac{\partial R}{\partial \xi'_2} + \dots + \xi'_\mu \frac{\partial R}{\partial \xi'_\mu} =$$
$$q'_1 \frac{\partial R}{\partial q'_1} + q'_2 \frac{\partial R}{\partial q'_2} + \dots + q'_m \frac{\partial R}{\partial q'_m}.$$

Unde expressa $R$ per $q_1, q_2, \dots q_m, q'_1, q'_2, \dots q'_m$, functio $H$ per easdem quantitates sic exhibetur:

$$H = q'_1 \frac{\partial R}{\partial q'_1} + q'_2 \frac{\partial R}{\partial q'_2} + \dots + q'_m \frac{\partial R}{\partial q'_m} - R.$$

Posito

$$p_1 = \frac{\partial R}{\partial q'_1}, \quad p_2 = \frac{\partial R}{\partial q'_2}, \quad \dots \quad p_m = \frac{\partial R}{\partial q'_m},$$

exprimatur $H$ per $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$; qua expressione differentiata secundum $p_1, p_2, \dots p_m$, vice versa obtinentur quantitates ipsis $q'_1, q'_2, \dots q'_m$ aequales sive habentur aequationes

$$q'_1 = \frac{\partial H}{\partial p_1}, \quad q'_2 = \frac{\partial H}{\partial p_2}, \quad \dots \quad q'_m = \frac{\partial H}{\partial p_m}.$$

Unde erit

$$R = v_1 \frac{\partial H}{\partial v_1} + v_2 \frac{\partial H}{\partial v_2} + \dots + v_\mu \frac{\partial H}{\partial v_\mu} - H$$
$$= p_1 \frac{\partial H}{\partial p_1} + p_2 \frac{\partial H}{\partial p_2} + \dots + p_m \frac{\partial H}{\partial p_m} - H.$$

His positis sint $\varphi$, $\psi$ binae functiones *quaecunque* ipsarum $\xi_1, \xi_2, \dots \xi_\mu, \xi'_1, \xi'_2, \dots \xi'_\mu$; quibus expressis per $q_1, q_2, \dots q_m, q'_1, q'_2, \dots q'_m$ ac deinde ope aequationum

$$p_1 = \frac{\partial R}{\partial q_1'}, \quad p_2 = \frac{\partial R}{\partial q_2'}, \quad \ldots \quad p_m = \frac{\partial R}{\partial q_m'}$$

iisdem expressis per $q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m$, formetur expressio

$$[\varphi, \psi] = \frac{\partial \varphi}{\partial q_1}\frac{\partial \psi}{\partial p_1} + \frac{\partial \varphi}{\partial q_2}\frac{\partial \psi}{\partial p_2} + \cdots + \frac{\partial \varphi}{\partial q_m}\frac{\partial \psi}{\partial p_m}$$
$$- \frac{\partial \varphi}{\partial p_1}\frac{\partial \psi}{\partial q_1} - \frac{\partial \varphi}{\partial p_2}\frac{\partial \psi}{\partial q_2} - \cdots - \frac{\partial \varphi}{\partial p_m}\frac{\partial \psi}{\partial q_m}.$$

Functiones $\varphi$, $\psi$ etiam per quantitates $\xi_1, \xi_2, \ldots \xi_\mu, v_1, v_2, \ldots v_\mu$ exhiberi possunt. Quibus cognitis expressionibus, quaeritur:

*„datis expressionibus trium functionum $H$, $\varphi$, $\psi$ per quantitates $\xi_1, \xi_2, \ldots \xi_\mu$, $v_1, v_2, \ldots v_\mu$ atque aequationibus, quae inter ipsas $\xi_1, \xi_2, \ldots \xi_\mu$ locum habent,*

$$F = 0, \quad \Phi = 0, \quad \text{etc.},$$

*neque vero cognitis relationibus, quibus quantitates $\xi_1, \xi_2, \ldots \xi_\mu$ per alias independentes $q_1, q_2, \ldots q_m$ determinantur, invenire valorem expressionis*

$$[\varphi, \psi]\text{“}.$$

Quod est problema propositum.

## 39.

Expositioni problematis antecedentis has addam dilucidationes. Quantitatum $\xi_i$, $\xi_i'$, $R$, $H$, $\varphi$, $\psi$ expressiones per $q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m$ sunt prorsus determinatae, simulac relationes datae sunt, quarum ope advocatis aequationibus $F = 0$, $\Phi = 0$, ... quantitates $\xi_1, \xi_2, \ldots \xi_\mu$ per $q_1$, $q_2, \ldots q_m$ exhiberi possunt. At expressiones ipsarum $q_i$, $p_i$, $\xi_i'$, $R$, $H$, $\varphi$, $\psi$ per $\xi_1, \xi_2, \ldots \xi_\mu, v_1, v_2, \ldots v_\mu$ ope aequationum $F = 0$, $\Phi = 0$, ... et quae ex iis differentiatione deducuntur, variis modis immutari possunt. Agamus primum de determinatione quantitatum $v_i$ per ipsas $\xi_i$, $\xi_i'$ atque de formatione functionis $H$ per ipsas $\xi_i$, $v_i$ exprimenda. Qua in re proficisci debemus a functione $R$, quae erat functio quaecunque ipsarum $\xi_i$, $\xi_i'$. Posito

$$F' = \frac{\partial F}{\partial \xi_1}\xi_1' + \frac{\partial F}{\partial \xi_2}\xi_2' + \cdots + \frac{\partial F}{\partial \xi_\mu}\xi_\mu',$$
$$\Phi' = \frac{\partial \Phi}{\partial \xi_1}\xi_1' + \frac{\partial \Phi}{\partial \xi_2}\xi_2' + \cdots + \frac{\partial \Phi}{\partial \xi_\mu}\xi_\mu',$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

functioni $R$ addi possunt expressiones $F$, $\Phi$, ... $F'$, $\Phi'$, ... per factores arbitrarios $\lambda$, $\mu$, ... $\lambda_1$, $\mu_1$, ... multiplicatae, quippe quae expressiones ex hypothesi sunt evanescentes. Et maxime distinguendum erit, an ipsi $R$ soli termini $\lambda F+\mu\Phi+\cdots$, an etiam termini $\lambda_1 F'+\mu_1\Phi'+\cdots$ addantur. Nam casu priore valores ipsarum

$$v_i = \frac{\partial R}{\partial \xi'_i}$$

prorsus iidem manent, seu potius alias non patiuntur mutationes, nisi quod iis termini in functiones evanescentes $F$, $\Phi$, ... multiplicati accedant. Unde etiam vice versa expressiones ipsarum $\xi'_i$ atque functionum $R$, $H$ per quantitates $\xi_i$, $v_i$ nonnisi easdem mutationes subeunt, scilicet termini per quantitates $F$, $\Phi$, ... multiplicati iis accedunt neque vero termini in $F'$, $\Phi'$, ... ducti.

At longe secus evenit, si functioni $R$ etiam termini $\lambda_1 F'+\mu_1\Phi'+\cdots$ addantur. Et functio quidem

$$H = \xi'_1\frac{\partial R}{\partial \xi'_1}+\xi'_2\frac{\partial R}{\partial \xi'_2}+\cdots+\xi'_\mu\frac{\partial R}{\partial \xi'_\mu}-R$$

valorem certe numericum non mutabit, quum sit identice:

$$\xi'_1\frac{\partial F'}{\partial \xi'_1}+\xi'_2\frac{\partial F'}{\partial \xi'_2}+\cdots+\xi'_\mu\frac{\partial F'}{\partial \xi'_\mu}-F' = 0,$$

$$\xi'_1\frac{\partial \Phi'}{\partial \xi'_1}+\xi'_2\frac{\partial \Phi'}{\partial \xi'_2}+\cdots+\xi'_\mu\frac{\partial \Phi'}{\partial \xi'_\mu}-\Phi' = 0,$$

$$\cdots\cdots\cdots\cdots\cdots$$

Sed quantitates $v_i$ non tantum formam additione expressionum evanescentium mutabunt, sed alios adeo valores numericos induunt. Quippe quae evadunt

$$v_i = \frac{\partial R}{\partial \xi'_i}+\lambda_1\frac{\partial F}{\partial \xi'_i}+\mu_1\frac{\partial \Phi}{\partial \xi'_i}+\cdots,$$

omissis terminis evanescentibus:

$$\frac{\partial \lambda}{\partial \xi'_i}F+\frac{\partial \mu}{\partial \xi'_i}\Phi+\cdots+\frac{\partial \lambda_1}{\partial \xi'_i}F'+\frac{\partial \mu_1}{\partial \xi'_i}\Phi'+\cdots.$$

Qua de re etiam forma functionis $H$ per ipsas $\xi_i$, $v_i$ expressa alias subire debet mutationes praeter accessum functionum evanescentium, quum valor ipsius $H$ immutatus manere debeat, dum quantitates $v_1$, $v_2$, ... $v_\mu$, quae functionem $H$ ingrediuntur, alios valores induant. Ut mutationes accuratius indicem, sit

$$l_i = \lambda_1\frac{\partial F}{\partial \xi'_i}+\mu_1\frac{\partial \Phi}{\partial \xi'_i}+\cdots,$$

$$L_i = \frac{\partial \lambda}{\partial \xi'_i}F+\frac{\partial \mu}{\partial \xi'_i}\Phi+\cdots+\frac{\partial \lambda_1}{\partial \xi'_i}F'+\frac{\partial \mu_1}{\partial \xi'_i}\Phi'+\cdots,$$

sint porro $v_i^0$ quantitates, in quas ipsae $v_i$ abeunt, si loco $R$ ponitur

$$R+\lambda F+\mu\Phi+\cdots+\lambda_1 F'+\mu_1\Phi'+\cdots.$$

Tum erit

$$v_i = v_i^0-l_i-L_i;$$

porro posito

$$\lambda^0=\Sigma\xi_i'\frac{\partial\lambda}{\partial\xi_i'}-\lambda,\qquad \mu^0=\Sigma\xi_i'\frac{\partial\mu}{\partial\xi_i'}-\mu,\quad \ldots$$

$$\lambda_1^0=\Sigma\xi_i'\frac{\partial\lambda_1}{\partial\xi_i'}\quad,\qquad \mu_1^0=\Sigma\xi_i'\frac{\partial\mu_1}{\partial\xi_i'}\quad,\quad \ldots$$

abit $H$ in expressionem

$$H^0 = H+\lambda^0 F+\mu^0\Phi+\cdots+\lambda_1^0 F'+\mu_1^0\Phi'+\cdots,$$

omissis quantitatibus se mutuo destruentibus

$$\lambda_1\left\{\Sigma_i\xi_i'\frac{\partial F'}{\partial\xi_i'}-F'\right\},\quad \mu_1\left\{\Sigma_i\xi_i'\frac{\partial\Phi'}{\partial\xi_i'}-\Phi'\right\},\quad \ldots$$

atque in termino primo $H$ loco ipsarum $v_i$ positis valoribus $v_i^0-l_i-L_i$. Quibus adhibitis expressionibus sine multo negotio invenitur, quod fieri debet,

$$\frac{\partial H^0}{\partial v_i^0}=\frac{\partial H}{\partial v_i}=\xi_i'.$$

Rejectis enim terminis evanescentibus fit

$$\frac{\partial H^0}{\partial v_i^0} = \frac{\partial H}{\partial v_i}-\Sigma_k\frac{\partial H}{\partial v_k}\left(\frac{\partial l_k}{\partial v_i^0}+\frac{\partial L_k}{\partial v_i^0}\right)+\lambda_1^0\frac{\partial F'}{\partial v_i^0}+\mu_1^0\frac{\partial\Phi'}{\partial v_i^0}+\cdots;$$

porro, quum omnes solarum $\xi_i$ functiones secundum $v_i^0$ differentiatae evanescant,

$$\Sigma_k\frac{\partial H}{\partial v_k}\frac{\partial l_k}{\partial v_i^0}=\Sigma_k\xi_k'\frac{\partial l_k}{\partial v_i^0}=\frac{\partial\lambda_1}{\partial v_i^0}\Sigma_k\xi_k'\frac{\partial F}{\partial\xi_k}+\frac{\partial\mu}{\partial v_i^0}\Sigma_k\xi_k'\frac{\partial\Phi}{\partial\xi_k}+\cdots=0,$$

$$\Sigma_k\frac{\partial H}{\partial v_k}\frac{\partial L_k}{\partial v_i^0}=\Sigma_k\xi_k'\frac{\partial L_k}{\partial v_i^0}=\frac{\partial F'}{\partial v_i^0}\Sigma_k\xi_k'\frac{\partial\lambda_1}{\partial\xi_k'}+\frac{\partial\Phi'}{\partial v_i^0}\Sigma_k\xi_k'\frac{\partial\mu_1}{\partial\xi_k'}+\cdots$$
$$=\lambda_1^0\frac{\partial F'}{\partial v_i^0}+\mu_1^0\frac{\partial\Phi'}{\partial v_i^0}+\cdots.$$

Unde rejectis terminis se mutuo destruentibus, prodit formula demonstranda:

$$\frac{\partial H^0}{\partial v_i^0}=\frac{\partial H}{\partial v_i},$$

quae valet pro quolibet indicis $i$ valore. Apposui antecedentia, quamquam ad propositi problematis solutionem non necessaria, sicuti innui, ad dilucidationem rei; prona enim est in hac quaestione ad errores via.

Etiam functiones $\varphi$ et $\psi$ variis modis mutari possunt addendo iis terminos in $F$, $\Phi$, ... $F'$, $\Phi'$, ... multiplicatos, sive expressis $\varphi$, $\psi$, sicuti requiritur, per quantitates $\xi_i$, $v_i$, addendo terminos multiplicatos in $F$, $\Phi$, ... $A$, $B$, ... siquidem per $A$, $B$, ... designamus valores ipsarum $F'$, $\Phi'$, ... per quantitates $\xi_i$, $v_i$ exhibitos, scilicet expressiones,

$$A = \frac{\partial F}{\partial \xi_1}\frac{\partial H}{\partial v_1}+\frac{\partial F}{\partial \xi_2}\frac{\partial H}{\partial v_2}+\cdots+\frac{\partial F}{\partial \xi_\mu}\frac{\partial H}{\partial v_\mu},$$

$$B = \frac{\partial \Phi}{\partial \xi_1}\frac{\partial H}{\partial v_1}+\frac{\partial \Phi}{\partial \xi_2}\frac{\partial H}{\partial v_2}+\cdots+\frac{\partial \Phi}{\partial \xi_\mu}\frac{\partial H}{\partial v_\mu},$$

. . . . . . . . . . . . . . .

Quae expressiones quum evanescere debeant, sunt $A=0$, $B=0$, ... aequationes conditionales, quae inter ipsas $\xi_i$, $v_i$ locum habent.

Ante omnia bene tenendum est, e variis quidem formis, quas functio $R$ per aequationes $F=0$, $\Phi=0$, ... $F'=0$, $\Phi'=0$, ... induere potest, quamcunque eligi posse; sed hac electa atque ratione praescripta inde deductis expressionibus ipsarum $v_i$ per quantitates $\xi_i$, $\xi'_i$ atque functionis $H$ per quantitates $\xi_i$, $v_i$, supponi, has expressiones per aequationes illas non denuo mutari. Alioqui enim in infinitos errores delaberemur.

*Expressionum quaesitarum formatio ad duarum summarum determinationem revocatur.*

40.

Adstruam primum aequationes, quibus quantitates $v_i$ determinantur per ipsas $q_i$ et $p_i$, siquidem data est expressio functionis $H$ per ipsas $\xi_i$ et $v_i$ atque expressiones quantitatum $\xi_i$ per ipsas $q_i$. Habemus

$$\delta\xi_i = \frac{\partial \xi_i}{\partial q_1}\delta q_1+\frac{\partial \xi_i}{\partial q_2}\delta q_2+\cdots+\frac{\partial \xi_i}{\partial q_m}\delta q_m,$$

ideoque

$$\Sigma_i\frac{\partial R}{\partial \xi'_i}\delta\xi_i = \delta q_1\Sigma_i\frac{\partial R}{\partial \xi'_i}\frac{\partial \xi_i}{\partial q_1}+\delta q_2\Sigma_i\frac{\partial R}{\partial \xi'_i}\frac{\partial \xi_i}{\partial q_2}+\cdots+\delta q_m\Sigma_i\frac{\partial R}{\partial \xi'_i}\frac{\partial \xi_i}{\partial q_m}.$$

Quum habeatur

$$\frac{\partial \xi'_i}{\partial q'_k} = \frac{\partial \xi_i}{\partial q_k},$$

quia in expressione ipsius $\xi'_i$ per quantitates $q_k$, $q'_k$ ipsa $q'_k$ tantum lineariter obvenit atque in $\frac{\partial \xi_i}{\partial q_k}$ ducta, fit:

$$\Sigma_i\frac{\partial R}{\partial \xi'_i}\frac{\partial \xi_i}{\partial q_k} = \Sigma_i\frac{\partial R}{\partial \xi'_i}\frac{\partial \xi'_i}{\partial q'_k} = \frac{\partial R}{\partial q'_k},$$

ideoque:

$$\Sigma_i \frac{\partial R}{\partial \xi'_i}\delta\xi_i = \frac{\partial R}{\partial \xi'_1}\delta\xi_1+\frac{\partial R}{\partial \xi'_2}\delta\xi_2+\cdots+\frac{\partial R}{\partial \xi'_\mu}\delta\xi_\mu$$

$$= \frac{\partial R}{\partial q'_1}\delta q_1+\frac{\partial R}{\partial q'_2}\delta q_2+\cdots+\frac{\partial R}{\partial q'_m}\delta q_m.$$

Quum vero positum sit,

$$\frac{\partial R}{\partial \xi'_i} = v_i, \qquad \frac{\partial R}{\partial q'_i} = p_i,$$

aequatio praecedens sic repraesentari potest:

$$(1.) \qquad v_1\delta\xi_1+v_2\delta\xi_2+\cdots+v_\mu\delta\xi_\mu = p_1\delta q_1+p_2\delta q_2+\cdots+p_m\delta q_m.$$

In aequatione (1.) variationes $\delta q_1$, $\delta q_2$, ... $\delta q_m$ prorsus arbitrariae sunt, dum inter variationes $\delta\xi_1$, $\delta\xi_2$, ... $\delta\xi_\mu$ locum habent $\mu-m$ conditiones:

$$\frac{\partial F}{\partial \xi_1}\delta\xi_1+\frac{\partial F}{\partial \xi_2}\delta\xi_2+\cdots+\frac{\partial F}{\partial \xi_\mu}\delta\xi_\mu = 0,$$

$$\frac{\partial \Phi}{\partial \xi_1}\delta\xi_1+\frac{\partial \Phi}{\partial \xi_2}\delta\xi_2+\cdots+\frac{\partial \Phi}{\partial \xi_\mu}\delta\xi_\mu = 0,$$

$$\cdots\cdots\cdots\cdots$$

Qua de re quantitates $p_i$ quidem per $\xi_i$, $v_i$, sed non quantitates $v_i$ per ipsas $p_i$ ex aequatione (1.) determinantur. Habentur enim e (1.) tantum $m$ aequationes:

$$p_1 = v_1\frac{\partial \xi_1}{\partial q_1}+v_2\frac{\partial \xi_2}{\partial q_1}+\cdots+v_\mu\frac{\partial \xi_\mu}{\partial q_1},$$

$$p_2 = v_1\frac{\partial \xi_1}{\partial q_2}+v_2\frac{\partial \xi_2}{\partial q_2}+\cdots+v_\mu\frac{\partial \xi_\mu}{\partial q_2},$$

$$\cdots\cdots\cdots\cdots$$

$$p_m = v_1\frac{\partial \xi_1}{\partial q_m}+v_2\frac{\partial \xi_2}{\partial q_m}+\cdots+v_\mu\frac{\partial \xi_\mu}{\partial q_m}.$$

Ad determinationem completam ipsarum $v_i$ praeter $m$ aequationes praecedentes adhiberi debent $\mu-m$ aequationes sequentes:

$$(2.) \quad \begin{cases} 0 = A = \dfrac{\partial F}{\partial \xi_1}\dfrac{\partial H}{\partial v_1}+\dfrac{\partial F}{\partial \xi_2}\dfrac{\partial H}{\partial v_2}+\cdots+\dfrac{\partial F}{\partial \xi_\mu}\dfrac{\partial H}{\partial v_\mu}, \\ 0 = B = \dfrac{\partial \Phi}{\partial \xi_1}\dfrac{\partial H}{\partial v_1}+\dfrac{\partial \Phi}{\partial \xi_2}\dfrac{\partial H}{\partial v_2}+\cdots+\dfrac{\partial \Phi}{\partial \xi_\mu}\dfrac{\partial H}{\partial v_\mu}, \\ \cdots\cdots\cdots\cdots \end{cases}$$

Conditis aequationibus (1.) et (2.), quibus quantitates $v_i$ determinantur, iam accedamus ad formationem propositam expressionis quantitatis $[\varphi,\psi]$ per ipsas $\xi_k$, $v_k$.

Fit

$$\frac{\partial\varphi}{\partial q_i} = \frac{\partial\varphi}{\partial\xi_1}\frac{\partial\xi_1}{\partial q_i}+\frac{\partial\varphi}{\partial\xi_2}\frac{\partial\xi_2}{\partial q_i}+\cdots+\frac{\partial\varphi}{\partial\xi_\mu}\frac{\partial\xi_\mu}{\partial q_i}$$

$$+\frac{\partial\varphi}{\partial v_1}\frac{\partial v_1}{\partial q_i}+\frac{\partial\varphi}{\partial v_2}\frac{\partial v_2}{\partial q_i}+\cdots+\frac{\partial\varphi}{\partial v_\mu}\frac{\partial v_\mu}{\partial q_i},$$

$$\frac{\partial\psi}{\partial p_i} = \frac{\partial\psi}{\partial v_1}\frac{\partial v_1}{\partial p_i}+\frac{\partial\psi}{\partial v_2}\frac{\partial v_2}{\partial p_i}+\cdots+\frac{\partial\psi}{\partial v_\mu}\frac{\partial v_\mu}{\partial p_i}.$$

His duabus expressionibus multiplicatis habetur valor ipsius $\frac{\partial\varphi}{\partial q_i}\frac{\partial\psi}{\partial p_i}$, e quo permutando $\varphi$ et $\psi$ valor ipsius $\frac{\partial\varphi}{\partial p_i}\frac{\partial\psi}{\partial q_i}$ prodit; quibus subductis, habetur valor expressionis

$$\frac{\partial\varphi}{\partial q_i}\frac{\partial\psi}{\partial p_i}-\frac{\partial\varphi}{\partial p_i}\frac{\partial\psi}{\partial q_i},$$

e quo, tribuendo ipsi $i$ valores 1, 2, ... $m$ et summando, prodit expressio ipsius $[\varphi, \psi]$ sequens:

$$(3.)\quad \begin{cases} [\varphi,\psi] = \Sigma\left\{\frac{\partial\varphi}{\partial\xi_k}\frac{\partial\psi}{\partial v_{k'}}-\frac{\partial\varphi}{\partial v_{k'}}\frac{\partial\psi}{\partial\xi_k}\right\}\frac{\partial\xi_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i}, \\ \qquad +\Sigma\left\{\frac{\partial\varphi}{\partial v_k}\frac{\partial\psi}{\partial v_{k'}}-\frac{\partial\varphi}{\partial v_{k'}}\frac{\partial\psi}{\partial v_k}\right\}\frac{\partial v_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i}. \end{cases}$$

In qua expressione sub signo summatorio tribuendi sunt indicibus $k$ et $k'$ valores 1, 2, ... $\mu$, atque indici $i$ valores 1, 2, ... $m$. Summam posteriorem sic quoque exhibere licet:

$$\Sigma\frac{\partial\varphi}{\partial v_k}\frac{\partial\psi}{\partial v_{k'}}\left\{\frac{\partial v_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i}-\frac{\partial v_{k'}}{\partial q_i}\frac{\partial v_k}{\partial p_i}\right\}.$$

Iam pro quibuslibet datis valoribus ipsarum $k$ et $k'$ investigemus valorem summarum simplicium:

$$\Sigma_i\frac{\partial\xi_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i} = \frac{\partial\xi_k}{\partial q_1}\frac{\partial v_{k'}}{\partial p_1}+\frac{\partial\xi_k}{\partial q_2}\frac{\partial v_{k'}}{\partial p_2}+\cdots+\frac{\partial\xi_k}{\partial q_m}\frac{\partial v_{k'}}{\partial p_m},$$

$$\Sigma_i\left\{\frac{\partial v_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i}-\frac{\partial v_{k'}}{\partial q_i}\frac{\partial v_k}{\partial p_i}\right\} = \frac{\partial v_k}{\partial q_1}\frac{\partial v_{k'}}{\partial p_1}+\frac{\partial v_k}{\partial q_2}\frac{\partial v_{k'}}{\partial p_2}+\cdots+\frac{\partial v_k}{\partial q_m}\frac{\partial v_{k'}}{\partial p_m}$$

$$-\frac{\partial v_{k'}}{\partial q_1}\frac{\partial v_k}{\partial p_1}-\frac{\partial v_{k'}}{\partial q_2}\frac{\partial v_k}{\partial p_2}-\cdots-\frac{\partial v_{k'}}{\partial q_m}\frac{\partial v_k}{\partial p_m}.$$

In quibus summis non ipsae $v_1$, $v_2$, ... $v_\mu$ sed tantum differentialia eorum partialia secundum quantitates $p_i$, $q_i$ sumta inveniuntur. Iam quaeram, quomodo binae summae per solas $\xi_1$, $\xi_2$, ... $\xi_\mu$, $v_1$, $v_2$, ... $v_\mu$ exhibeantur.

*Summarum propositarum prior determinatur.*

41.

Sunt quantitates $\xi_k$ prorsus determinatae functiones ipsarum $q_k$; nam etsi $\xi_k$ maiore sint numero ipsis $q_k$ quae per illas expressae supponuntur, possunt tamen advocatis aequationibus $F=0$, $\Phi=0$ etc. inter ipsas $\xi_k$ datis, vice versa ipsae $\xi_k$ per $q_k$ modo prorsus determinato exprimi. Quae expressiones ipsas $p_k$ nullo modo involvunt. Contra sunt ipsae $v_k$ functiones ipsarum $q_k$ et $p_k$ per (1.) et (2.) determinatae. Quibus observatis, differentietur (1.) secundum $p_i$; prodit:

$$(4.)\qquad \delta q_i = \frac{\partial v_1}{\partial p_i}\delta\xi_1 + \frac{\partial v_2}{\partial p_i}\delta\xi_2 + \cdots + \frac{\partial v_\mu}{\partial p_i}\delta\xi_\mu.$$

Ex hac aequatione et ex aequationibus (2.) et ipsis secundum $p_i$ differentiatis nanciscimur aequationes sequentes, quarum numerus est $\mu$ et e quibus ipsarum $\frac{\partial v_1}{\partial p_i}, \frac{\partial v_2}{\partial p_i}, \ldots \frac{\partial v_\mu}{\partial p_i}$ valores determinari possunt:

$$(5.)\qquad \begin{cases} 0 = \frac{\partial v_1}{\partial p_i}\frac{\partial \xi_1}{\partial q_1} + \frac{\partial v_2}{\partial p_i}\frac{\partial \xi_2}{\partial q_1} + \cdots + \frac{\partial v_\mu}{\partial p_i}\frac{\partial \xi_\mu}{\partial q_1}, \\ 0 = \frac{\partial v_1}{\partial p_i}\frac{\partial \xi_1}{\partial q_2} + \frac{\partial v_2}{\partial p_i}\frac{\partial \xi_2}{\partial q_2} + \cdots + \frac{\partial v_\mu}{\partial p_i}\frac{\partial \xi_\mu}{\partial q_2}, \\ \cdots\cdots\cdots\cdots \\ 1 = \frac{\partial v_1}{\partial p_i}\frac{\partial \xi_1}{\partial q_i} + \frac{\partial v_2}{\partial p_i}\frac{\partial \xi_2}{\partial q_i} + \cdots + \frac{\partial v_\mu}{\partial p_i}\frac{\partial \xi_\mu}{\partial q_i}, \\ \cdots\cdots\cdots\cdots \\ 0 = \frac{\partial v_1}{\partial p_i}\frac{\partial \xi_1}{\partial q_m} + \frac{\partial v_2}{\partial p_i}\frac{\partial \xi_2}{\partial q_m} + \cdots + \frac{\partial v_\mu}{\partial p_i}\frac{\partial \xi_\mu}{\partial q_m}, \\ 0 = \frac{\partial v_1}{\partial p_i}\frac{\partial A}{\partial v_1} + \frac{\partial v_2}{\partial p_i}\frac{\partial A}{\partial v_2} + \cdots + \frac{\partial v_\mu}{\partial p_i}\frac{\partial A}{\partial v_\mu}, \\ 0 = \frac{\partial v_1}{\partial p_i}\frac{\partial B}{\partial v_1} + \frac{\partial v_2}{\partial p_i}\frac{\partial B}{\partial v_2} + \cdots + \frac{\partial v_\mu}{\partial p_i}\frac{\partial B}{\partial v_\mu}, \\ \cdots\cdots\cdots\cdots \end{cases}$$

Eiusmodi aequationum linearium eruimus $m$ systemata ponendo loco $i$ numeros $1, 2, \ldots m$; quae systemata multiplicemus respective per

$$\frac{\partial \xi_k}{\partial q_1}, \quad \frac{\partial \xi_k}{\partial q_2}, \quad \ldots \quad \frac{\partial \xi_k}{\partial q_m}$$

et post multiplicationem factam instituamus additionem. Tum adhibita notatione sequente, in qua $k$ et $k'$ inter se diversi accipiuntur:

$$(6.)\quad \begin{cases} \dfrac{\partial \xi_k}{\partial q_1}\dfrac{\partial v_{k'}}{\partial p_1}+\dfrac{\partial \xi_k}{\partial q_2}\dfrac{\partial v_{k'}}{\partial p_2}+\cdots+\dfrac{\partial \xi_k}{\partial q_m}\dfrac{\partial v_{k'}}{\partial p_m} = k_{k'}, \\ \dfrac{\partial \xi_k}{\partial q_1}\dfrac{\partial v_k}{\partial p_1}+\dfrac{\partial \xi_k}{\partial q_2}\dfrac{\partial v_k}{\partial p_2}+\cdots+\dfrac{\partial \xi_k}{\partial q_m}\dfrac{\partial v_k}{\partial p_m} = 1+k_k, \end{cases}$$

invenimus:

$$(7.)\quad \begin{cases} 0 = \dfrac{\partial \xi_1}{\partial q_1}k_1+\dfrac{\partial \xi_2}{\partial q_1}k_2+\cdots+\dfrac{\partial \xi_\mu}{\partial q_1}k_\mu, \\ 0 = \dfrac{\partial \xi_1}{\partial q_2}k_1+\dfrac{\partial \xi_2}{\partial q_2}k_2+\cdots+\dfrac{\partial \xi_\mu}{\partial q_2}k_\mu, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ 0 = \dfrac{\partial \xi_1}{\partial q_m}k_1+\dfrac{\partial \xi_2}{\partial q_m}k_2+\cdots+\dfrac{\partial \xi_\mu}{\partial q_m}k_\mu, \\ -\dfrac{\partial A}{\partial v_k} = \dfrac{\partial A}{\partial v_1}k_1+\dfrac{\partial A}{\partial v_2}k_2+\cdots+\dfrac{\partial A}{\partial v_\mu}k_\mu, \\ -\dfrac{\partial B}{\partial v_k} = \dfrac{\partial B}{\partial v_1}k_1+\dfrac{\partial B}{\partial v_2}k_2+\cdots+\dfrac{\partial B}{\partial v_\mu}k_\mu, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{cases}$$

Harum aequationum resolutio revocari potest ad aliarum, quarum numerus tantum est $\mu-m$ sive idem atque aequationum conditionalium $F=0$, $\Phi=0$, etc. Habetur enim:

$$(8.)\quad \begin{cases} k_1 = \lambda_1^{(k)}\dfrac{\partial F}{\partial \xi_1}+\lambda_2^{(k)}\dfrac{\partial \Phi}{\partial \xi_1}+\cdots, \\ k_2 = \lambda_1^{(k)}\dfrac{\partial F}{\partial \xi_2}+\lambda_2^{(k)}\dfrac{\partial \Phi}{\partial \xi_2}+\cdots, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ k_\mu = \lambda_1^{(k)}\dfrac{\partial F}{\partial \xi_\mu}+\lambda_2^{(k)}\dfrac{\partial \Phi}{\partial \xi_\mu}+\cdots, \end{cases}$$

multiplicatoribus $\lambda_1^{(k)}$, $\lambda_2^{(k)}$, ..., quorum numerus est $\mu-m$, determinatis per aequationes:

$$(9.)\quad \begin{cases} -\dfrac{\partial A}{\partial v_k} = a_1\lambda_1^{(k)}+a_2\lambda_2^{(k)}+\cdots, \\ -\dfrac{\partial B}{\partial v_k} = b_1\lambda_1^{(k)}+b_2\lambda_2^{(k)}+\cdots, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{cases}$$

siquidem:

$$(10.)\quad\begin{cases} a_1 = \dfrac{\partial F}{\partial \xi_1}\dfrac{\partial A}{\partial v_1}+\dfrac{\partial F}{\partial \xi_2}\dfrac{\partial A}{\partial v_2}+\cdots+\dfrac{\partial F}{\partial \xi_\mu}\dfrac{\partial A}{\partial v_\mu}, \\ a_2 = b_1 = \dfrac{\partial \Phi}{\partial \xi_1}\dfrac{\partial A}{\partial v_1}+\dfrac{\partial \Phi}{\partial \xi_2}\dfrac{\partial A}{\partial v_2}+\cdots+\dfrac{\partial \Phi}{\partial \xi_\mu}\dfrac{\partial A}{\partial v_\mu} \\ \qquad = \dfrac{\partial F}{\partial \xi_1}\dfrac{\partial B}{\partial v_1}+\dfrac{\partial F}{\partial \xi_2}\dfrac{\partial B}{\partial v_2}+\cdots+\dfrac{\partial F}{\partial \xi_\mu}\dfrac{\partial B}{\partial v_\mu}, \\ b_2 = \dfrac{\partial \Phi}{\partial \xi_1}\dfrac{\partial B}{\partial v_1}+\dfrac{\partial \Phi}{\partial \xi_2}\dfrac{\partial B}{\partial v_2}+\cdots+\dfrac{\partial \Phi}{\partial \xi_\mu}\dfrac{\partial B}{\partial v_\mu}, \\ \cdots\cdots\cdots\cdots \end{cases}$$

Aequalitas coefficientium $a_2$ et $b_1$ facile patet ex expressionibus ipsarum $A$ et $B$ §. 40 (2.) propositis. Invenitur enim utriusque expressionis idem valor:

$$a_2 = b_1 = \Sigma_{k,k'}\frac{\partial F}{\partial \xi_k}\frac{\partial \Phi}{\partial \xi_{k'}}\frac{\partial^2 H}{\partial v_k \partial v_{k'}},$$

ipsis $k$ et $k'$ valoribus 1, 2, ... $\mu$ tributis. Generaliter aequationes lineares (9.), ad quarum resolutionem investigatio ipsarum $k_1$, $k_2$, ... $k_\mu$ reducta est, ita comparatae sunt, ut series verticales et horizontales coefficientium eaedem sint. Quae porro coefficientes tantum ab ipsis functionibus $F$, $\Phi$, etc. neque ab indice $k$ vel $k'$ pendent; index tamen $k$ afficit aequationum (9.) partes constantes. Posito igitur:

$$(11.)\quad\begin{cases} -\lambda_1^{(k)} = \varDelta_{1,1}\dfrac{\partial A}{\partial v_k}+\varDelta_{1,2}\dfrac{\partial B}{\partial v_k}+\cdots \\ -\lambda_2^{(k)} = \varDelta_{2,1}\dfrac{\partial A}{\partial v_k}+\varDelta_{2,2}\dfrac{\partial B}{\partial v_k}+\cdots \\ \cdots\cdots\cdots\cdots \end{cases}$$

eruis $m$ systemata eiusmodi formularum tribuendo ipsi $k$ valores 1, 2, ... $m$, ipsis coefficientibus $\varDelta$ immutatis manentibus. Ceterum e noto theoremate algebraico fit

$$\varDelta_{a,b} = \varDelta_{b,a},$$

sive etiam in aequationibus (11.) coefficientium series horizontales eaedem sunt atque verticales.

*Agitur de altera summa determinanda.*

### 42.

E duabus summis simplicibus

$$\Sigma_i \frac{\partial \xi_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i}, \qquad \Sigma_i\left\{\frac{\partial v_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i}-\frac{\partial v_{k'}}{\partial q_i}\frac{\partial v_k}{\partial p_i}\right\},$$

quarum valores §. 40. vidimus investigandos esse, alteram antecedentibus determinavi seu certe ad alias revocavi quantitates $\lambda_1^{(k)}$, $\lambda_2^{(k)}$, ... quae per resolutionem $\mu-m$ aequationum linearium inveniuntur. Iam alteram investigemus summam simplicem

$$\Sigma_i \left\{ \frac{\partial v_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i} - \frac{\partial v_{k'}}{\partial q_i}\frac{\partial v_k}{\partial p_i} \right\},$$

cuius complicatior fit expressio.

Statuamus, e sequentibus $\mu$ aequationibus linearibus:

$$(12.)\quad \begin{cases} M_1 = \dfrac{\partial \xi_1}{\partial q_1}u_1 + \dfrac{\partial \xi_2}{\partial q_1}u_2 + \cdots + \dfrac{\partial \xi_\mu}{\partial q_1}u_\mu, \\ M_2 = \dfrac{\partial \xi_1}{\partial q_2}u_1 + \dfrac{\partial \xi_2}{\partial q_2}u_2 + \cdots + \dfrac{\partial \xi_\mu}{\partial q_2}u_\mu, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ M_m = \dfrac{\partial \xi_1}{\partial q_m}u_1 + \dfrac{\partial \xi_2}{\partial q_m}u_2 + \cdots + \dfrac{\partial \xi_\mu}{\partial q_m}u_\mu, \\ N_1 = \dfrac{\partial A}{\partial v_1}u_1 + \dfrac{\partial A}{\partial v_2}u_2 + \cdots + \dfrac{\partial A}{\partial v_\mu}u_\mu, \\ N_2 = \dfrac{\partial B}{\partial v_1}u_1 + \dfrac{\partial B}{\partial v_2}u_2 + \cdots + \dfrac{\partial B}{\partial v_\mu}u_\mu, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{cases}$$

obtineri resolutione facta incognitarum $u_1$, $u_2$, ... $u_\mu$ valores sequentes:

$$\begin{aligned} u_1 &= C_{1,1}M_1 + C_{1,2}M_2 + \cdots + C_{1,m}M_m + D_{1,1}N_1 + D_{1,2}N_2 + \cdots, \\ u_2 &= C_{2,1}M_1 + C_{2,2}M_2 + \cdots + C_{2,m}M_m + D_{2,1}N_1 + D_{2,2}N_2 + \cdots, \\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ u_\mu &= C_{\mu,1}M_1 + C_{\mu,2}M_2 + \cdots + C_{\mu,m}M_m + D_{\mu,1}N_1 + D_{\mu,2}N_2 + \cdots. \end{aligned}$$

Si in aequationibus (12.) ponitur $M_i = 1$, reliquae autem omnes $M_1$, $M_2$, ... $M_m$, $N_1$, $N_2$, ... praeter $M_i$ evanescunt, aequationes illae eaedem fiunt atque aequationes (5.), e quibus valores ipsarum $\frac{\partial v_1}{\partial p_i}$, $\frac{\partial v_2}{\partial p_i}$, ... $\frac{\partial v_\mu}{\partial p_i}$ petendi sunt, sive fit

$$\frac{\partial v_1}{\partial p_i} = C_{1,i},\quad \frac{\partial v_2}{\partial p_i} = C_{2,i},\quad \ldots\quad \frac{\partial v_\mu}{\partial p_i} = C_{\mu,i},$$

vel generaliter

$$\frac{\partial v_k}{\partial p_i} = C_{k,i}.$$

*Unde facta aequationum* (12.) *resolutione obtinentur incognitarum* $u_1$, $u_2$, ... $u_\mu$

*valores sequentes:*

$$(13.)\quad\begin{cases} u_1 = \frac{\partial v_1}{\partial p_1}M_1+\frac{\partial v_1}{\partial p_2}M_2+\cdots+\frac{\partial v_1}{\partial p_m}M_m+D_{1,1}N_1+D_{1,2}N_2+\cdots, \\ u_2 = \frac{\partial v_2}{\partial p_1}M_1+\frac{\partial v_2}{\partial p_2}M_2+\cdots+\frac{\partial v_2}{\partial p_m}M_m+D_{2,1}N_1+D_{2,2}N_2+\cdots, \\ \cdots\cdots\cdots\cdots \\ u_\mu = \frac{\partial v_\mu}{\partial p_1}M_1+\frac{\partial v_\mu}{\partial p_2}M_2+\cdots+\frac{\partial v_\mu}{\partial p_m}M_m+D_{\mu,1}N_1+D_{\mu,2}N_2+\cdots. \end{cases}$$

His praeparatis, differentiemus aequationes binas, quae ex (1.) (§. 40.) sequuntur:

$$p_i = v_1\frac{\partial\xi_1}{\partial q_i}+v_2\frac{\partial\xi_2}{\partial q_i}+\cdots+v_\mu\frac{\partial\xi_\mu}{\partial q_i},$$
$$p_{i'} = v_1\frac{\partial\xi_1}{\partial q_{i'}}+v_2\frac{\partial\xi_2}{\partial q_{i'}}+\cdots+v_\mu\frac{\partial\xi_\mu}{\partial q_{i'}},$$

priorem secundum $q_{i'}$, posteriorem secundum $q_i$. Id quod licet, quum substitutis ipsorum $\xi_k$, $v_k$ valoribus per $p_k$, $q_k$ expressis, aequationes illae identicae evadere debent. Facta differentiatione, partes ad laevam ut ab ipsis $q_i$, $q_{i'}$ vacuae evanescunt, partes ad dextram commune nanciscuntur aggregatum

$$v_1\frac{\partial^2\xi_1}{\partial q_i\partial q_{i'}}+v_2\frac{\partial^2\xi_2}{\partial q_i\partial q_{i'}}+\cdots+v_\mu\frac{\partial^2\xi_\mu}{\partial q_i\partial q_{i'}}.$$

Duabus expressionibus evanescentibus aequiparatis et aggregato communi reiecto, nanciscimur:

$$(14.)\quad\begin{cases} \frac{\partial\xi_1}{\partial q_i}\frac{\partial v_1}{\partial q_{i'}}+\frac{\partial\xi_2}{\partial q_i}\frac{\partial v_2}{\partial q_{i'}}+\cdots+\frac{\partial\xi_\mu}{\partial q_i}\frac{\partial v_\mu}{\partial q_{i'}} = \\ \frac{\partial\xi_1}{\partial q_{i'}}\frac{\partial v_1}{\partial q_i}+\frac{\partial\xi_2}{\partial q_{i'}}\frac{\partial v_2}{\partial q_i}+\cdots+\frac{\partial\xi_\mu}{\partial q_{i'}}\frac{\partial v_\mu}{\partial q_i}. \end{cases}$$

Ex hac aequatione obtinemus numerum $m$ aequationum tribuendo ipsi $i'$ valores $1, 2, \ldots m$, quarum aequationum una valori $i=i'$ respondens adeo identica est. Porro differentiando aequationes $A=0$, $B=0$, etc. secundum $q_i$, obtinemus:

$$(15.)\quad\begin{cases} -\frac{\partial A}{\partial\xi_1}\frac{\partial\xi_1}{\partial q_i}-\frac{\partial A}{\partial\xi_2}\frac{\partial\xi_2}{\partial q_i}-\cdots-\frac{\partial A}{\partial\xi_\mu}\frac{\partial\xi_\mu}{\partial q_i} = \\ \frac{\partial A}{\partial v_1}\frac{\partial v_1}{\partial q_i}+\frac{\partial A}{\partial v_2}\frac{\partial v_2}{\partial q_i}+\cdots+\frac{\partial A}{\partial v_\mu}\frac{\partial v_\mu}{\partial q_i}, \\ -\frac{\partial B}{\partial\xi_1}\frac{\partial\xi_1}{\partial q_i}-\frac{\partial B}{\partial\xi_2}\frac{\partial\xi_2}{\partial q_i}-\cdots-\frac{\partial B}{\partial\xi_\mu}\frac{\partial\xi_\mu}{\partial q_i} = \\ \frac{\partial B}{\partial v_1}\frac{\partial v_1}{\partial q_i}+\frac{\partial B}{\partial v_2}\frac{\partial v_2}{\partial q_i}+\cdots+\frac{\partial B}{\partial v_\mu}\frac{\partial v_\mu}{\partial q_i}, \\ \cdots\cdots\cdots\cdots \end{cases}$$

Si ponimus

$$M_{i'} = \frac{\partial \xi_1}{\partial q_i}\frac{\partial v_1}{\partial q_{i'}} + \frac{\partial \xi_2}{\partial q_i}\frac{\partial v_2}{\partial q_{i'}} + \cdots + \frac{\partial \xi_\mu}{\partial q_i}\frac{\partial v_\mu}{\partial q_{i'}}$$

atque

$$N_1^{(i)} = -\frac{\partial A}{\partial \xi_1}\frac{\partial \xi_1}{\partial q_i} - \frac{\partial A}{\partial \xi_2}\frac{\partial \xi_2}{\partial q_i} - \cdots - \frac{\partial A}{\partial \xi_\mu}\frac{\partial \xi_\mu}{\partial q_i},$$

$$N_2^{(i)} = -\frac{\partial B}{\partial \xi_1}\frac{\partial \xi_1}{\partial q_i} - \frac{\partial B}{\partial \xi_2}\frac{\partial \xi_2}{\partial q_i} - \cdots - \frac{\partial B}{\partial \xi_\mu}\frac{\partial \xi_\mu}{\partial q_i},$$

. . . . . . . . . . . . . . . . .

systema $m$ aequationum (14.) iunctum systemati aequationum (15.) convenit cum aequationibus (12.), siquidem $\mu$ quantitates $\frac{\partial v_1}{\partial q_i}, \frac{\partial v_2}{\partial q_i}, \ldots \frac{\partial v_\mu}{\partial q_i}$ pro incognitis $u_1, u_2, \ldots u_\mu$ habentur. Hinc secundum formulas (13.) nanciscimur:

$$\frac{\partial v_k}{\partial q_i} = u_k = \frac{\partial v_k}{\partial p_1}M_1 + \frac{\partial v_k}{\partial p_2}M_2 + \cdots + \frac{\partial v_k}{\partial p_m}M_m$$
$$+ D_{k,1}N_1^{(i)} + D_{k,2}N_2^{(i)} + \cdots.$$

Statuamus

$$(16.)\qquad \frac{\partial v_{k'}}{\partial q_1}\frac{\partial v_k}{\partial p_1} + \frac{\partial v_{k'}}{\partial q_2}\frac{\partial v_k}{\partial p_2} + \cdots + \frac{\partial v_{k'}}{\partial q_m}\frac{\partial v_k}{\partial p_m} = (k')_k,$$

ita ut $(k)_{k'} - (k')_k$ sit altera summa §. 40 investigatu proposita. Qua adhibita notatione, poterit aequatio praecedens hoc modo repraesentari:

$$(17.)\qquad \frac{\partial v_k}{\partial q_i} = \frac{\partial \xi_1}{\partial q_i}(1)_k + \frac{\partial \xi_2}{\partial q_i}(2)_k + \cdots + \frac{\partial \xi_\mu}{\partial q_i}(\mu)_k$$
$$+ D_{k,1}N_1^{(i)} + D_{k,2}N_2^{(i)} + \cdots.$$

Substitutis ipsorum $N_1^{(i)}$, $N_2^{(i)}$, etc. valoribus positoque

$$(18.)\qquad \begin{cases} w_1^{(k)} = (1)_k - D_{k,1}\frac{\partial A}{\partial \xi_1} - D_{k,2}\frac{\partial B}{\partial \xi_1} - \cdots, \\ w_2^{(k)} = (2)_k - D_{k,1}\frac{\partial A}{\partial \xi_2} - D_{k,2}\frac{\partial B}{\partial \xi_2} - \cdots, \\ \cdots\cdots\cdots\cdots \\ w_\mu^{(k)} = (\mu)_k - D_{k,1}\frac{\partial A}{\partial \xi_\mu} - D_{k,2}\frac{\partial B}{\partial \xi_\mu} - \cdots, \end{cases}$$

aequatio praecedens in hanc abit:

$$(19.)\qquad \frac{\partial v_k}{\partial q_i} = \frac{\partial \xi_1}{\partial q_i}w_1^{(k)} + \frac{\partial \xi_2}{\partial q_i}w_2^{(k)} + \cdots + \frac{\partial \xi_\mu}{\partial q_i}w_\mu^{(k)}.$$

E qua nanciscimur $m$ formulas tribuendo ipsi $i$ valores 1, 2, ... $m$. E quibus ut obtineatur rursus systema aequationum linearium formae aequationum (12.) investigemus adhuc valorem expressionum

$$N_1 = \frac{\partial A}{\partial v_1} w_1^{(k)} + \frac{\partial A}{\partial v_2} w_2^{(k)} + \cdots + \frac{\partial A}{\partial v_\mu} w_\mu^{(k)},$$

$$N_2 = \frac{\partial B}{\partial v_1} w_1^{(k)} + \frac{\partial B}{\partial v_2} w_2^{(k)} + \cdots + \frac{\partial B}{\partial v_\mu} w_\mu^{(k)},$$

$$\cdots\cdots\cdots\cdots$$

Quod hoc modo fieri potest.

## 43.

Habetur

$$\frac{\partial A}{\partial v_1}(1)_k + \frac{\partial A}{\partial v_2}(2)_k + \cdots + \frac{\partial A}{\partial v_\mu}(\mu)_k = \Sigma_{k'} \frac{\partial A}{\partial v_{k'}}(k')_k = \Sigma_{k'} \Sigma_i \frac{\partial A}{\partial v_{k'}} \frac{\partial v_{k'}}{\partial q_i} \frac{\partial v_k}{\partial p_i}.$$

Differentiando autem aequationem $A = 0$ secundum $q_i$ fit:

$$\Sigma_{k'} \frac{\partial A}{\partial v_{k'}} \frac{\partial v_{k'}}{\partial q_i} = -\Sigma_{k'} \frac{\partial A}{\partial \xi_{k'}} \frac{\partial \xi_{k'}}{\partial q_i},$$

unde

$$\Sigma_{k'} \frac{\partial A}{\partial v_{k'}}(k')_k = -\frac{\partial A}{\partial \xi_k} - \Sigma_{k'} \frac{\partial A}{\partial \xi_{k'}} k'_k.$$

Erat autem

$$k'_k = \lambda_1^{(k')} \frac{\partial F}{\partial \xi_k} + \lambda_2^{(k')} \frac{\partial \Phi}{\partial \xi_k} + \cdots.$$

Ponendo igitur:

$$(20.)\quad \begin{cases} A_1 = \Sigma_{k'} \frac{\partial A}{\partial \xi_{k'}} \lambda_1^{(k')} = \frac{\partial A}{\partial \xi_1} \lambda_1^{(1)} + \frac{\partial A}{\partial \xi_2} \lambda_1^{(2)} + \cdots + \frac{\partial A}{\partial \xi_\mu} \lambda_1^{(\mu)}, \\ A_2 = \Sigma_{k'} \frac{\partial A}{\partial \xi_{k'}} \lambda_2^{(k')} = \frac{\partial A}{\partial \xi_1} \lambda_2^{(1)} + \frac{\partial A}{\partial \xi_2} \lambda_2^{(2)} + \cdots + \frac{\partial A}{\partial \xi_\mu} \lambda_2^{(\mu)}, \\ \cdots\cdots\cdots\cdots \end{cases}$$

fit

$$\Sigma_{k'} \frac{\partial A}{\partial v_{k'}}(k')_k = -\frac{\partial A}{\partial \xi_k} - A_1 \frac{\partial F}{\partial \xi_k} - A_2 \frac{\partial \Phi}{\partial \xi_k} - \cdots.$$

Eodem modo ponendo

$$(21.)\quad \begin{cases} B_1 = \frac{\partial B}{\partial \xi_1} \lambda_1^{(1)} + \frac{\partial B}{\partial \xi_2} \lambda_1^{(2)} + \cdots + \frac{\partial B}{\partial \xi_\mu} \lambda_1^{(\mu)}, \\ B_2 = \frac{\partial B}{\partial \xi_1} \lambda_2^{(1)} + \frac{\partial B}{\partial \xi_2} \lambda_2^{(2)} + \cdots + \frac{\partial B}{\partial \xi_\mu} \lambda_2^{(\mu)}, \\ \cdots\cdots\cdots\cdots \end{cases}$$

fit

$$\Sigma_{k'} \frac{\partial B}{\partial v_{k'}} (k')_k = -\frac{\partial B}{\partial \xi_k} - B_1 \frac{\partial F}{\partial \xi_k} - B_2 \frac{\partial \Phi}{\partial \xi_k} - \cdots$$

et similes aequationes pro qualibet aequatione conditionali obtinentur.

Statuamus

$$(22.)\quad \begin{cases} \frac{\partial A}{\partial \xi_1}\frac{\partial A}{\partial v_1} + \frac{\partial A}{\partial \xi_2}\frac{\partial A}{\partial v_2} + \cdots + \frac{\partial A}{\partial \xi_\mu}\frac{\partial A}{\partial v_\mu} = \alpha_1, \\ \frac{\partial A}{\partial \xi_1}\frac{\partial B}{\partial v_1} + \frac{\partial A}{\partial \xi_2}\frac{\partial B}{\partial v_2} + \cdots + \frac{\partial A}{\partial \xi_\mu}\frac{\partial B}{\partial v_\mu} = \alpha_2, \\ \cdots\cdots\cdots\cdots \end{cases}$$

porro

$$(23.)\quad \begin{cases} \frac{\partial B}{\partial \xi_1}\frac{\partial A}{\partial v_1} + \frac{\partial B}{\partial \xi_2}\frac{\partial A}{\partial v_2} + \cdots + \frac{\partial B}{\partial \xi_\mu}\frac{\partial A}{\partial v_\mu} = \beta_1, \\ \frac{\partial B}{\partial \xi_1}\frac{\partial B}{\partial v_1} + \frac{\partial B}{\partial \xi_2}\frac{\partial B}{\partial v_2} + \cdots + \frac{\partial B}{\partial \xi_\mu}\frac{\partial B}{\partial v_\mu} = \beta_2, \\ \cdots\cdots\cdots\cdots \end{cases}$$

Tum multiplicando aequationes (9.) §. 41 per $\frac{\partial A}{\partial \xi_k}$, $\frac{\partial B}{\partial \xi_k}$, ... et summationem instituendo respectu indicis $k$, nanciscimur aequationum linearium systemata, quibus valores ipsarum $A_1$, $A_2$, ..., $B_1$, $B_2$, ... determinantur:

$$(24.)\quad \begin{cases} -\alpha_1 = a_1 A_1 + a_2 A_2 + \cdots, \\ -\alpha_2 = b_1 A_1 + b_2 A_2 + \cdots, \\ \cdots\cdots\cdots \\ -\beta_1 = a_1 B_1 + a_2 B_2 + \cdots, \\ -\beta_2 = b_1 B_1 + b_2 B_2 + \cdots, \\ \cdots\cdots\cdots \end{cases}$$

Quodlibet systema tot continet aequationes lineares totque incognitas quot datae sunt inter quantitates $\xi_1$, $\xi_2$, ... $\xi_\mu$ aequationes conditionales $F=0$, $\Phi=0$, .... Resolutione facta nanciscimur ipsorum $A_1$, $A_2$, ..., $B_1$, $B_2$, ... valores:

$$(25.)\quad \begin{cases} -A_1 = \varDelta_{1,1}\alpha_1 + \varDelta_{1,2}\alpha_2 + \cdots, \\ -A_2 = \varDelta_{2,1}\alpha_1 + \varDelta_{2,2}\alpha_2 + \cdots, \\ \cdots\cdots\cdots \\ -B_1 = \varDelta_{1,1}\beta_1 + \varDelta_{1,2}\beta_2 + \cdots, \\ -B_2 = \varDelta_{2,1}\beta_1 + \varDelta_{2,2}\beta_2 + \cdots, \\ \cdots\cdots\cdots \end{cases}$$

quibus in formulis coefficientes $\mathcal{A}_{i,i'}$ iidem sunt atque in (11.) §. 41. Antequam ulterius progrediar, valores ipsorum $D$ et ipsi ad quantitates $\mathcal{A}$ revocandi sunt. Eum in finem in aequationibus (12.) pono

$$u_1 = \frac{\partial F}{\partial \xi_1}, \quad u_2 = \frac{\partial F}{\partial \xi_2}, \quad \dots \quad u_\mu = \frac{\partial F}{\partial \xi_\mu},$$

unde fit

$$M_1 = 0, \quad M_2 = 0, \quad \dots \quad M_i = 0,$$
$$N_1 = a_1, \quad N_2 = b_1, \quad \dots,$$

quibus substitutis ex aequationibus (13.) obtinetur:

$$(26.) \qquad \frac{\partial F}{\partial \xi_k} = a_1 D_{k,1} + b_1 D_{k,2} + \cdots.$$

Eodemque modo fit:

$$(27.) \qquad \begin{cases} \dfrac{\partial \Phi}{\partial \xi_k} = a_2 D_{k,1} + b_2 D_{k,2} + \cdots \\ \dots\dots\dots\dots \end{cases}$$

Aequationibus (26.), (27.) resolutis prodeunt valores quaesiti:

$$(28.) \qquad \begin{cases} D_{k,1} = \mathcal{A}_{1,1} \dfrac{\partial F}{\partial \xi_k} + \mathcal{A}_{1,2} \dfrac{\partial \Phi}{\partial \xi_k} + \cdots, \\ D_{k,2} = \mathcal{A}_{2,1} \dfrac{\partial F}{\partial \xi_k} + \mathcal{A}_{2,2} \dfrac{\partial \Phi}{\partial \xi_k} + \cdots, \\ \dots\dots\dots\dots \end{cases}$$

His valoribus simulque ipsorum $A_1$, $A_2$, ... valoribus (25.) substitutis in aequatione:

$$\Sigma_{k'} \frac{\partial A}{\partial v_{k'}} (k')_k = -\frac{\partial A}{\partial \xi_k} - A_1 \frac{\partial F}{\partial \xi_k} - A_2 \frac{\partial \Phi}{\partial \xi_k} - \cdots,$$

simulque revocando, quod supra §. 41 adnotavi, esse

$$\mathcal{A}_{a,b} = \mathcal{A}_{b,a},$$

eruimus:

$$(29.) \qquad \Sigma_{k'} \frac{\partial A}{\partial v_{k'}} (k')_k = -\frac{\partial A}{\partial \xi_k} + D_{k,1}\alpha_1 + D_{k,2}\alpha_2 + \cdots,$$

eodemque modo fit:

$$(30.) \qquad \Sigma_{k'} \frac{\partial B}{\partial v_{k'}} (k')_k = -\frac{\partial B}{\partial \xi_k} + D_{k,1}\beta_1 + D_{k,2}\beta_2 + \cdots.$$

Hinc, substituto e (18.) valore:

$$w_{k'}^{(k)} = (k')_k - D_{k,1} \frac{\partial A}{\partial \xi_{k'}} - D_{k,2} \frac{\partial B}{\partial \xi_{k'}} - \cdots,$$

prodit:

$$\begin{aligned}
\Sigma_{k'} \frac{\partial A}{\partial v_{k'}} w_{k'}^{(k)} &= \frac{\partial A}{\partial v_1} w_1^{(k)} + \frac{\partial A}{\partial v_2} w_2^{(k)} + \cdots + \frac{\partial A}{\partial v_\mu} w_\mu^{(k)} \\
&= -\frac{\partial A}{\partial \xi_k} + D_{k,1}\alpha_1 + D_{k,2}\alpha_2 + \cdots \\
&\qquad - D_{k,1}\alpha_1 - D_{k,2}\beta_1 - \cdots, \\
\Sigma_{k'} \frac{\partial B}{\partial v_{k'}} w_{k'}^{(k)} &= \frac{\partial B}{\partial v_1} w_1^{(k)} + \frac{\partial B}{\partial v_2} w_2^{(k)} + \cdots + \frac{\partial B}{\partial v_\mu} w_\mu^{(k)} \\
&= -\frac{\partial B}{\partial \xi_k} + D_{k,1}\beta_1 + D_{k,2}\beta_2 + \cdots \\
&\qquad - D_{k,1}\alpha_2 - D_{k,2}\beta_2 - \cdots \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

*Statuamus, quaecunque sint $A$ et $B$ variabilium $\xi_1, \xi_2, \ldots \xi_\mu, v_1, v_2, \ldots v_\mu$ functiones,*

$$(31.)\quad \begin{cases} [A, B]' = \dfrac{\partial A}{\partial \xi_1}\dfrac{\partial B}{\partial v_1} + \dfrac{\partial A}{\partial \xi_2}\dfrac{\partial B}{\partial v_2} + \cdots + \dfrac{\partial A}{\partial \xi_\mu}\dfrac{\partial B}{\partial v_\mu} \\ \qquad - \dfrac{\partial A}{\partial v_1}\dfrac{\partial B}{\partial \xi_1} - \dfrac{\partial A}{\partial v_2}\dfrac{\partial B}{\partial \xi_2} - \cdots - \dfrac{\partial A}{\partial v_\mu}\dfrac{\partial B}{\partial \xi_\mu}, \end{cases}$$

qua in notatione plagulam superposui, ut expressionem distinguam a similiter respectu variabilium $q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m$ formata.

Eruentur hac nova notatione ad formulas praecedentes applicata expressiones quaesitae:

$$(32.)\quad \begin{cases} N_1 = \dfrac{\partial A}{\partial v_1} w_1^{(k)} + \dfrac{\partial A}{\partial v_2} w_2^{(k)} + \cdots + \dfrac{\partial A}{\partial v_\mu} w_\mu^{(k)} \\ \quad = -\dfrac{\partial A}{\partial \xi_k} + D_{k,2}[A, B]' + \cdots, \\ N_2 = \dfrac{\partial B}{\partial v_1} w_1^{(k)} + \dfrac{\partial B}{\partial v_2} w_2^{(k)} + \cdots + \dfrac{\partial B}{\partial v_\mu} w_\mu^{(k)} \\ \quad = -\dfrac{\partial B}{\partial \xi_k} + D_{k,1}[B, A]' + \cdots, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Quae aequationes iunctae $m$ aequationibus, quae tribuendo ipsi $i$ valores $1, 2, \ldots m$ obtinentur e (19.), suppeditant systema aequationum linearium ipsis (12.) simile.

Quarum resolutio suppeditat e (13.):

$$w_{k'}^{(k)} = (k)_{k'} + D_{k',1}N_1 + D_{k',2}N_2 + \cdots$$

Unde substitutis praecedentibus ipsorum $N_1, N_2, \ldots$ valoribus (32.) atque e

(18.) valore

$$w_{k'}^{(k)} = (k')_k - D_{k,1}\frac{\partial A}{\partial \xi_{k'}} - D_{k,2}\frac{\partial B}{\partial \xi_{k'}} - \cdots,$$

prodit altera summa §. 40 investigatu proposita

$$(33.)\quad \begin{cases} (k')_k - (k)_{k'} = D_{k,1}\frac{\partial A}{\partial \xi_{k'}} - D_{k',1}\frac{\partial A}{\partial \xi_k} + D_{k,2}\frac{\partial B}{\partial \xi_{k'}} - D_{k',2}\frac{\partial B}{\partial \xi_k} + \cdots \\ \qquad + (D_{k',1}D_{k,2} - D_{k,1}D_{k',2})[A, B]' + \cdots. \end{cases}$$

Quae est expressio quaesita, e qua si ipsorum $D$ valores (28.) substituis variabiles $q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m$ prorsus, quod postulabatur, abierunt. Observo in formula (33.) terminos

$$(D_{k',1}D_{k,2} - D_{k,1}D_{k',2})[A, B]'$$

inveniri tot, quot binarum aequationum conditionalium habentur combinationes, sive numerum $\frac{(\mu-m)(\mu-m-1)}{2}$. Unde si *unica* tantum aequatio conditionalis datur, eiusmodi non habentur termini. Eo casu habetur, si $F=0$ est aequatio conditionalis:

$$a.k_{k'} = -\frac{\partial F}{\partial \xi_{k'}}\frac{\partial A}{\partial v_k},$$

$$a.\{(k')_k - (k)_{k'}\} = \frac{\partial F}{\partial \xi_k}\frac{\partial A}{\partial \xi_{k'}} - \frac{\partial F}{\partial \xi_{k'}}\frac{\partial A}{\partial \xi_k},$$

ubi

$$A = \Sigma\frac{\partial F}{\partial \xi_k}\frac{\partial H}{\partial v_k}, \qquad a = \Sigma\frac{\partial F}{\partial \xi_k}\cdot\frac{\partial F}{\partial \xi_{k'}}\frac{\partial^2 H}{\partial v_k \partial v_{k'}},$$

siquidem in altera summa ipsi $k$, in altera ipsis $k$ et $k'$ valores $1, 2, \ldots \mu$ tribuuntur.

*Formulae antecedentes applicantur ad casum, quo ipsae $\xi_1, \xi_2, \ldots \xi_\mu$ coordinatas orthogonales punctorum materialium significant.*

## 44.

Sit $\mu = 3n$ simulque quantitates $\xi_1, \xi_2, \ldots \xi_{3n}$ designent $3n$ coordinatas punctorum motorum orthogonales, sitque puncti, cuius una coordinata per $\xi_k$ denotatur, massa $m_k$, ita ut quantitatum $m_1, m_2, \ldots m_\mu$ ternae ad idem punctum pertinentes inter se aequales existant. Tum erit, designante $U$ solarum $\xi_k$ functionem ab ipsis $\xi'_k$ vacuam atque $T$ semissem virium vivarum,

$$H = T - U = \tfrac{1}{2}\Sigma m_k \xi'_k \xi'_k - U,$$

$$v_k = \frac{\partial T}{\partial \xi'_k} = m_k \xi'_k,$$

atque

$$A = \Sigma \frac{\partial F}{\partial \xi_k} \cdot \xi'_k, \qquad B = \Sigma \frac{\partial \Phi}{\partial \xi_k} \cdot \xi'_k, \qquad \ldots$$

$$a_1 = \Sigma \frac{1}{m_k} \left( \frac{\partial F}{\partial \xi_k} \right)^2, \quad a_2 = b_1 = \Sigma \frac{1}{m_k} \frac{\partial F}{\partial \xi_k} \frac{\partial \Phi}{\partial \xi_k}, \quad b_2 = \Sigma \frac{1}{m_k} \left( \frac{\partial \Phi}{\partial \xi_k} \right)^2, \quad \ldots$$

$$m_k \lambda_{k'}^{(k)} = A_{k',1} \frac{\partial F}{\partial \xi_k} + A_{k',2} \frac{\partial \Phi}{\partial \xi_k} + \cdots,$$

$$m_k k_{k'} = m_{k'} k'_k = A_{1,1} \frac{\partial F}{\partial \xi_k} \frac{\partial F}{\partial \xi_{k'}} + A_{1,2} \left\{ \frac{\partial F}{\partial \xi_k} \frac{\partial \Phi}{\partial \xi_{k'}} + \frac{\partial F}{\partial \xi_{k'}} \frac{\partial \Phi}{\partial \xi_k} \right\}$$

$$+ A_{2,2} \frac{\partial \Phi}{\partial \xi_k} \frac{\partial \Phi}{\partial \xi_{k'}} + \cdots$$

. . . . . . . . . . . . . . . .

Adnoto data occasione, fieri pro assignata ipsarum $\xi_k$ significatione

$$m_k . k_{k'} = m_{k'} . k'_k$$

sive

$$\Sigma_i \frac{\partial \xi_k}{\partial q_i} \frac{\partial \xi'_{k'}}{\partial p_i} = \Sigma_i \frac{\partial \xi_{k'}}{\partial q_i} \frac{\partial \xi'_k}{\partial p_i}$$

quod facile hoc modo intelligitur. Est

$$\xi'_{k'} = \Sigma_{i'} \frac{\partial \xi_{k'}}{\partial q_{i'}} \cdot q'_{i'}, \qquad \xi'_k = \Sigma_i \frac{\partial \xi_k}{\partial q_{i'}} \cdot q'_{i'},$$

unde

$$\Sigma_i \frac{\partial \xi_k}{\partial q_i} \frac{\partial \xi'_{k'}}{\partial p_i} = \Sigma_{i,i'} \frac{\partial \xi_k}{\partial q_i} \frac{\partial \xi_{k'}}{\partial q_{i'}} \cdot \frac{\partial q'_{i'}}{\partial p_i},$$

$$\Sigma_i \frac{\partial \xi_{k'}}{\partial q_i} \frac{\partial \xi'_k}{\partial p_i} = \Sigma_{i,i'} \frac{\partial \xi_{k'}}{\partial q_i} \frac{\partial \xi_k}{\partial q_{i'}} \cdot \frac{\partial q'_{i'}}{\partial p_i}.$$

Est autem

$$q'_i = \frac{\partial H}{\partial p_i}, \qquad q'_{i'} = \frac{\partial H}{\partial p_{i'}},$$

unde

$$\frac{\partial q'_{i'}}{\partial p_i} = \frac{\partial q'_i}{\partial p_{i'}} = \frac{\partial^2 H}{\partial p_i \partial p_{i'}},$$

sive expressio $\frac{\partial q'_{i'}}{\partial p_i}$ indicibus $i$ et $i'$ commutatis immutata manet. Unde altera summarum appositarum duplicium, scribendo $i'$ loco $i$ atque $i$ loco $i'$, in alteram abit, sive binae inter se aequales existunt, q. d. e.

*De usu functionum $\Delta$ in determinandis multiplicatoribus Lagrangianis.*

45.

Quantitatibus $\Delta_{a,b}$, quibus antecedentibus usi sumus, etiam *mnltiplicatores Lagrangiani* determinantur, qui formandis aequationibus differentialibus dynamicis inserviunt, quoties inter variabiles, quae punctorum materialium positionem determinant, aequationes conditionales habentur. Ad quas formandas aequationes differentiales, adhibeo formulam symbolicam §. 37 (1.) propositam, in qua, ut $q_1, q_2, \ldots q_m$ semper variabiles independentes designent, loco $q_1, q_2, \ldots q_m$, $p_1, p_2, \ldots p_m$ scribo $\xi_1, \xi_2, \ldots \xi_\mu, v_1, v_2, \ldots v_\mu$. Quo facto aequatio illa haec fit:

$$(1.)\quad 0 = \left\{\frac{\partial H}{\partial \xi_1}+\frac{dv_1}{dt}\right\}\delta\xi_1+\left\{\frac{\partial H}{\partial \xi_2}+\frac{dv_2}{dt}\right\}\delta\xi_2+\cdots+\left\{\frac{\partial H}{\partial \xi_\mu}+\frac{dv_\mu}{dt}\right\}\delta\xi_\mu .$$

Inter variationes $\delta\xi_1$, $\delta\xi_2$, etc. habentur aequationes:

$$\begin{aligned}
\frac{\partial F}{\partial \xi_1}\delta\xi_1+\frac{\partial F}{\partial \xi_2}\delta\xi_2+\cdots+\frac{\partial F}{\partial \xi_\mu}\delta\xi_\mu &= 0,\\
\frac{\partial \Phi}{\partial \xi_1}\delta\xi_1+\frac{\partial \Phi}{\partial \xi_2}\delta\xi_2+\cdots+\frac{\partial \Phi}{\partial \xi_\mu}\delta\xi_\mu &= 0,\\
\ldots\ldots\ldots\ldots\ldots\ldots &,
\end{aligned}$$

siquidem rursus $F=0$, $\Phi=0$, etc. sunt aequationes conditionales inter quantitates $\xi_1, \xi_2, \ldots \xi_\mu$ propositae. Per regulam notam aequationes praecedentes in multiplicatores $\lambda_1, \lambda_2, \ldots$ ductas aequationis (1.) alteri parti addo et terminos in singulas variationes ductos evanescere statuo. Quo facto aequationes differentiales inter variabiles $t, \xi_1, \xi_2, \ldots \xi_\mu, v_1, v_2, \ldots v_\mu$ obtinentur sequentes, insuper aequationibus $\xi_i' = \frac{\partial H}{\partial v_i}$ advocatis:

$$(2.)\quad \left\{\begin{array}{ll}
\frac{d\xi_1}{dt} = \frac{\partial H}{\partial v_1}, & \frac{dv_1}{dt} = -\frac{\partial H}{\partial \xi_1}-\lambda_1\frac{\partial F}{\partial \xi_1}-\lambda_2\frac{\partial \Phi}{\partial \xi_1}-\cdots,\\
\frac{d\xi_2}{dt} = \frac{\partial H}{\partial v_2}, & \frac{dv_2}{dt} = -\frac{\partial H}{\partial \xi_2}-\lambda_1\frac{\partial F}{\partial \xi_2}-\lambda_2\frac{\partial \Phi}{\partial \xi_2}-\cdots,\\
\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\\
\frac{d\xi_\mu}{dt} = \frac{\partial H}{\partial v_\mu}, & \frac{dv_\mu}{dt} = -\frac{\partial H}{\partial \xi_\mu}-\lambda_1\frac{\partial F}{\partial \xi_\mu}-\lambda_2\frac{\partial \Phi}{\partial \xi_\mu}-\cdots.
\end{array}\right.$$

Quibus aequationibus adiungendae sunt ipsae aequationes conditionales

$$F=0,\quad \Phi=0,\quad \ldots$$

et quae ex earum differentiatione sequuntur:

$$A=0,\quad B=0,\quad \ldots .$$

His postremis iterum differentiatis et substitutis e (2.) ipsorum $\frac{d\xi_i}{dt}$, $\frac{dv_i}{dt}$ valoribus obtinemus:

$$\begin{aligned}
&\frac{\partial A}{\partial \xi_1}\frac{\partial H}{\partial v_1}+\frac{\partial A}{\partial \xi_2}\frac{\partial H}{\partial v_2}+\cdots+\frac{\partial A}{\partial \xi_\mu}\frac{\partial H}{\partial v_\mu} \\
&-\frac{\partial A}{\partial v_1}\frac{\partial H}{\partial \xi_1}-\frac{\partial A}{\partial v_2}\frac{\partial H}{\partial \xi_2}-\cdots-\frac{\partial A}{\partial v_\mu}\frac{\partial H}{\partial \xi_\mu} = a_1\lambda_1+a_2\lambda_2+\cdots, \\
&\frac{\partial B}{\partial \xi_1}\frac{\partial H}{\partial v_1}+\frac{\partial B}{\partial \xi_2}\frac{\partial H}{\partial v_2}+\cdots+\frac{\partial B}{\partial \xi_\mu}\frac{\partial H}{\partial \xi_\mu} \\
&-\frac{\partial B}{\partial v_1}\frac{\partial H}{\partial \xi_1}-\frac{\partial B}{\partial v_2}\frac{\partial H}{\partial \xi_2}-\cdots-\frac{\partial B}{\partial v_\mu}\frac{\partial H}{\partial \xi_\mu} = b_1\lambda_1+b_2\lambda_2+\cdots, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

siquidem hic $a_1, a_2, \ldots, b_1, b_2, \ldots$ eaedem sunt quantitates atque §. 41., (10.). Unde, si advocamus notationem §. 43., (31.) propositam, eruimus valores multiplicatorum sequentes:

$$(3.)\quad \begin{cases} \lambda_1 = A_{1,1}[A, H]'+A_{1,2}[B, H]'+\cdots, \\ \lambda_2 = A_{2,1}[A, H]'+A_{2,2}[B, H]'+\cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Unde e §. 43., (28.) aequationes differentiales dynamicae fiunt:

$$(4.)\quad \begin{cases} \frac{dv_1}{dt} = -\frac{\partial H}{\partial \xi_1}-D_{1,1}[A, H]'-D_{1,2}[B, H]'-\cdots, \\ \frac{dv_2}{dt} = -\frac{\partial H}{\partial \xi_2}-D_{2,1}[A, H]'-D_{2,2}[B, H]'-\cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \frac{dv_\mu}{dt} = -\frac{\partial H}{\partial \xi_\mu}-D_{\mu,1}[A, H]'-D_{\mu,2}[B, H]'-\cdots \end{cases}$$

e quibus iam multiplicatores sunt eliminati.

*Ope summarum supra inventarum ipsa expressio $[\varphi, \psi]$ formatur.*

## 46.

Formulam (3.), §. 40:

$$[\varphi, \psi] = \Sigma\left\{\frac{\partial \varphi}{\partial \xi_k}\frac{\partial \psi}{\partial v_{k'}}-\frac{\partial \varphi}{\partial v_{k'}}\frac{\partial \psi}{\partial \xi_k}\right\}\frac{\partial \xi_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i}+\Sigma\frac{\partial \varphi}{\partial v_k}\frac{\partial \psi}{\partial v_{k'}}\left\{\frac{\partial v_k}{\partial q_i}\frac{\partial v_{k'}}{\partial p_i}-\frac{\partial v_{k'}}{\partial q_i}\frac{\partial v_k}{\partial p_i}\right\}$$

e notationibus supra adhibitis §. 41., (6.) atque §. 42., (16.) sic exhibere possumus:

$$
(1.)\quad \left\{\begin{array}{c}
[\varphi, \psi] = \\
\Sigma_k \left\{ \frac{\partial\varphi}{\partial\xi_k}\frac{\partial\psi}{\partial v_k} - \frac{\partial\varphi}{\partial v_k}\frac{\partial\psi}{\partial\xi_k} \right\} + \Sigma_{k,k'} \left\{ \frac{\partial\varphi}{\partial\xi_k}\frac{\partial\psi}{\partial v_{k'}} - \frac{\partial\varphi}{\partial v_{k'}}\frac{\partial\psi}{\partial\xi_k} \right\} k_{k'} \\
+\Sigma_{k,k'} \frac{\partial\varphi}{\partial v_k}\frac{\partial\psi}{\partial v_{k'}} \{(k)_{k'} - (k')_k\}.
\end{array}\right.
$$

Qua aequatione, si ipsorum $k_{k'}$ atque $(k)_{k'} - (k')_k$ valores supra inventi substituuntur, prodit ipsius $[\varphi, \psi]$ expressio investigatu proposita, in qua variabilium $q_1, q_2, \ldots q_m$ vestigia nulla inveniuntur.

Summarum, e quibus expressio antecedens componitur, secundae et tertiae transformationes sequentes adiungo.

Habetur e §. 41., (8.):

$$
\begin{aligned}
&\Sigma_{k,k'} \left\{ \frac{\partial\varphi}{\partial\xi_k}\frac{\partial\psi}{\partial v_{k'}} - \frac{\partial\varphi}{\partial v_{k'}}\frac{\partial\psi}{\partial\xi_k} \right\} k_{k'} = \\
&\Sigma_k \frac{\partial\varphi}{\partial\xi_k} \left\{ k_1 \frac{\partial\psi}{\partial v_1} + k_2 \frac{\partial\psi}{\partial v_2} + \cdots + k_\mu \frac{\partial\psi}{\partial v_\mu} \right\} \\
&- \Sigma_k \frac{\partial\psi}{\partial\xi_k} \left\{ k_1 \frac{\partial\varphi}{\partial v_1} + k_2 \frac{\partial\varphi}{\partial v_2} + \cdots + k_\mu \frac{\partial\varphi}{\partial v_\mu} \right\} \\
&= \left\{ \frac{\partial F}{\partial\xi_1}\frac{\partial\psi}{\partial v_1} + \frac{\partial F}{\partial\xi_2}\frac{\partial\psi}{\partial v_2} + \cdots + \frac{\partial F}{\partial\xi_\mu}\frac{\partial\psi}{\partial v_\mu} \right\} \Sigma_k \lambda_1^{(k)} \frac{\partial\varphi}{\partial\xi_k} \\
&- \left\{ \frac{\partial F}{\partial\xi_1}\frac{\partial\varphi}{\partial v_1} + \frac{\partial F}{\partial\xi_2}\frac{\partial\varphi}{\partial v_2} + \cdots + \frac{\partial F}{\partial\xi_\mu}\frac{\partial\varphi}{\partial v_\mu} \right\} \Sigma_k \lambda_1^{(k)} \frac{\partial\psi}{\partial\xi_k} \\
&+ \left\{ \frac{\partial\Phi}{\partial\xi_1}\frac{\partial\psi}{\partial v_1} + \frac{\partial\Phi}{\partial\xi_2}\frac{\partial\psi}{\partial v_2} + \cdots + \frac{\partial\Phi}{\partial\xi_\mu}\frac{\partial\psi}{\partial v_\mu} \right\} \Sigma_k \lambda_2^{(k)} \frac{\partial\varphi}{\partial\xi_k} \\
&- \left\{ \frac{\partial\Phi}{\partial\xi_1}\frac{\partial\varphi}{\partial v_1} + \frac{\partial\Phi}{\partial\xi_2}\frac{\partial\varphi}{\partial v_2} + \cdots + \frac{\partial\Phi}{\partial\xi_\mu}\frac{\partial\varphi}{\partial v_\mu} \right\} \Sigma_k \lambda_2^{(k)} \frac{\partial\psi}{\partial\xi_k} \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

sive etiam:

$$
(2.)\quad \left\{\begin{array}{l}
\Sigma_{k,k'} \left\{ \frac{\partial\varphi}{\partial\xi_k}\frac{\partial\psi}{\partial v_{k'}} - \frac{\partial\varphi}{\partial v_{k'}}\frac{\partial\psi}{\partial\xi_k} \right\} k_{k'} = \\
\Sigma_k \frac{\partial\varphi}{\partial\xi_k} \{ \lambda_1^{(k)} [F, \psi]' + \lambda_2^{(k)} [\Phi, \psi]' + \cdots \} \\
-\Sigma_k \frac{\partial\psi}{\partial\xi_k} \{ \lambda_1^{(k)} [F, \varphi]' + \lambda_2^{(k)} [\Phi, \varphi]' + \cdots \}.
\end{array}\right.
$$

Habetur porro e §. 43., (33.):

$$(3.)\quad\begin{cases}\Sigma_{k,k'}\dfrac{\partial\varphi}{\partial v_k}\dfrac{\partial\psi}{\partial v_{k'}}\{(k')_k-(k)_{k'}\} = \\ \Sigma_k\dfrac{\partial\varphi}{\partial v_k}D_{k,1}.\Sigma_k\dfrac{\partial\psi}{\partial v_k}\dfrac{\partial A}{\partial\xi_k}-\Sigma_k\dfrac{\partial\psi}{\partial v_k}D_{k,1}.\Sigma_k\dfrac{\partial\varphi}{\partial v_k}\dfrac{\partial A}{\partial\xi_k} \\ +\Sigma_k\dfrac{\partial\varphi}{\partial v_k}D_{k,2}.\Sigma_k\dfrac{\partial\psi}{\partial v_k}\dfrac{\partial B}{\partial\xi_k}-\Sigma_k\dfrac{\partial\psi}{\partial v_k}D_{k,2}.\Sigma_k\dfrac{\partial\varphi}{\partial v_k}\dfrac{\partial B}{\partial\xi_k} \\ \dots\dots\dots\dots\dots\dots \\ -[A,B]'\left\{\Sigma_k\dfrac{\partial\varphi}{\partial v_k}D_{k,1}.\Sigma_k\dfrac{\partial\psi}{\partial v_k}D_{k,2}-\Sigma_k\dfrac{\partial\varphi}{\partial v_k}D_{k,2}.\Sigma_k\dfrac{\partial\psi}{\partial v_k}\cdot D_{k,1}\right\} \\ \dots\dots\dots\dots\dots\dots\end{cases}$$

Fit autem e §. 43., (28.):

$$\Sigma_k D_{k,k'}\frac{\partial\varphi}{\partial v_k} = A_{k',1}\Sigma_k\frac{\partial F}{\partial\xi_k}\frac{\partial\varphi}{\partial v_k}+A_{k',2}\Sigma_k\frac{\partial\Phi}{\partial\xi_k}\frac{\partial\varphi}{\partial v_k}+\cdots,$$

$$\Sigma_k D_{k,k'}\frac{\partial\psi}{\partial v_k} = A_{k',1}\Sigma_k\frac{\partial F}{\partial\xi_k}\frac{\partial\psi}{\partial v_k}+A_{k',2}\Sigma_k\frac{\partial\Phi}{\partial\xi_k}\frac{\partial\psi}{\partial v_k}+\cdots$$

sive

$$(4.)\quad\begin{cases}\Sigma_k D_{k,k'}\dfrac{\partial\varphi}{\partial v_k} = A_{k',1}[F,\varphi]'+A_{k',2}[\Phi,\varphi]'+\cdots, \\ \Sigma_k D_{k,k'}\dfrac{\partial\psi}{\partial v_k} = A_{k',1}[F,\psi]'+A_{k',2}[\Phi,\psi]'+\cdots.\end{cases}$$

His formulis si utimur et advocatis (11.) §. 41. aequationem (2.) sic repraesentare licet:

$$(5.)\quad\begin{cases}\Sigma_{k,k'}\left\{\dfrac{\partial\varphi}{\partial\xi_k}\dfrac{\partial\psi}{\partial v_{k'}}-\dfrac{\partial\varphi}{\partial v_{k'}}\dfrac{\partial\psi}{\partial\xi_k}\right\}k_{k'} = \\ -\Sigma_k D_{k,1}\dfrac{\partial\psi}{\partial v_k}\cdot\Sigma_k\dfrac{\partial\varphi}{\partial\xi_k}\dfrac{\partial A}{\partial v_k}-\Sigma_k D_{k,2}\dfrac{\partial\psi}{\partial v_k}\cdot\Sigma_k\dfrac{\partial\varphi}{\partial\xi_k}\dfrac{\partial B}{\partial v_k}-\cdots \\ +\Sigma_k D_{k,1}\dfrac{\partial\varphi}{\partial v_k}\cdot\Sigma_k\dfrac{\partial\psi}{\partial\xi_k}\dfrac{\partial A}{\partial v_k}+\Sigma_k D_{k,2}\dfrac{\partial\varphi}{\partial v_k}\cdot\Sigma_k\dfrac{\partial\psi}{\partial\xi_k}\dfrac{\partial B}{\partial v_k}+\cdots.\end{cases}$$

Formulis (3.) et (5.) substitutis in (1.) prodit:

$$(6.)\quad\begin{cases}[\varphi,\psi]-[\varphi,\psi]' = \\ -[\varphi,A]'\Sigma_k D_{k,1}\dfrac{\partial\psi}{\partial v_k}-[\varphi,B]'\Sigma_k D_{k,2}\dfrac{\partial\psi}{\partial v_k}-\cdots \\ +[\psi,A]'\Sigma_k D_{k,1}\dfrac{\partial\varphi}{\partial v_k}+[\psi,B]'\Sigma_k D_{k,2}\dfrac{\partial\varphi}{\partial v_k}+\cdots \\ +[A,B]'\left\{\Sigma_k D_{k,1}\dfrac{\partial\varphi}{\partial v_k}\cdot\Sigma_k D_{k,2}\dfrac{\partial\psi}{\partial v_k}-\Sigma_k D_{k,2}\dfrac{\partial\varphi}{\partial v_k}\cdot\Sigma_k D_{k,1}\dfrac{\partial\psi}{\partial v_k}\right\} \\ \dots\dots\dots\dots\dots\dots\end{cases}$$

Quae formula generalis satis difficilis erat investigatu.

*Expressio inventa per varias eius proprietates verificatur.*

## 47.

Quantitas per quam in §. antecedente ipsam $[\varphi, \psi]$ expressi et quam denotabo per

$$(1.)\quad \begin{cases} \Xi = [\varphi, \psi]' - [\varphi, A]'\Sigma_k D_{k,1}\dfrac{\partial\psi}{\partial v_k} - [\varphi, B]'\Sigma_k D_{k,2}\dfrac{\partial\psi}{\partial v_k} - \cdots \\ \qquad + [\psi, A]'\Sigma_k D_{k,1}\dfrac{\partial\varphi}{\partial v_k} + [\psi, B]'\Sigma_k D_{k,2}\dfrac{\partial\varphi}{\partial v_k} + \cdots \\ \qquad + [A, B]'\left\{\Sigma_k D_{k,1}\dfrac{\partial\varphi}{\partial v_k}\cdot\Sigma_k D_{k,2}\dfrac{\partial\psi}{\partial v_k} - \Sigma_k D_{k,1}\dfrac{\partial\psi}{\partial v_k}\cdot\Sigma_k D_{k,2}\dfrac{\partial\varphi}{\partial v_k}\right\} \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

variis gaudere debet proprietatibus memorabilibus, quae simul varias expressionis inventae verificationes suppeditant. Ac primum non mutetur eius valor necesse est, si in locum functionum $\varphi$, $\psi$ ponatur

$$\varphi + \lambda F + \mu \Phi + \cdots + \lambda' A + \mu' B + \cdots,$$
$$\psi + \lambda_1 F + \mu_1 \Phi + \cdots + \lambda_1' A + \mu_1' B + \cdots,$$

designantibus $\lambda$, $\mu$, $\lambda'$, $\mu'$, $\lambda_1$, $\mu_1$, $\lambda_1'$, $\mu_1'$, ... quascunque ipsarum $\xi_i$, $v_i$ functiones. Valor enim quantitatis $[\varphi, \psi]$, cui expressio $\Xi$ aequalis inventa est, ea mutatione nullo modo afficitur. Quae expressionis $\Xi$ proprietas ex ipsa eius formatione facile patebit, si haec alia propositio antea demonstrata erit, *expressionem $\Xi$, posita in locum alterutrius functionum $\varphi$, $\psi$ una e functionibus $F$, $\Phi$, ..., $A$, $B$, ..., quaecunque sit altera functio, evanescere.* Quae propositio tantum probanda erit pro casibus, quibus $\varphi = F$ atque $\varphi = A$ ponitur, functione $\psi$ arbitraria manente. Reliqui enim casus, quibus $\varphi$ functionibus $\Phi$, ..., $B$, ... aequiparatur, sive quibus $\varphi$ arbitraria manet atque $\psi$ alicui e functionibus $F$, $\Phi$, ..., $A$, $B$, ... aequalis ponitur, prorsus eodem modo tractari possunt.

Posito $\varphi = F$, evanescunt termini

$$\Sigma_k D_{k,1}\frac{\partial\varphi}{\partial v_k}, \quad \Sigma_k D_{k,2}\frac{\partial\varphi}{\partial v_k}, \quad \ldots$$

quum functio $F$ solas $\xi_k$ involvat neque quantitates $v_k$. Hinc posito $\varphi = F$ eruimus:

$$\begin{aligned} \Xi &= [F, \psi]' - [F, A]'\Sigma_k D_{k,1}\frac{\partial\psi}{\partial v_k} - [F, B]'\Sigma_k D_{k,2}\frac{\partial\psi}{\partial v_k} - \cdots \\ &= [F, \psi]' - \Sigma_k\{[F, A]'D_{k,1} + [F, B]'D_{k,2} + \cdots\}\frac{\partial\psi}{\partial v_k}. \end{aligned}$$

At e formulis (26.), (27.) §. 43, in quibus est

$$a_1 = [F, A]', \quad b_1 = [F, B]', \quad \dots$$
$$a_2 = [\Phi, A]', \quad b_2 = [\Phi, B]', \quad \dots$$

habetur:

$$(2.) \quad \begin{cases} \dfrac{\partial F}{\partial \xi_k} = [F, A]' D_{k,1} + [F, B]' D_{k,2} + \cdots, \\ \dfrac{\partial \Phi}{\partial \xi_k} = [\Phi, A]' D_{k,1} + [\Phi, B]' D_{k,2} + \cdots, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

Quarum formularum ope abit expressio ipsius $\Xi$ antecedens in hanc:

$$\Xi = [F, \psi]' - \Sigma_k \frac{\partial F}{\partial \xi_k} \frac{\partial \psi}{\partial v_k} = 0.$$

Evanescit igitur $\Xi$ posito $\varphi = F$, q. d. e. Eodem modo demonstratur evanescere $\Xi$ ponendo $\varphi = \Phi$, vel ponendo $\psi = F$, sive $\psi = \Phi$.

Ponamus iam in expressione (1.) $\varphi = A$; quaerendi sunt ante omnia valores quantitatum

$$E_1 = \Sigma_k D_{k,1} \frac{\partial A}{\partial v_k}, \quad E_2 = \Sigma_k D_{k,2} \frac{\partial A}{\partial v_k}, \quad \dots .$$

Ad quos inveniendos multiplicentur (2.) per $\dfrac{\partial A}{\partial v_k}$ atque positis loco $k$ valoribus 1, 2, ... $\mu$ instituatur pro singulis aequationibus (2.) summatio; provenit:

$$\Sigma_k \frac{\partial A}{\partial v_k} \frac{\partial F}{\partial \xi_k} = [F, A]' = [F, A]' E_1 + [F, B]' E_2 + \cdots,$$
$$\Sigma_k \frac{\partial A}{\partial v_k} \frac{\partial \Phi}{\partial \xi_k} = [\Phi, A]' = [\Phi, A]' E_1 + [\Phi, B]' E_2 + \cdots,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Unde obtinetur

$$(3.) \quad E_1 = \Sigma_k D_{k,1} \frac{\partial A}{\partial v_k} = 1, \quad E_2 = \Sigma_k D_{k,2} \frac{\partial A}{\partial v_2} = 0, \quad \dots$$

ac si aequationes $F = 0$, $\Phi = 0$, ... plures duabus datae sunt, evanescunt reliquae omnes similes expressiones $\Sigma_k D_{k,3} \dfrac{\partial A}{\partial v_k}$, $\Sigma_k D_{k,4} \dfrac{\partial A}{\partial v_k}$, .... Eodem modo probatur fieri

$$\Sigma_k D_{k,2} \frac{\partial B}{\partial v_k} = 1,$$

atque evanescere reliquas omnes quantitates $\Sigma_k D_{k,1} \dfrac{\partial B}{\partial v_k}$, $\Sigma_k D_{k,3} \dfrac{\partial B}{\partial v_k}$, .... Per aequationes (3.) videmus, ipsam $\Xi$ posito $\varphi = A$ evanescere, quum sit $[A, A]' = 0$, $[A, \psi]' + [\psi, A]' = 0$, ac facile pateat, quemadmodum ea substitutione facta termini in $[A, B]'$ ducti evanescunt, ita si aequationes conditionales plures duabus datae sint, evanescere terminos ductos in expressiones similes, quarum numerus idem est atque numerus combinationum binarum aequationum conditionalium. Eodem modo demonstratur, evanescere $\Xi$ ponendo $\varphi = B$ vel ponendo $\psi = A$ sive $\psi = B$.

Designemus iam expressionem (1.) per

$$\Xi = [\varphi, \psi]'',$$

ac ponamus

$$\begin{aligned} \varphi^0 &= \varphi + \lambda\, F + \mu\, \Phi + \cdots + \lambda' A + \mu' B + \cdots, \\ \psi^0 &= \psi + \lambda_1 F + \mu_1 \Phi + \cdots + \lambda'_1 A + \mu'_1 B + \cdots. \end{aligned}$$

Patet e formatione expressionis $[\varphi, \psi]''$, rejiciendo post differentiationes partiales transactas expressiones per quantitates evanescentes $F$, $\Phi$, ..., $A$, $B$, ... multiplicatas, fieri:

$$\begin{aligned} [\varphi^0, \psi^0]'' &= [\varphi^0, \psi + \lambda_1 F + \mu_1 \Phi + \cdots + \lambda'_1 A + \mu'_1 B + \cdots]'' \\ &= [\varphi^0, \psi]'' + \lambda_1 [\varphi^0, F]'' + \mu_1 [\varphi^0, \Phi]'' + \cdots \\ &\quad + \lambda'_1 [\varphi^0, A]'' + \mu'_1 [\varphi^0, B]'' + \cdots. \end{aligned}$$

At demonstravi modo, quaecunque sit $\varphi^0$ functio, haberi

$$\begin{aligned} &[\varphi^0, F]'' = 0, \quad [\varphi^0, \Phi]'' = 0, \quad \ldots, \\ &[\varphi^0, A]'' = 0, \quad [\varphi^0, B]'' = 0, \quad \ldots; \end{aligned}$$

unde fit

$$[\varphi^0, \psi^0]'' = [\varphi^0, \psi]''.$$

Eodem modo probatur, reiectis post differentiationes partiales transactas expressionibus per quantitates evanescentes $F$, $\Phi$, ..., $A$, $B$, ... multiplicatis, fieri

$$\begin{aligned} [\varphi^0, \psi]'' &= [\varphi + \lambda F + \mu \Phi + \cdots + \lambda' A + \mu' B + \cdots, \psi]'' \\ &= [\varphi, \psi]'' + \lambda\, [F, \psi]'' + \mu\, [\Phi, \psi]'' + \cdots \\ &\quad + \lambda' [A, \psi]'' + \mu' [B, \psi]'' + \cdots. \end{aligned}$$

Unde quum probatum sit, quaecunque sit $\psi$ functio, haberi

$$0 = [F, \psi]'' = [\Phi, \psi]'' = \cdots = [A, \psi]'' = [B, \psi]'' = \cdots,$$

fit

$$[\varphi^0, \psi^0]'' = [\varphi^0, \psi]'' = [\varphi, \psi]'',$$

quod est theorema demonstrandum.

Deinde etiam non mutari debet expressionis $\Xi$ valor, si loco functionis $R$ ponitur $R+\lambda F+\mu\Phi+\cdots+\lambda_1 F'+\mu_1\Phi'+\cdots$, atque de hac nova functione deducuntur valores ipsarum $v_i$ a praecedentibus diversi et forma valde discrepans functionis $H$, sicuti §. 39 praecepi. Sed verificatio huius proprietatis, ex ipsa quantitatis $\Xi$ formatione petita, quum molestissima esse videatur, sufficiat rem examinare, si ipsi $R$ tantum termini $\lambda F+\mu\Phi+\cdots$ addantur, quo casu ibidem vidimus, ipsarum $v_i$ valores non mutari, atque functioni $H$ similes tantum terminos accedere.

Demonstremus igitur, quantitatem $\Xi$ non mutare valorem, si in eo loco ipsius $H$ ponatur $H+\lambda F+\mu\Phi+\cdots$, designantibus $\lambda$, $\mu$, ... quascunque ipsarum $\xi_i$, $v_i$ expressiones. Qua mutatione functionis $H$ facile patet, etiam ipsas $A$, $B$, ... similes tantum mutationes subire, ideoque etiam ipsarum $A$, $B$, ... differentialia partialia secundum quantitates $v_i$ sumta, nec non expressiones $[F, A]'$, $[F, B]'$, ... $[\Phi, A]'$, $[\Phi, B]'$, ...; unde, sicuti e formulis (2.) elucet, etiam quantitates $D_{k,1}$, $D_{k,2}$, etc. alias non mutationes subeunt. Qua de re omnium harum quantitatum valores immutati manebunt. Sed mutabunt valorem expressiones $[\varphi, A]'$, $[A, B]'$, ... ac similes. Quae tamen mutationes eae esse debent, ut ipsius $\Xi$ valor immutatus maneat. Quod facile patebit, ubi probatum erit, expressionis $\Xi$ terminorum, qui functione $A$ affecti sunt, aggregatum evanescere, si loco $A$ in iis ponatur $F$. Tum enim similes quoque propositiones locum habebunt, evanescere idem aggregatum, si loco $A$ ponatur $\Phi$, vel evanescere aggregatum terminorum, qui functione $B$ affecti sunt, ponendo $F$ sive $\Phi$ loco $B$, etc. Quibus iunctis observationi, expressiones huiusmodi $[A, B]'$ evanescere, ubi simul loco $A$ atque $B$ ponantur quaecunque sive eaedem sive diversae e functionibus $F$, $\Phi$, ..., sponte elucet, valorem ipsius $\Xi$ non mutari. Propositio autem, evanescere terminorum ipsius $\Xi$ functione $A$ affectorum aggregatum, si loco $A$ substituatur $F$, sequitur absque magno negotio ex aequationibus (2.).

*De functionibus quibuslibet $\varphi$, $\psi$, per aequationes datas conditionales $F=0$, $\Phi=0$, ... ita transformandis, ut fiat $[\varphi, \psi] = [\varphi, \psi]'$.*

## 48.

Formas, quas functio $\varphi$ induere potest propter aequationes, quae locum habent, conditionales, semper ita determinare licet, ut *per has ipsas aequationes conditionales* evanescant valores expressionum

$$[F,\varphi]' = \frac{\partial F}{\partial \xi_1}\frac{\partial \varphi}{\partial v_1}+\frac{\partial F}{\partial \xi_2}\frac{\partial \varphi}{\partial v_2}+\cdots+\frac{\partial F}{\partial \xi_\mu}\frac{\partial \varphi}{\partial v_\mu},$$

$$[\Phi,\varphi]' = \frac{\partial \Phi}{\partial \xi_1}\frac{\partial \varphi}{\partial v_1}+\frac{\partial \Phi}{\partial \xi_2}\frac{\partial \varphi}{\partial v_2}+\cdots+\frac{\partial \Phi}{\partial \xi_\mu}\frac{\partial \varphi}{\partial v_\mu},$$

. . . . . . . . . . . . . . . . . . . . . .

ac similiter functioni $\psi$ eam formam conciliare licet, ut per aequationes conditionales evanescant valores expressionum $[F,\psi]'$, $[\Phi,\psi]'$, .... Sit ipsius $\varphi$ expressio adhibenda $\varphi+\lambda' A+\mu' B+\cdots$; multiplicatores $\lambda'$, $\mu'$, ... semper ita determinare licet, ut evanescant valores quantitatum

$$[F,\varphi+\lambda' A+\mu' B+\cdots]',\qquad [\Phi,\varphi+\lambda' A+\mu' B+\cdots]',$$

seu reiectis terminis in $A$, $B$, ... ductis ut evanescentibus, ut fiat

$$[F,\varphi]'+\lambda'[F,A]'+\mu'[F,B]'+\cdots = 0,$$
$$[\Phi,\varphi]'+\lambda'[\Phi,A]'+\mu'[\Phi,B]'+\cdots = 0,$$

. . . . . . . . . . . . . . . .

Per formulas similes determinantur multiplicatores $\lambda_1'$, $\mu_1'$, ... ita, ut quantitates

$$[F,\psi+\lambda_1' A+\mu_1' B+\cdots]',\qquad [\Phi,\psi+\lambda_1' A+\mu_1' B+\cdots]'$$

evanescant. Quibus expressionibus $\varphi+\lambda' A+\mu' B+\cdots$, $\psi+\lambda_1' A+\mu_1' B+\cdots$, quod licet, loco $\varphi$, $\psi$ positis, habemus ipsarum $\varphi$, $\psi$ formas tales, pro quibus fiat

$$(1.)\quad \begin{cases}[F,\varphi]'=0, & [\Phi,\varphi]'=0, \quad \ldots,\\ [F,\psi]'=0, & [\Phi,\psi]'=0, \quad \ldots,\end{cases}$$

quod propositum erat.

Inventis ipsarum $\varphi$, $\psi$ formis, pro quibus aequationes antecedentes (1.) locum habent, statim sequitur e (28.), §. 43., fieri etiam:

$$(2.)\quad \begin{cases}\Sigma_k D_{k,1}\dfrac{\partial \varphi}{\partial v_k}=0, & \Sigma_k D_{k,2}\dfrac{\partial \varphi}{\partial v_k}=0, \quad \ldots,\\[2ex] \Sigma_k D_{k,1}\dfrac{\partial \psi}{\partial v_k}=0, & \Sigma_k D_{k,2}\dfrac{\partial \psi}{\partial v_k}=0 \quad \ldots.\end{cases}$$

Unde in expressione ipsius $\Xi$ termini omnes praeter $[\varphi,\psi]'$ evanescunt, sive *fit, quoties* $[F,\varphi]'=0$, $[\Phi,\varphi]'=0$, ..., $[F,\psi]'=0$, $[\Phi,\psi]'=0$, ... *haec aequatio:*

$$(3.)\quad [\varphi,\psi] = [\varphi,\psi]'.$$

Quum per aequationes conditionales functiones $\varphi$, $\psi$ semper ita transformare liceat, ut conditionibus illis satisfiat, sequitur *datis ipsarum* $\xi_1$, $\xi_2$, ... $\xi_\mu$, $v_1$, $v_2$, ... $v_\mu$ *functionibus binis quibuscunque* $\varphi$ *et* $\psi$, *semper per aequationes* *conditionales, quae inter quantitates illas locum habent, formam talem iis conciliari posse, ut fiat:*

$$\begin{aligned}
&\frac{\partial\varphi}{\partial q_1}\frac{\partial\psi}{\partial p_1}+\frac{\partial\varphi}{\partial q_2}\frac{\partial\psi}{\partial p_2}+\cdots+\frac{\partial\varphi}{\partial q_m}\frac{\partial\psi}{\partial p_m}\\
&-\frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial q_1}-\frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_2}-\cdots-\frac{\partial\varphi}{\partial p_m}\frac{\partial\psi}{\partial q_m}\\
=&\frac{\partial\varphi}{\partial \xi_1}\frac{\partial\psi}{\partial v_1}+\frac{\partial\varphi}{\partial \xi_2}\frac{\partial\psi}{\partial v_2}+\cdots+\frac{\partial\varphi}{\partial \xi_\mu}\frac{\partial\psi}{\partial v_\mu}\\
&-\frac{\partial\varphi}{\partial v_1}\frac{\partial\psi}{\partial \xi_1}-\frac{\partial\varphi}{\partial v_2}\frac{\partial\psi}{\partial \xi_2}-\cdots-\frac{\partial\varphi}{\partial v_\mu}\frac{\partial\psi}{\partial \xi_\mu}.
\end{aligned}$$

Ex aequationibus (2.) §. antec. facile deduco sequentes:

$$(4.)\quad \begin{cases}
[F,\ \varphi]' = [F,\ A]'\Sigma_k D_{k,1}\frac{\partial\varphi}{\partial v_k}+[F,\ B]'\Sigma_k D_{k,2}\frac{\partial\varphi}{\partial v_k}+\cdots,\\
[\Phi,\ \varphi]' = [\Phi,\ A]'\Sigma_k D_{k,1}\frac{\partial\varphi}{\partial v_k}+[\Phi,\ B]'\Sigma_k D_{k,2}\frac{\partial\varphi}{\partial v_k}+\cdots,\\
\cdots\cdots\cdots\cdots\cdots
\end{cases}$$

Quibus comparatis cum iis, quibus antecedentibus multiplicatorum $\lambda'$, $\mu'$, ... valores determinabantur, fit:

$$\lambda' = -\Sigma_k D_{k,1}\frac{\partial\varphi}{\partial v_k},\quad \mu' = -\Sigma_k D_{k,2}\frac{\partial\varphi}{\partial v_k},\quad \ldots.$$

Unde *quaecunque sit $\varphi$ functio,* habemus e (1.):

$$(5.)\quad \begin{cases}
\left[F,\ \varphi-\Sigma_k(D_{k,1}A+D_{k,2}B+\cdots)\frac{\partial\varphi}{\partial v_k}\right]' = 0,\\
\left[\Phi,\ \varphi-\Sigma_k(D_{k,1}A+D_{k,2}B+\cdots)\frac{\partial\varphi}{\partial v_k}\right]' = 0,\\
\cdots\cdots\cdots\cdots\cdots
\end{cases}$$

Qua de re etiam habetur:

$$\begin{aligned}
&\left[F,\ \psi-\Sigma_k(D_{k,1}A+D_{k,2}B+\cdots)\frac{\partial\psi}{\partial v_k}\right]' = 0,\\
&\left[\Phi,\ \psi-\Sigma_k(D_{k,1}A+D_{k,2}B+\cdots)\frac{\partial\psi}{\partial v_k}\right]' = 0,\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Erit igitur e (3.):

$$(6.)\quad [\varphi,\psi] = \Xi =$$

$$\left[\varphi-\Sigma_k(D_{k,1}A+D_{k,2}B+\cdots)\frac{\partial\varphi}{\partial v_k},\ \psi-\Sigma_k(D_{k,1}A+D_{k,2}B+\cdots)\frac{\partial\psi}{\partial v_k}\right]'.$$

Quae expressio nova ipsius $\Xi$ facile convenit cum illa §. 47., (1.), ad quam supra pervenimus, ubi reputas, reiectis terminis in $A$, $B$, ... ductis ut evanescentibus, fieri pro multiplicatoribus $\lambda'$, $\mu'$, ... $\lambda'_1$, $\mu'_1$, ... quibuscunque

$$\begin{aligned}[][\varphi+\lambda'A+\mu'B\ldots,\ \psi+\lambda'_1A+\mu'_1B\ldots]' = [\varphi,\psi]'+\lambda'_1[\varphi,A]'+\mu'_1[\varphi,B]'-\cdots \\ +\lambda'[\psi,A]'+\mu'[\psi,B]'-\cdots \\ +(\lambda'\mu'_1-\lambda'_1\mu')[A,B]'+\cdots\end{aligned}$$

Considerationibus antecedentibus superstrui potest nova methodus, qua expressio ipsius $\Xi$, supra via satis prolixa inventa, indagetur; quae methodus huic toti quaestioni magnam lucem affundet.

*E considerationibus antecedentibus alia via petitur expressionem propositam ipsius $[\varphi,\psi]$ derivandi.*

49.

Quantitates $q_1$, $q_2$, ... $q_m$ non sunt functiones prorsus determinatae ipsarum $\xi_1$, $\xi_2$, ... $\xi_\mu$, quippe quibus addi possunt functiones $F$, $\Phi$, ... in factores arbitrarios ductae. Sic etiam $p_1$, $p_2$, ... $p_m$ non sunt functiones prorsus determinatae ipsarum $\xi_1$, $\xi_2$, ... $\xi_\mu$, $v_1$, $v_2$, ... $v_\mu$, quippe valoribus ipsarum $p_i$ §. 40 traditis addi possunt functiones $F$, $\Phi$, ..., $A$, $B$, ... in factores arbitrarios ductae. Hinc, si datur expressio functionis alicuius $\varphi$ per quantitates $\xi_i$, $v_i$, simulque habetur expressio eiusdem functionis $\varphi$ per quantitates $q_i$, $p_i$, quaeri potest, quaenam e variis illis formis valorum quantitatum $q_i$, $p_i$ eligendae sint, ut ex hac ipsius $\varphi$ expressione post factas substitutiones illa data proveniat. Iam dico, *si in expressione functionis cuiuslibet $\varphi$ per quantitates $q_i$, $p_i$ ipsarum quidem $q_i$ valores formas assumant, quascunque per aequationes $F=0$, $\Phi=0$, ... assumere possunt; ipsarum vero $p_i$ valoribus ea ipsa forma tribuatur, qua in formis §. 40 propositis gaudent, neque ullo modo forma illa mutatur auxilio aequationum $F=0$, $\Phi=0$, ..., $A=0$, $B=0$, ... fore ut ea forma functionis $\varphi$ proveniat, pro qua habetur:*

$$[F,\varphi]'=0,\qquad [\Phi,\varphi]'=0,\qquad \ldots .$$

Fit enim

$$[F,\varphi]'=\Sigma_k\frac{\partial F}{\partial\xi_k}\frac{\partial\varphi}{\partial v_k}=\Sigma_{k,i}\frac{\partial F}{\partial\xi_k}\frac{\partial\varphi}{\partial p_i}\frac{\partial p_i}{\partial v_k}.$$

At quum supponatur, *identice* positum esse e §. 40:

$$p_i = v_1\frac{\partial\xi_1}{\partial q_i}+v_2\frac{\partial\xi_2}{\partial q_i}+\cdots+v_\mu\frac{\partial\xi_\mu}{\partial q_i},$$

illa suppositione fit

$$\frac{\partial p_i}{\partial v_k} = \frac{\partial \xi_k}{\partial q_i}.$$

Quibus substitutis obtinemus:

$$[F, \varphi]' = \Sigma_{k,i} \frac{\partial F}{\partial \xi_k} \frac{\partial \varphi}{\partial p_i} \frac{\partial \xi_k}{\partial q_i} = \Sigma_i \Big( \frac{\partial \varphi}{\partial p_i} \Sigma_k \frac{\partial F}{\partial \xi_k} \frac{\partial \xi_k}{\partial q_i} \Big).$$

Iam vero habetur

$$\Sigma_k \frac{\partial F}{\partial \xi_k} \frac{\partial \xi_k}{\partial q_i} = \frac{\partial F}{\partial \xi_1} \frac{\partial \xi_1}{\partial q_i} + \frac{\partial F}{\partial \xi_2} \frac{\partial \xi_2}{\partial q_i} + \cdots + \frac{\partial F}{\partial \xi_\mu} \frac{\partial \xi_\mu}{\partial q_i} = \frac{\partial F}{\partial q_i} = 0,$$

quum functio $F$, substitutis ipsarum $\xi_i$ valoribus per quantitates $q_i$ expressis, identice evanescere debeat. Unde, substitutis in ipsis $\frac{\partial \xi_k}{\partial q_i}$ ipsarum $q_i$ valoribus assumtis per quantitates $\xi_i$ expressis, abire debet expressio

$$\Sigma_k \frac{\partial F}{\partial \xi_k} \frac{\partial \xi_k}{\partial q_i}$$

in aggregatum terminorum in $F$, $\Phi$, ... ductorum. Unde etiam e formula antecedente sequitur expressionem $[F, \varphi]'$ in tale aggregatum abire, hoc est, *si in functione aliqua $\varphi$ per ipsas $q_1, q_2, \ldots q_m, p_1, p_1, \ldots p_m$ expressa substituuntur loco ipsarum $p_i$ valores:*

$$p_i = v_1 \frac{\partial \xi_1}{\partial q_i} + v_2 \frac{\partial \xi_2}{\partial q_i} + \cdots + v_\mu \frac{\partial \xi_\mu}{\partial q_i},$$

*ac deinde loco ipsarum $q_i$ quaecunque ponuntur functiones ipsarum $\xi_i$, denique adjumento $\mu - m$ aequationum $F = 0$, $\Phi = 0$, ... exprimuntur etiam quantitates $\frac{\partial \xi_k}{\partial q_i}$ per ipsas $\xi_i$, abit functio $\varphi$ in talem expressionem ipsarum $\xi_i$, $v_i$, ut quantitas*

$$[F, \varphi]' = \frac{\partial F}{\partial \xi_1} \frac{\partial \varphi}{\partial v_1} + \frac{\partial F}{\partial \xi_2} \frac{\partial \varphi}{\partial v_2} + \cdots + \frac{\partial F}{\partial \xi_\mu} \frac{\partial \varphi}{\partial v_\mu}$$

*evadat aggregatum terminorum in $F$, $\Phi$, ... ductorum ideoque eius valor evanescat.* Eodem modo demonstratur, expressiones $[\Phi, \varphi]'$, $[F, \psi]'$, $[\Phi, \psi]'$, etc. in eiusmodi aggregata abire ideoque evanescere. Electis functionibus ipsarum $\xi_i$, quae in locum ipsarum $q_i$ ponantur, habentur quotientes differentiales partiales $\frac{\partial \xi_k}{\partial q_i}$ per ipsas $\xi_i$ expressae ope aequationum linearium:

$$0 = \frac{\partial q_1}{\partial \xi_1}\frac{\partial \xi_1}{\partial q_i}+\frac{\partial q_1}{\partial \xi_2}\frac{\partial \xi_2}{\partial q_i}+\cdots+\frac{\partial q_1}{\partial \xi_\mu}\frac{\partial \xi_\mu}{\partial q_i},$$

$$0 = \frac{\partial q_2}{\partial \xi_1}\frac{\partial \xi_1}{\partial q_i}+\frac{\partial q_2}{\partial \xi_2}\frac{\partial \xi_2}{\partial q_i}+\cdots+\frac{\partial q_2}{\partial \xi_\mu}\frac{\partial \xi_\mu}{\partial q_i},$$

$$\cdots\cdots\cdots\cdots,$$

$$1 = \frac{\partial q_i}{\partial \xi_1}\frac{\partial \xi_1}{\partial q_i}+\frac{\partial q_i}{\partial \xi_2}\frac{\partial \xi_2}{\partial q_i}+\cdots+\frac{\partial q_i}{\partial \xi_\mu}\frac{\partial \xi_\mu}{\partial q_i},$$

$$\cdots\cdots\cdots\cdots,$$

$$0 = \frac{\partial q_m}{\partial \xi_1}\frac{\partial \xi_1}{\partial q_i}+\frac{\partial q_m}{\partial \xi_2}\frac{\partial \xi_2}{\partial q_i}+\cdots+\frac{\partial q_m}{\partial \xi_\mu}\frac{\partial \xi_\mu}{\partial q_i},$$

$$0 = \frac{\partial F}{\partial \xi_1}\frac{\partial \xi_1}{\partial q_i}+\frac{\partial F}{\partial \xi_2}\frac{\partial \xi_2}{\partial q_i}+\cdots+\frac{\partial F}{\partial \xi_\mu}\frac{\partial \xi_\mu}{\partial q_i},$$

$$0 = \frac{\partial \Phi}{\partial \xi_1}\frac{\partial \xi_1}{\partial q_i}+\frac{\partial \Phi}{\partial \xi_2}\frac{\partial \xi_2}{\partial q_i}+\cdots+\frac{\partial \Phi}{\partial \xi_\mu}\frac{\partial \xi_\mu}{\partial q_i},$$

$$\cdots\cdots\cdots\cdots$$

Si quantitates $\frac{\partial \xi_k}{\partial q_i}$ per aequationes $F=0$, $\Phi=0$, ... in tales expressiones rediguntur, ut *identice* sit pro quolibet ipsius $i$ valore

$$\Sigma_k \frac{\partial F}{\partial \xi_k}\frac{\partial \xi_k}{\partial q_i}=0, \qquad \Sigma_k \frac{\partial \Phi}{\partial \xi_k}\frac{\partial \xi_k}{\partial q_i}=0, \qquad \ldots$$

functionibus $\varphi$ talis forma conciliata erit, pro quibus expressiones

$$[F,\varphi]', \qquad [\Phi,\psi]', \qquad \ldots$$

adeo *identice* evanescant. Generaliter autem data quaecunque functio $\varphi$ per aequationes $A=0$, $B=0$, ... in formam redigitur, pro qua expressiones $[F,\varphi]'$, $[\Phi,\varphi]'$, ... identice evanescunt, si eliguntur $m$ expressiones a se invicem independentes:

$$w_i = \alpha_i' v_1+\alpha_i'' v_2+\cdots+\alpha_i^{(\mu)} v_\mu$$

respectu ipsarum $v_k$ lineares, in qua coefficientes $\alpha_i^{(k)}$ ut tales ipsarum $\xi_k$ functiones determinantur, pro quibus identice fiat

$$0 = \frac{\partial F}{\partial \xi_1}\alpha_i'+\frac{\partial F}{\partial \xi_2}\alpha_i''+\cdots+\frac{\partial F}{\partial \xi_\mu}\alpha_i^{(\mu)},$$

$$0 = \frac{\partial \Phi}{\partial \xi_1}\alpha_i'+\frac{\partial \Phi}{\partial \xi_2}\alpha_i''+\cdots+\frac{\partial \Phi}{\partial \xi_\mu}\alpha_i^{(\mu)},$$

$$\cdots\cdots\cdots\cdots$$

Ex. gr., si $\mu - m = 2$ sive si duae tantum adsunt aequationes conditionales sive functiones $F$, $\Phi$, assumere licet

$$\begin{aligned} w_1 &= \alpha_1 v_1 + \beta_1 v_2 + v_3, \\ w_2 &= \alpha_2 v_1 + \beta_2 v_2 + v_4, \\ &\quad \cdots\cdots\cdots \\ w_{\mu-2} &= \alpha_{\mu-2} v_1 + \beta_{\mu-2} v_2 + v_\mu, \end{aligned}$$

determinatis $\alpha_k$, $\beta_k$ per duas aequationes

$$\begin{aligned} \alpha_k \frac{\partial F}{\partial \xi_1} + \beta_k \frac{\partial F}{\partial \xi_2} + \frac{\partial F}{\partial \xi_{k+2}} &= 0, \\ \alpha_k \frac{\partial \Phi}{\partial \xi_1} + \beta_k \frac{\partial \Phi}{\partial \xi_2} + \frac{\partial \Phi}{\partial \xi_{k+2}} &= 0. \end{aligned}$$

Quae facile ad quemlibet numerum aequationum conditionalium sive functionum $F$, $\Phi$, ... extenduntur. Determinatis functionibus linearibus $w_i$ ita ut dictis conditionibus satisfaciant, eliminari possunt per $\mu - m$ aequationes $A = 0$, $B = 0$, ... quantitates $v_1$, $v_2$, ... $v_\mu$ e functione $\varphi$, ita ut solarum $\xi_i$, $w_i$ functio evadat, quae erit expressio quaesita.

Ut eruatur expressio ipsius $\Xi$ §. 47 proposita, tantum opus est, ut demonstretur, *quoties*

$$\begin{aligned} &(1.) \qquad [F, \varphi]' = 0, \quad [\Phi, \varphi]' = 0, \quad \ldots \\ &(2.) \qquad [F, \psi]' = 0, \quad [\Phi, \psi]' = 0, \quad \ldots \end{aligned}$$

*fieri*

$$[\varphi, \psi] = [\varphi, \psi]'.$$

Vocemus enim $\varphi^0$, $\psi^0$ eas expressiones ipsarum $\varphi$, $\psi$, pro quibus aequationes (1.), (2.) locum habent, sitque

$$[\varphi, \psi] = [\varphi^0, \psi^0]',$$

sequitur e §. 48., fore

$$\begin{aligned} \varphi^0 &= \varphi + \lambda' A + \mu' B + \cdots, \\ \psi^0 &= \psi + \lambda'_1 A + \mu'_1 B + \cdots, \end{aligned}$$

ubi

$$\begin{aligned} \lambda' &= -\Sigma_k D_{k,1} \frac{\partial \varphi}{\partial v_k}, \quad \mu' = -\Sigma_k D_{k,2} \frac{\partial \varphi}{\partial v_k}, \quad \ldots \\ \lambda'_1 &= -\Sigma_k D_{k,1} \frac{\partial \psi}{\partial v_k}, \quad \mu'_1 = -\Sigma_k D_{k,2} \frac{\partial \varphi}{\partial v_k}, \quad \ldots \end{aligned}$$

neque alias formas induere posse functiones $\varphi^0$, $\psi^0$, nisi quod iis adhuc addi possint termini in $F$, $\Phi$, ... ducti. Facile autem patet, quum aequationes (1.), (2.) posito $\varphi = \varphi^0$, $\psi = \psi^0$ locum habeant, expressionem $[\varphi^0, \psi^0]'$ eiusmodi terminis ipsis $\varphi^0$, $\psi^0$ additis valorem non mutare. Unde eruetur

$$[\varphi, \psi] = [\varphi+\lambda' A+\mu' B+\cdots, \psi+\lambda_1' A+\mu_1' B+\cdots]' = \Xi$$

q. e. d. Eodem modo probatur, si solae (1.) locum habeant, fore

$$[\varphi, \psi] = [\varphi, \psi+\lambda_1' A+\mu_1' B+\cdots]'.$$

Propositio autem illa, quoties aequationes (1.), (2.) locum habeant, fore

$$[\varphi, \psi] = [\varphi, \psi]',$$

sic demonstrari potest.

*Eadem continuantur. Demonstratur, integrale tertium, quod e binis aequationum dynamicarum integralibus conflare licet, nullo modo pendere a variabilium electione.*

50.

Antecedentibus probavi, *pro omnibus formis functionum $\varphi$, $\psi$, pro quibus aequationes* (1.) §. praec. *locum habeant, quantitatem $[\varphi, \psi]'$ eundem valorem servare.* Unde supponere licet, $\varphi$, $\psi$ eas esse functiones, quae ex earum expressionibus per quantitates $q_k$, $p_k$ prodeunt ponendo loco $p_k$ expressionem

$$p_k = v_1 \frac{\partial \xi_1}{\partial q_k} + v_2 \frac{\partial \xi_2}{\partial q_k} + \cdots + v_\mu \frac{\partial \xi_\mu}{\partial q_k},$$

quippe quibus proprietatem illam suppetere §. antec. vidimus. Pro illis autem functionibus $\varphi$, $\psi$ habetur

$$\frac{\partial \varphi}{\partial \xi_i} = \Sigma_k \frac{\partial \varphi}{\partial q_k} \frac{\partial q_k}{\partial \xi_i} + \Sigma_k \frac{\partial \varphi}{\partial p_k} \frac{\partial p_k}{\partial \xi_i},$$

$$\frac{\partial \varphi}{\partial v_i} = \Sigma_k \frac{\partial \varphi}{\partial p_k} \frac{\partial p_k}{\partial v_i} = \Sigma_k \frac{\partial \varphi}{\partial p_k} \frac{\partial \xi_i}{\partial q_k},$$

similesque formulae pro functione $\psi$ locum habent. Unde fit

$$\frac{\partial \varphi}{\partial \xi_i} \frac{\partial \psi}{\partial v_i} - \frac{\partial \varphi}{\partial v_i} \frac{\partial \psi}{\partial \xi_i} =$$

$$\Sigma_{k,k'} \left( \frac{\partial \varphi}{\partial q_k} \frac{\partial \psi}{\partial p_{k'}} - \frac{\partial \psi}{\partial q_k} \frac{\partial \varphi}{\partial p_{k'}} \right) \frac{\partial q_k}{\partial \xi_i} \frac{\partial \xi_i}{\partial q_{k'}} +$$

$$\Sigma_{k,k'} \left( \frac{\partial \varphi}{\partial p_k} \frac{\partial \psi}{\partial p_{k'}} - \frac{\partial \psi}{\partial p_k} \frac{\partial \varphi}{\partial p_{k'}} \right) \frac{\partial p_k}{\partial \xi_i} \frac{\partial \xi_i}{\partial q_{k'}}.$$

In qua expressione indici $i$ valores $1, 2, \ldots \mu$ tribuendi sunt atque nova summatio instituenda; fit autem

$$\Sigma_i \frac{\partial q_k}{\partial \xi_i} \frac{\partial \xi_i}{\partial q_{k'}} = \frac{\partial q_k}{\partial q_{k'}} = 0, \qquad \Sigma_i \frac{\partial p_k}{\partial \xi_i} \frac{\partial \xi_i}{\partial q_{k'}} = \frac{\partial p_k}{\partial q_{k'}} = 0,$$

excepto casu, quo in priore formula fit $k = k'$, quo casu illa in unitatem abit; unde evanescunt nova illa summatione termini omnes praeter

$$\Sigma_k\left(\left(\frac{\partial\varphi}{\partial q_k}\frac{\partial\psi}{\partial p_k}-\frac{\partial\varphi}{\partial p_k}\frac{\partial\psi}{\partial q_k}\right)\Sigma_i\frac{\partial q_k}{\partial\xi_i}\frac{\partial\xi_i}{\partial q_k}\right) = \Sigma_k\left(\frac{\partial\varphi}{\partial q_k}\frac{\partial\psi}{\partial p_k}-\frac{\partial\varphi}{\partial p_k}\frac{\partial\psi}{\partial q_k}\right).$$

Unde prodit

$$\Sigma_i\left(\frac{\partial\varphi}{\partial\xi_i}\frac{\partial\psi}{\partial v_i}-\frac{\partial\varphi}{\partial v_i}\frac{\partial\psi}{\partial\xi_i}\right) = \Sigma_k\left(\frac{\partial\varphi}{\partial q_k}\frac{\partial\psi}{\partial p_k}-\frac{\partial\varphi}{\partial p_k}\frac{\partial\psi}{\partial q_k}\right),$$

sive

$$[\varphi, \psi]' = [\varphi, \psi],$$

q. d. e.

Prorsus eadem demonstratione facile probatur, *si aequationes conditionales inter ipsas* $\xi_i$ *omnino non habeantur ideoque* $\mu = m$, *semper fieri*

$$[\varphi, \psi] = [\varphi, \psi]'$$

sive

$$\begin{aligned}&\frac{\partial\varphi}{\partial q_1}\frac{\partial\psi}{\partial p_1}+\frac{\partial\varphi}{\partial q_2}\frac{\partial\psi}{\partial p_2}+\cdots+\frac{\partial\varphi}{\partial q_m}\frac{\partial\psi}{\partial p_m}\\ &-\frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial q_1}-\frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_2}-\cdots-\frac{\partial\varphi}{\partial p_m}\frac{\partial\psi}{\partial q_m}\\ &=\\ &\frac{\partial\varphi}{\partial\xi_1}\frac{\partial\psi}{\partial v_1}+\frac{\partial\varphi}{\partial\xi_2}\frac{\partial\psi}{\partial v_2}+\cdots+\frac{\partial\varphi}{\partial\xi_\mu}\frac{\partial\psi}{\partial v_\mu}\\ &-\frac{\partial\varphi}{\partial v_1}\frac{\partial\psi}{\partial\xi_1}-\frac{\partial\varphi}{\partial v_2}\frac{\partial\psi}{\partial\xi_2}-\cdots-\frac{\partial\varphi}{\partial v_\mu}\frac{\partial\psi}{\partial\xi_\mu}.\end{aligned}$$

Unde patet, *quantitatem* $[\varphi, \psi]$ *nullo modo pendere a variabilium* $q_i$ *electione, sed tantum a natura intima functionum* $\varphi$ *et* $\psi$. Unde etiam, si

$$\varphi = \text{Const.}, \qquad \psi = \text{Const.}$$

sunt bina integralia systematis aequationum differentialium vulgarium

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

earundem aequationum differentialium integrale *tertium, quod datur per aequationem:*

$$[\varphi, \psi] = \text{Const.},$$

*nullo modo pendet a variabilium electione.*

*Theorema de tertio integrali e binis inveniendo extenditur ad casum quo aequationes conditionales inter variabiles intercedunt. — De relationibus quae locum habent inter integralia principium conservationis virium vivarum et principium conservationis centri gravitatis concernentia.*

51.

Statuamus, aequationem

$$\varphi = \text{Const.}$$

esse integrale aequationum differentialium vulgarium (2.) §. 45 propositarum,

$$\frac{d\xi_i}{dt} = \frac{\partial H}{\partial v_i}, \quad \frac{dv_i}{dt} = -\frac{\partial H}{\partial \xi_i} - \lambda_1 \frac{\partial F}{\partial \xi_i} - \lambda_2 \frac{\partial \Phi}{\partial \xi_i} - \cdots,$$

atque insuper *ita comparatam esse functionem* $\varphi$, *ut identice habeatur*

$$[\varphi, H]' = 0, \quad [\varphi, F]' = 0, \quad [\varphi, \Phi]' = 0, \quad \ldots$$

*dico fore, ut habeatur*

$$[\varphi, \psi] = [\varphi, \psi]',$$

*quaecunque sit* $\psi$ *functio.* Nam e theoremate V. §. 26 identice fit, quaecunque sint $F$, $H$, $\varphi$ functiones:

$$[F, [\varphi, H]']' + [\varphi, [H, F]']' + [H, [F, \varphi]']' = 0,$$

unde casu proposito identice erit:

$$[\varphi, [H, F]']' = 0 \quad \text{sive} \quad [\varphi, A]' = 0.$$

eodemque modo obtinetur identice:

$$[\varphi, B]' = 0.$$

Probavi autem §. 48., quoties

$$[\varphi, F]' = 0, \quad [\varphi, \Phi]' = 0, \quad \ldots$$

fieri

$$[\varphi, \psi] = [\varphi, \psi + \lambda_1' A + \lambda_2' B + \cdots]',$$

unde sequitur

$$[\varphi, \psi] = [\varphi, \psi]' + \lambda_1'[\varphi, A]' + \lambda_2'[\varphi, B]' + \cdots$$

Hinc casu proposito, quo vidimus evanescere $[\varphi, A]'$, $[\varphi, B]'$, ... fit

$$[\varphi, \psi] = [\varphi, \psi]'$$

q. d. e.

Ope propositionis antecedentis deduci potest e theoremate VI. hoc theorema:

### Theorema VII.

*„Sint $F$, $\Phi$, ... quaecunque quantitatum $\xi_1$, $\xi_2$, ... $\xi_\mu$ functiones, atque sit identice:*

$$\frac{\partial F}{\partial \xi_1}\frac{\partial \varphi}{\partial v_1}+\frac{\partial F}{\partial \xi_2}\frac{\partial \varphi}{\partial v_2}+\cdots+\frac{\partial F}{\partial \xi_\mu}\frac{\partial \varphi}{\partial v_\mu} = 0,$$

$$\frac{\partial \Phi}{\partial \xi_1}\frac{\partial \varphi}{\partial v_1}+\frac{\partial \Phi}{\partial \xi_2}\frac{\partial \varphi}{\partial v_2}+\cdots+\frac{\partial \Phi}{\partial \xi_\mu}\frac{\partial \varphi}{\partial v_\mu} = 0,$$

. . . . . . . . . . . . . . . .

*porro identice habeatur:*

$$\frac{\partial H}{\partial \xi_1}\frac{\partial \varphi}{\partial v_1}+\frac{\partial H}{\partial \xi_2}\frac{\partial \varphi}{\partial v_2}+\cdots+\frac{\partial H}{\partial \xi_\mu}\frac{\partial \varphi}{\partial v_\mu}$$
$$-\frac{\partial H}{\partial v_1}\frac{\partial \varphi}{\partial \xi_1}-\frac{\partial H}{\partial v_2}\frac{\partial \varphi}{\partial \xi_2}-\cdots-\frac{\partial H}{\partial v_\mu}\frac{\partial \varphi}{\partial \xi_\mu} = 0,$$

*unde $\varphi = Const.$ fit integrale systematis aequationum differentialium vulgarium*

$$\frac{d\xi_i}{dt}=\frac{\partial H}{\partial v_i},\quad \frac{dv_i}{dt}=-\frac{\partial H}{\partial \xi_i}-\lambda_1\frac{\partial F}{\partial \xi_i}-\lambda_2\frac{\partial \Phi}{\partial \xi_i}-\cdots,$$

*in quibus supponamus quantitates $\xi_1$, $\xi_2$, ... $\xi_\mu$ subiectas esse aequationibus $F=0$, $\Phi=0$, ...; sit denique $\psi = Const.$ aliud earundem aequationum integrale quodcunque, erit etiam aequatio sequens:*

$$\frac{\partial \varphi}{\partial \xi_1}\frac{\partial \psi}{\partial v_1}+\frac{\partial \varphi}{\partial \xi_2}\frac{\partial \psi}{\partial v_2}+\cdots+\frac{\partial \varphi}{\partial \xi_\mu}\frac{\partial \psi}{\partial v_\mu}$$
$$-\frac{\partial \varphi}{\partial v_1}\frac{\partial \psi}{\partial \xi_1}-\frac{\partial \varphi}{\partial v_2}\frac{\partial \psi}{\partial \xi_2}-\cdots-\frac{\partial \varphi}{\partial v_\mu}\frac{\partial \psi}{\partial \xi_\mu} = \text{Const.}$$

*aequationum differentialium vulgarium propositarum integrale.“*

Theorematis praecedentis applicationem faciam ad integralia, quae principia *conservationis arearum* et *conservationis centri gravitatis* concernent.

Designantibus $x_i$, $y_i$, $z_i$ coordinatas orthogonales puncti, cuius massa $m_i$, habentur tria integralia, quae *principium conservationis arearum* concernunt:

$$\text{Const.} = \varphi_1 = \Sigma m_i(y_i z_i' - z_i y_i'),$$
$$\text{Const.} = \varphi_2 = \Sigma m_i(z_i x_i' - x_i z_i'),$$
$$\text{Const.} = \varphi_3 = \Sigma m_i(x_i y_i' - y_i x_i').$$

Quae notum est semper locum habere, si vires puncta systematis sollicitantes sint attractiones vel repulsiones sive mutuae sive versus initium coordinatarum directae, atque insuper systema per conditiones, quibus subiectum est, nullo modo impediatur, quin libere circa initium coordinatarum rotetur. Quoties $\xi_k$ unam e quantitatibus $x_i$, $y_i$, $z_i$ designat, loco $v_k$ (cf. §. 44) respective ponendum erit $m_i x'_i$, $m_i y'_i$, $m_i z'_i$. Unde designante $\psi$ aliam quamcunque ipsarum $x_i$, $y_i$, $z_i$, $x'_i$, $y'_i$, $z'_i$ functionem, fit

$$\begin{aligned}[\varphi_1, \psi]' &= \Sigma \frac{1}{m_i}\left\{\frac{\partial\varphi_1}{\partial y_i}\frac{\partial\psi}{\partial y'_i}+\frac{\partial\varphi_1}{\partial z_i}\frac{\partial\psi}{\partial z'_i}-\frac{\partial\varphi_1}{\partial y'_i}\frac{\partial\psi}{\partial y_i}-\frac{\partial\varphi_1}{\partial z'_i}\frac{\partial\psi}{\partial z_i}\right\} \\ &= \Sigma\left\{z'_i\frac{\partial\psi}{\partial y'_i}-y'_i\frac{\partial\psi}{\partial z'_i}+z_i\frac{\partial\psi}{\partial y_i}-y_i\frac{\partial\psi}{\partial z_i}\right\},\end{aligned}$$

ac formulae similes respectu functionum $\varphi_2$, $\varphi_3$ obtinentur. Casu, quem consideramus, valent conditiones, quae in theoremate VII. postulantur, siquidem pro functione $\varphi$ unam e functionibus $\varphi_1$, $\varphi_2$, $\varphi_3$ accipimus. Quoties igitur $\psi = \text{Const.}$ et ipsum integrale quodcunque problematis est, e theoremate illo eruitur:

$$(1.)\quad \begin{cases} \text{Const.} = [\varphi_1, \psi]' = \Sigma\left\{z'_i\frac{\partial\psi}{\partial y'_i}-y'_i\frac{\partial\psi}{\partial z'_i}+z_i\frac{\partial\psi}{\partial y_i}-y_i\frac{\partial\psi}{\partial z_i}\right\}, \\ \text{Const.} = [\varphi_2, \psi]' = \Sigma\left\{x'_i\frac{\partial\psi}{\partial z'_i}-z'_i\frac{\partial\psi}{\partial x'_i}+x_i\frac{\partial\psi}{\partial z_i}-z_i\frac{\partial\psi}{\partial x_i}\right\}, \\ \text{Const.} = [\varphi_3, \psi]' = \Sigma\left\{y'_i\frac{\partial\psi}{\partial x'_i}-x'_i\frac{\partial\psi}{\partial y'_i}+y_i\frac{\partial\psi}{\partial x_i}-x_i\frac{\partial\psi}{\partial y_i}\right\}. \end{cases}$$

Si in his formulis statuimus, quod licet, functionem $\psi$ esse unam ex ipsis functionibus $\varphi_1$, $\varphi_2$, $\varphi_3$, facile invenitur:

$$(2.)\quad \begin{cases} [\varphi_2, \varphi_3]' = \varphi_1, \\ [\varphi_3, \varphi_1]' = \varphi_2, \\ [\varphi_1, \varphi_2]' = \varphi_3. \end{cases}$$

Quoties in problemate mechanico principium conservationis arearum locum habet, satisfit aequationibus identicis, quas in theoremate VII. statuimus, siquidem in theoremate illo loco $\varphi$ ponitur una e functionibus $\varphi_1$, $\varphi_2$, $\varphi_3$. Nam aequationes illae identicae in theoremate VII. propositae hunc ipsum constituunt characterem *conservationis*, e quo principium mechanicum suam traxit appellationem. Hinc formulis praecedentibus theorema VII. applicare possumus, sive designante $\varphi = \text{Const.}$ integrale, quod ad principium conservationis arearum pertinet, atque $\psi = \text{Const.}$ aliud quodcunque integrale problematis mechanici, in quo principium illud valet, erit

$$(3.)\quad [\varphi, \psi] = [\varphi, \psi]'.$$

Unde e tribus formulis praecedentibus (2.) fluunt etiam tres sequentes:

$$(4.)\quad [\varphi_2, \varphi_3] = \varphi_1, \quad [\varphi_3, \varphi_1] = \varphi_2, \quad [\varphi_1, \varphi_2] = \varphi_3.$$

In formula (3.) functio $\varphi$ sive unam e functionibus $\varphi_1$, $\varphi_2$, $\varphi_3$ sive etiam earum functionem quamlibet designare potest.

Videmus e formulis (4.), si regula generalis, secundum quam vidimus e duobus integralibus formari posse tertium, applicetur ad tria integralia, quae principium conservationis arearum suppeditat, haec integralia tantum sese ipsa generare neque in illo casu ea regula ad integralia nova perveniri. Animadverti autem potest, quum secundum regulam illam trium illorum integralium bina quaelibet tertium procreent, eam demonstrare fieri non posse, ut in ullo problemate mechanico duo tantum locum habeant, tertium integrale locum non habeat. Quod hic per propositiones mere analyticas absque ullo considerationum geometricarum auxilio evincitur.

Statuamus

$$\chi_1 = \Sigma m_i x'_i, \qquad \chi_2 = \Sigma m_i y'_i, \qquad \chi_3 = \Sigma m_i z'_i;$$

constituunt tria integralia

$$\chi_1 = \text{Const.}, \qquad \chi_2 = \text{Const.}, \qquad \chi_3 = \text{Const.}$$

principium *conservationis centri gravitatis*. Invenitur autem, si loco $\psi$ ponitur in (1.) successive $\chi_1$, $\chi_2$, $\chi_3$:

$$(5.)\quad \begin{cases} [\varphi_1, \chi_1]' = 0, & [\varphi_1, \chi_2]' = \chi_3, & [\varphi_1, \chi_3]' = -\chi_2, \\ [\varphi_2, \chi_1]' = -\chi_3, & [\varphi_2, \chi_2]' = 0, & [\varphi_2, \chi_3]' = \chi_1, \\ [\varphi_3, \chi_1]' = \chi_2, & [\varphi_3, \chi_2]' = -\chi_1, & [\varphi_3, \chi_3]' = 0. \end{cases}$$

Sequitur ex his formulis, quod etiam considerationibus geometricis probari potest, quoties principium conservationis arearum valeat, trium integralium, quae principium conservationis centri gravitatis concernunt, unum quodcunque necessario duo reliqua secum ducere. Si unicum valet integrale $\varphi_1 = \text{Const.}$ e tribus, quae principium conservationis arearum concernunt, hoc et integrale $\chi_1 = \text{Const.}$ aliud non generatur; sed integrale $\varphi_1 = \text{Const.}$ et alterum integralium $\chi_2 = \text{Const.}$, $\chi_3 = \text{Const.}$ alterum procreat. Secundum theorema VII. formulae (5.) etiam valent, si plagulae superscriptae rejiciuntur.

*Formulae perturbationum simplicissimae, quae e systemate integralium proposito obtinentur.*

52.

Redeamus ad systema aequationum differentialium vulgarium:

$$(1.)\quad \begin{cases} \dfrac{dq_1}{dt} = \dfrac{\partial f}{\partial p_1}, & \dfrac{dq_2}{dt} = \dfrac{\partial f}{\partial p_2}, & \dots & \dfrac{dq_m}{dt} = \dfrac{\partial f}{\partial p_m}, \\ \dfrac{dp_1}{dt} = -\dfrac{\partial f}{\partial q_1}, & \dfrac{dp_2}{dt} = -\dfrac{\partial f}{\partial q_2}, & \dots & \dfrac{dp_m}{dt} = -\dfrac{\partial f}{\partial q_m}. \end{cases}$$

Quarum integralia

$$(2.)\quad \begin{cases} f = H = a, & H_1 = a_1, \quad H_2 = a_2, \quad \dots \quad H_{m-1} = a_{m-1}, \\ H' = b + t, & H_1' = b_1, \quad H_2' = b_2, \quad \dots \quad H_{m-1}' = b_{m-1} \end{cases}$$

sub forma tali invenire docui §. 34, ut identice sit

$$(3.)\quad [H_i, H_k] = 0, \quad [H_i, H_k'] = 0, \quad [H_i', H_k'] = 0,$$

excepto casu, quo in expressione $[H_i, H_k']$ fit $i = k$; quippe habetur

$$(4.)\quad [H_i, H_i'] = -1, \quad \text{sive} \quad [H_i', H_i] = +1.$$

Quibus aequationibus fit, ut pro forma sub qua integralia invenimus, etiam formulae, quae problema perturbatum concernunt, formam simplicissimam induant.

Consideremus enim in integrabilibus inventis quantitates $a, a_1, a_2, \dots a_{m-1}$, $b, b_1, b_2, \dots b_{m-1}$ ut functiones ipsius $t$ tales, ut integralia iam satisfaciant aequationibus differentialibus:

$$(5.)\quad \begin{cases} \frac{dq_1}{dt} = \frac{\partial f}{\partial p_1} + \frac{\partial \Omega}{\partial p_1}, & \frac{dp_1}{dt} = -\frac{\partial f}{\partial q_1} - \frac{\partial \Omega}{\partial q_1}, \\ \frac{dq_2}{dt} = \frac{\partial f}{\partial p_2} + \frac{\partial \Omega}{\partial p_2}, & \frac{dp_2}{dt} = -\frac{\partial f}{\partial q_2} - \frac{\partial \Omega}{\partial q_2}, \\ \dots\dots & \dots\dots \\ \frac{dq_m}{dt} = \frac{\partial f}{\partial p_m} + \frac{\partial \Omega}{\partial p_m}, & \frac{dp_m}{dt} = -\frac{\partial f}{\partial q_m} - \frac{\partial \Omega}{\partial q_m}, \end{cases}$$

designante $\Omega$ functionem ipsarum $t, q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ quamcunque. Quae est extensio formularum vulgarium perturbationum, primum ab ill. *Hamilton* in medium prolata, dum vulgo functionem perturbatricem $\Omega$ quantitates $p_1, p_2, \dots p_m$ non implicare supponitur. Quoties enim functio $\Omega$ ipsas $p_i$ non continet, fit e (5.), sicuti in (1.):

$$\frac{dq_i}{dt} = \frac{\partial f}{\partial p_i},$$

sive *differentialia* prima $\frac{dq_1}{dt}, \frac{dq_2}{dt}, \dots \frac{dq_m}{dt}$ eodem modo in problemate perturbato atque non perturbato per $t, q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ exprimuntur. Unde, quum in utroque problemate ipsae $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ eodem modo a $t$ atque elementis $a, a_1, a_2, \dots a_{m-1}, b, b_1, b_2, \dots b_{m-1}$ pendeant, quae tantum in posteriore problemate ut variabiles spectantur, etiam differentialia prima $\frac{dq_1}{dt}, \frac{dq_2}{dt}, \dots \frac{dq_m}{dt}$ in perturbato atque non perturbato problemate per tempus et elementa iisdem formulis exhibentur. Quae est suppositio vulgaris. Sed in ea, quam secundum ill. *Hamilton* proposui, extensione variabiles quidem

$q_i$, $p_i$ omnes eodem modo per tempus et elementa in utroque problemate exhibentur, sed differentialia prima diversa ratione per $q_i$ et $p_i$ exprimuntur ideoque etiam diversa ratione per tempus et elementa.

Differentiando (2.) et substituendo (5.) obtinetur:

$$\frac{da_i}{dt} = [H_i, f] + [H_i, \Omega],$$

$$\frac{db_i}{dt} = [H'_i, f] + [H'_i, \Omega],$$

excepta tantum formula, quae pro elemento $b$ invenitur:

$$\frac{db}{dt} + 1 = [H', f] + [H', \Omega].$$

Sed habemus e (3.), (4.):

$$[H_i, f] = [H_i, H] = 0, \qquad [H'_i, f] = [H'_i, H] = 0,$$

praeter

$$[H', f] = [H', H] = +1;$$

unde pro *quolibet* ipsius $i$ valore fit:

$$(6.) \quad \begin{cases} \dfrac{da_i}{dt} = [H_i, \Omega], \\ \dfrac{db_i}{dt} = [H'_i, \Omega]. \end{cases}$$

Si in his formulis post expressiones ad dextram ope aequationum (2.) formatas loco variabilium $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ introducuntur ut variabiles ipsae $a, a_1, \dots a_{m-1}, b, b_1, \dots b_{m-1}$, evadunt illae formulae (6.) inter has et ipsum $t$ aequationes $2m$ differentiales vulgares, quibus elementa $a_i$, $b_i$ ut functiones ipsius $t$ determinanda sunt. Habetur autem, si functionem $\Omega$ in expressionibus ad dextram, per elementa $a_i$, $b_i$ atque $t$ expressam supponimus, quum sit $a_i = H_i$, $b_i = H'_i$:

$$[H_i, \Omega] = \Sigma_k \frac{\partial \Omega}{\partial a_k} [H_i, H_k] + \Sigma_k \frac{\partial \Omega}{\partial b_k} [H_i, H'_k],$$

$$[H'_i, \Omega] = \Sigma_k \frac{\partial \Omega}{\partial a_k} [H'_i, H_k] + \Sigma_k \frac{\partial \Omega}{\partial b_k} [H'_i, H'_k].$$

Evanescunt autem e (3.) termini in differentialia partialia $\frac{\partial \Omega}{\partial a_k}$, $\frac{\partial \Omega}{\partial b_k}$ ducti omnes, praeter

$$[H_i, H'_i] = -1,$$

unde fit,

$$[H_i, \Omega] = -\frac{\partial \Omega}{\partial b_i},$$

$$[H'_i, \Omega] = \frac{\partial \Omega}{\partial a_i}.$$

Hinc abeunt formulae (6.) in sequentes:

$$(7.) \quad \begin{cases} \dfrac{da_i}{dt} = -\dfrac{\partial \Omega}{\partial b_i}, \\[2ex] \dfrac{db_i}{dt} = \dfrac{\partial \Omega}{\partial a_i}, \end{cases}$$

sive

$$\begin{array}{ll} \dfrac{da}{dt} = -\dfrac{\partial \Omega}{\partial b}, & \dfrac{db}{dt} = \dfrac{\partial \Omega}{\partial a}, \\[2ex] \dfrac{da_1}{dt} = -\dfrac{\partial \Omega}{\partial b_1}, & \dfrac{db_1}{dt} = \dfrac{\partial \Omega}{\partial a_1}, \\[2ex] \dfrac{da_2}{dt} = -\dfrac{\partial \Omega}{\partial b_2}, & \dfrac{db_2}{dt} = \dfrac{\partial \Omega}{\partial a_2}, \\ \cdots\cdots & \cdots\cdots \\ \dfrac{da_{m-1}}{dt} = -\dfrac{\partial \Omega}{\partial b_{m-1}}, & \dfrac{db_{m-1}}{dt} = \dfrac{\partial \Omega}{\partial a_{m-1}}. \end{array}$$

Quae formulae pro differentialibus elementorum perturbatorum inventae, sunt egregiae simplicitatis.

E quibus patet insequens theorema:

*) Problema quoddam approximatum huiuscemodi aequationibus contineatur:

$$\frac{dq_1}{dt} = \frac{\partial f}{\partial p_1}, \quad \frac{dq_2}{dt} = \frac{\partial f}{\partial p_2}, \quad \ldots \quad \frac{dq_m}{dt} = \frac{\partial f}{\partial p_m},$$

$$\frac{dp_1}{dt} = -\frac{\partial f}{\partial q_1}, \quad \frac{dp_2}{dt} = -\frac{\partial f}{\partial q_2}, \quad \ldots \quad \frac{dp_m}{dt} = -\frac{\partial f}{\partial q_m},$$

designante $f$ quamlibet ipsarum $p_i$, $q_i$ functionem. Cuius systematis secundum methodum supra propositam inventa sint integralia:

$$f = H = a, \quad H_1 = a_1, \quad \ldots \quad H_{m-1} = a_{m-1},$$

ubi $a, a_1, \ldots a_{m-1}$ constantes arbitrarias denotent, quae in functionibus $H, H_1, \ldots H_{m-1}$ non occurrant, et ubi functiones $H, H_1, \ldots H_{m-1}$ aequationibus

$$0 = [H_i, H_k] = \frac{\partial H_i}{\partial q_1}\frac{\partial H_k}{\partial p_1} + \frac{\partial H_i}{\partial q_2}\frac{\partial H_k}{\partial p_2} + \cdots + \frac{\partial H_i}{\partial q_m}\frac{\partial H_k}{\partial p_m}$$
$$- \frac{\partial H_i}{\partial p_1}\frac{\partial H_k}{\partial q_1} - \frac{\partial H_i}{\partial p_2}\frac{\partial H_k}{\partial q_2} - \cdots - \frac{\partial H_i}{\partial p_m}\frac{\partial H_k}{\partial q_m}$$

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*) Abhinc usque ad initium §. 53 lacuna in manuscripto invenitur, quam illo argumento quod sine dubio *Jacobi* eo loco tractandum sibi proposuerat explere conatus sum. C.

identice satisfaciant. Deinde si ope aequationum

$$H = a, \quad H_1 = a_1, \quad \dots \quad H_{m-1} = a_{m-1}$$

ipsarum $p_i$ valores per $q_1, q_2, \dots q_m$ et per constantes arbitrarias $a, a_1, \dots a_{m-1}$ exhibentur, erit

$$V = \int (p_1 dq_1 + p_2 dq_2 + \dots + p_m dq_m)$$

expressio integrabilis, atque erunt

$$\frac{\partial V}{\partial a} = b + t, \quad \frac{\partial V}{\partial a_1} = b_1, \quad \dots \quad \frac{\partial V}{\partial a_{m-1}} = b_{m-1}$$

aequationes finitae problematis approximati, designantibus $b, b_1, b_2, \dots b_{m-1}$ constantes novas arbitrarias. Jam si problema perturbatum contineatur aequationibus his:

$$\frac{dq_1}{dt} = \frac{\partial(f+\Omega)}{\partial p_1}, \quad \frac{dq_2}{dt} = \frac{\partial(f+\Omega)}{\partial p_2}, \quad \dots \quad \frac{dq_m}{dt} = \frac{\partial(f+\Omega)}{\partial p_m},$$
$$\frac{dp_1}{dt} = -\frac{\partial(f+\Omega)}{\partial q_1}, \quad \frac{dp_2}{dt} = -\frac{\partial(f+\Omega)}{\partial q_2}, \quad \dots \quad \frac{dp_m}{dt} = -\frac{\partial(f+\Omega)}{\partial q_m},$$

in quibus functio perturbatrix $\Omega$ functionem quamlibet ipsarum $t, q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ denotet, exprimantur ex aequationibus integralibus problematis approximati ipsae $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$, nec non functio $\Omega$, per $a, a_1, \dots a_{m-1}, b, b_1, \dots b_{m-1}, t$. Tum introductis quantitatibus $a, a_1, \dots a_{m-1}, b, b_1, \dots b_{m-1}$ loco ipsarum $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ tamquam variabilibus, aequationes differentiales problematis perturbati abeunt in has:

$$\frac{da}{dt} = -\frac{\partial\Omega}{\partial b}, \quad \frac{da_1}{dt} = -\frac{\partial\Omega}{\partial b_1}, \quad \dots \quad \frac{da_{m-1}}{dt} = -\frac{\partial\Omega}{\partial b_{m-1}},$$
$$\frac{db}{dt} = \frac{\partial\Omega}{\partial a}, \quad \frac{db_1}{dt} = \frac{\partial\Omega}{\partial a_1}, \quad \dots \quad \frac{db_{m-1}}{dt} = \frac{\partial\Omega}{\partial a_{m-1}},$$

quarum forma aequationibus propositis prorsus est similis.

*Formulae perturbationum et theorema de tertio integrali e binis inveniendo extenduntur ad casum, quo functio f ipsam t explicite continet.*

53.

Formulae perturbatoriae §. antec. traditae nullo modo mutantur, si functio $f$ ipsam $t$ etiam explicite involvit. Factis enim in §. antec. mutationibus indicatis, invenimus, datis aequationibus differentialibus perturbatis:

$$\frac{dq_1}{dt} = \frac{\partial(f+\Omega)}{\partial p_1}, \quad \frac{dq_2}{dt} = \frac{\partial(f+\Omega)}{\partial p_2}, \quad \dots \quad \frac{dq_m}{dt} = \frac{\partial(f+\Omega)}{\partial p_m},$$
$$\frac{dp_1}{dt} = -\frac{\partial(f+\Omega)}{\partial q_1}, \quad \frac{dp_2}{dt} = -\frac{\partial(f+\Omega)}{\partial q_2}, \quad \dots \quad \frac{dp_m}{dt} = -\frac{\partial(f+\Omega)}{\partial q_m},$$

fieri formulas differentiales elementorum perturbatorum:

$$\frac{da_1}{dt} = -\frac{\partial \Omega'}{\partial b_1}, \quad \frac{da_2}{dt} = -\frac{\partial \Omega}{\partial b_2}, \quad \dots \quad \frac{da_m}{dt} = -\frac{\partial \Omega}{\partial b_m},$$

$$\frac{db_1}{dt} = \frac{\partial \Omega}{\partial a_1}, \quad \frac{db_2}{dt} = \frac{\partial \Omega}{\partial a_2}, \quad \dots \quad \frac{db_m}{dt} = \frac{\partial \Omega}{\partial a_m}.$$

Addam, etiam theorema VI. §. 27 valere, si functio $f$ ipsam $t$ involvat, sive, *designantibus*

$$\varphi = \text{Const.}, \qquad \psi = \text{Const.}$$

*bina integralia quaecunque aequationum*

$$\frac{dq_1}{dt} = \frac{\partial f}{\partial p_1}, \quad \frac{dq_2}{dt} = \frac{\partial f}{\partial p_2}, \quad \dots \quad \frac{dp_m}{dt} = \frac{\partial f}{\partial p_m},$$

$$\frac{dp_1}{dt} = -\frac{\partial f}{\partial q_1}, \quad \frac{dp_2}{dt} = -\frac{\partial f}{\partial q_2}, \quad \dots \quad \frac{dp_m}{dt} = -\frac{\partial f}{\partial q_m},$$

*fieri tertium integrale*

$$[\varphi, \psi] = \text{Const.}$$

Ut enim $\varphi = \text{Const.}$, $\psi = \text{Const.}$ integralia sint aequationum differentialium propositarum, earum ope identice fieri debet $\frac{d\varphi}{dt} = 0$, $\frac{d\psi}{dt} = 0$, sive

$$\frac{\partial \varphi}{\partial t} + [\varphi, f] = 0, \qquad \frac{\partial \psi}{\partial t} + [\psi, f] = 0.$$

Unde aequatio identica, quae e theoremate V. §. 26 habetur,

$$[[\varphi, \psi], f] + [[\psi, f], \varphi] + [[f, \varphi], \psi] = 0,$$

substitutis aequationibus *identicis:*

$$[\psi, f] = -\frac{\partial \psi}{\partial t}, \qquad [f, \varphi] = \frac{\partial \varphi}{\partial t},$$

in hanc abit:

$$[[\varphi, \psi], f] + \left[\varphi, \frac{\partial \psi}{\partial t}\right] + \left[\frac{\partial \varphi}{\partial t}, \psi\right] =$$

$$[[\varphi, \psi], f] + \frac{\partial [\varphi, \psi]}{\partial t} = 0.$$

Quae ope aequationum differentialium propositarum convenit cum aequatione

$$\frac{d[\varphi, \psi]}{dt} = 0,$$

quae demonstranda erat. Propositionem praecedentem ea, qua eam exhibuimus extensione, iam ill. *Poisson* olim tradidit.

*De integrali cuius variatione aequationes dynamicae derivantur, etiam casu quo functio f vel U ipsam t explicite involvat.*

54.

Si in problematis mechanicis functio $f$ adhuc ipsam $t$ *explicite* involvit, quem casum antecedentibus consideravimus, principia generalia de conservatione virium vivarum, arearum, centri gravitatis valere desinunt. Tantum in locum principii minimae actionis aliud proponere licet simile, quod etiam hoc casu valet. Quoties enim ipsa $t$ ut variabilis independens non variatur, sed solae functiones eius $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$, atque statuuntur aequationes:

$$\frac{dq_1}{dt} = \frac{\partial f}{\partial p_1}, \quad \frac{dq_2}{dt} = \frac{\partial f}{\partial p_2}, \quad \dots \quad \frac{dq_m}{dt} = \frac{\partial f}{\partial p_m},$$

quibus determinentur $p_1, p_2, \dots p_m$ per $t, q_1, q_2, \dots q_m, \frac{dq_1}{dt}, \frac{dq_2}{dt}, \dots \frac{dq_m}{dt}$: aequationes differentiales reliquas

$$\frac{dp_1}{dt} = -\frac{\partial f}{\partial q_1}, \quad \frac{dp_2}{dt} = -\frac{\partial f}{\partial q_2}, \quad \dots \quad \frac{dp_m}{dt} = -\frac{\partial f}{\partial q_m}$$

amplectitur una aequatio symbolica:

$$(1.) \quad \begin{cases} \delta \left\{ p_1 \frac{\partial f}{\partial p_1} + p_2 \frac{\partial f}{\partial p_2} + \dots + p_m \frac{\partial f}{\partial p_m} - f \right\} = \\ \frac{d.\{ p_1 \delta q_1 + p_2 \delta q_2 + \dots + p_m \delta q_m \}}{dt}. \end{cases}$$

Quod etiam locum habet, si $f$ ipsam $t$ explicite continet, quippe quae invariata manet. Ex integratione aequationis praecedentis prodit, si pro limitibus ipsius $t$ evanescunt variationes omnes $\delta q_i$ sive quantitates $q_i$ datos valores induere debent:

$$(2.) \quad \delta \int \left\{ p_1 \frac{\partial f}{\partial p_1} + p_2 \frac{\partial f}{\partial p_2} + \dots + p_m \frac{\partial f}{\partial p_m} - f \right\} dt = 0.$$

Formula (1.) eadem est atque supra §. 37 exhibita, si tantum $H$ loco $f$ scribitur; illo tamen loco suppositum erat, $H$ sive $f$ ipsam $t$ explicite non continere. Ibidem vidimus in applicationibus mechanicis, expressionem in (2.) sub signo integrali contentam, esse

$$p_1 \frac{\partial f}{\partial p_1} + p_2 \frac{\partial f}{\partial p_2} + \dots + p_m \frac{\partial f}{\partial p_m} - f = T + U,$$

semper designante $T$ semissem summae virium vivarum atque $U$ functionem coordinatarum $x_i, y_i, z_i$, cuius differentialia partialia secundum $x_i, y_i, z_i$ sumta exprimunt vires motrices, quibus massa $m_i$ secundum directiones axium coordinatarum sollicitatur. Quae functio $U$ casu, quem consideramus, ipsam $t$ etiam explicite involvit. In problematis igitur mechanicis, etiamsi $U$ ipsam $t$ explicite continet, valebit aequatio:

$$(3.)\qquad \delta\int(T+U)\,dt = 0,$$

quae eo casu quodammodo principii minimae actionis locum tenet. Quam aequationem (3.) primum video ab ill. *Hamilton* in Commentationibus iam saepius laudatis adhibitam esse. Quae adeo facilius sese accommodat ad aequationes differentiales dynamicas inde derivandas quam principium illud. Neque hoc est, uti opinabantur Mathematici, sed illa aequatio, quae respondet principio statico *quietis*. Neque vero de integrali $\int(T+U)\,dt$ valet, quod de principio minimae actionis probari potest, integrale cuius evanescit variatio, *semper* fieri *minimum*, dummodo ne per nimium intervallum extendatur. Nam illud integrale etiam pro angustissimis intervallis, aliis casibus minimum, aliis maximum, aliis neutrum fit.

*De combinatione quadam principii conservationis virium vivarum cum principio conservationis arearum, quae certis casibus etiam valet, si functio U ipsam t explicite continet.*

## 55.

Quoties $U$ ipsam $t$ involvit, quum neque principium conservationis virium vivarum neque conservationis arearum valeat, videamus, an non casibus quibusdam earum combinatio locum habere possit. Sint aequationes propositae:

$$(1.)\qquad \begin{cases} m_i\dfrac{d^2x_i}{dt^2} = \dfrac{\partial U}{\partial x_i}+\lambda_1\dfrac{\partial F}{\partial x_i}+\lambda_2\dfrac{\partial \Phi}{\partial x_i}+\cdots, \\[2ex] m_i\dfrac{d^2y_i}{dt^2} = \dfrac{\partial U}{\partial y_i}+\lambda_1\dfrac{\partial F}{\partial y_i}+\lambda_2\dfrac{\partial \Phi}{\partial y_i}+\cdots, \\[2ex] m_i\dfrac{d^2z_i}{dt^2} = \dfrac{\partial U}{\partial z_i}+\lambda_1\dfrac{\partial F}{\partial z_i}+\lambda_2\dfrac{\partial \Phi}{\partial z_i}+\cdots, \end{cases}$$

designantibus $F=0$, $\Phi=0$, ... aequationes conditionales atque ipsi $i$ tributis valoribus $1, 2, \dots n$, siquidem $n$ est numerus punctorum materialium systematis. Quae sunt notae formulae dynamicae, in quibus iam suppono, functionem $U$ etiam ipsam $t$ continere. Multiplicatis (1.) per $\frac{dx_i}{dt}$, $\frac{dy_i}{dt}$, $\frac{dz_i}{dt}$ et summatione facta, prodit:

$$(2.)\qquad \frac{d(T-U)}{dt}+\frac{\partial U}{\partial t} = 0,$$

terminis in $\lambda_1$, $\lambda_2$, ... ductis per aequationes conditionales evanescentibus.

Ut respectu plani coordinatarum $x$, $y$ principium conservationis arearum locum habeat, primum aequationes conditionales ita comparatae esse debent, ut sit identice:

$$(3.)\quad \begin{cases} \Sigma_i\left\{y_i\frac{\partial F}{\partial x_i}-x_i\frac{\partial F}{\partial y_i}\right\} = 0, \\ \Sigma_i\left\{y_i\frac{\partial \Phi}{\partial x_i}-x_i\frac{\partial \Phi}{\partial y_i}\right\} = 0, \\ \dots\dots\dots\dots \end{cases}$$

Deinde etiam functio $U$, a qua vires sollicitantes pendent, ita comparata esse debet, ut identice sit:

$$\Sigma_i\left\{y_i\frac{\partial U}{\partial x_i}-x_i\frac{\partial U}{\partial y_i}\right\} = 0.$$

Sed ut obtineatur integrale aliquod casu quem consideramus, non opus est ut expressio ad laevam aequationis praecedentis evanescat. Nam quum e (1.) sequatur:

$$(4.)\quad \Sigma m_i\left\{y_i\frac{dx_i}{dt}-x_i\frac{dy_i}{dt}\right\} = \int\Sigma\left\{y_i\frac{\partial U}{\partial x_i}-x_i\frac{\partial U}{\partial y_i}\right\}dt$$

atque e (2.) expressionis $\frac{\partial U}{\partial t}$ integrale obtineatur, tantum necesse est, ut identice habeatur:

$$(5.)\quad \Sigma\left\{y_i\frac{\partial U}{\partial x_i}-x_i\frac{\partial U}{\partial y_i}\right\} = \alpha\frac{\partial U}{\partial t},$$

designante $\alpha$ constantem. Quippe quo casu e (2.) et (3.) eruetur integrale aequationum differentialium propositarum:

$$(6.)\quad \alpha(T-U)+\Sigma m_i\left\{y_i\frac{dx_i}{dt}-x_i\frac{dy_i}{dt}\right\} = \text{Const.},$$

sive *certa combinatio principiorum conservationis virium vivarum et arearum locum habebit.*

Restat, ut functio $U$ ita determinetur, ut aequationi (4). identice satisfaciat, et indagetur, quaenam sint problemata mechanica, quae functioni $U$ ita determinatae respondeant.

Docent praecepta nota integrationis aequationum differentialium partialium linearium, $U$ designare posse quamcunque functionem integralium systematis aequationum differentialium vulgarium:

$$dt:\ dx_1:\ dx_2:\dots:\ dx_n:dy_1:dy_2:\dots:dy_n =$$
$$\alpha:\ -y_1:-y_2:\dots:-y_n:\ x_1\ :\ x_2:\dots:x_n,$$

hoc est functionum, quae in integratione harum aequationum constantibus arbitrariis aequales existunt. Aequationibus differentialibus

$$dx_1 : dx_2 : \ldots : dx_n : dy_1 : dy_2 : \ldots : dy_n =$$
$$-y_1 : -y_2 : \ldots : -y_n : x_1 : x_2 : \ldots : x_n$$

satisfit aequationibus:

$$x_i = \alpha_i \cos(\varphi+\beta_i), \qquad y_i = \alpha_i \sin(\varphi+\beta_i),$$

in quibus $\alpha_i$, $\beta_i$ sunt constantes arbitrariae atque $\varphi$ designare potest functionem quamcunque ipsius $t$. Quae functio $\varphi$ determinatur per proportionem

$$dt : \alpha = dx_1 : -y_1 = d\varphi : 1,$$

unde

$$\alpha\varphi = t.$$

Si loco coordinatarum orthogonalium $x_i$, $y_i$ polares introducuntur, ponendo

$$x_i = r_i \cos v_i, \qquad y_i = r_i \sin v_i,$$

atque loco constantis $\frac{1}{\alpha}$ ponitur $\gamma$, fit:

$$\alpha_i = r_i, \qquad \beta_i = v_i - \gamma t.$$

Unde iam est forma maxime generalis functionis $U$, quae aequationi (5.) identice satisfacit, functio arbitraria ipsarum $r_i$ atque $v_i - \gamma t = v_i - \frac{t}{\alpha}$, hoc est distantiarum punctorum materialium ab initio coordinatarum proiectarum in ipsarum $x$, $y$ planum, et angulorum, quos distantiae proiectae faciunt cum recta, *quae in plano illo uniformiter circa initium coordinatarum rotatur.* Insuper functio $U$ etiam quantitates $z_i$ quocunque modo continere potest.

Aequationibus (3.) satisfieri constat, si $F$ et $\Phi$ sunt functiones ipsarum $r_i$ atque differentiarum ipsarum $v_i$. Unde habemus propositionem:

„*Statuamus in aequationibus differentialibus dynamicis* (1.), *posito* $x_i = r_i \cos v_i$, $y_i = r_i \sin v_i$, *functiones* $F$, $\Phi$, ... *praeter quantitates* $z_i$, $r_i$ *tantum differentias ipsarum* $v_i$ *continere, porro ipsam* $U$ *esse functionem quamlibet quantitatum* $z_i$, $r_i$ *atque* $v_i - \gamma t$, *designante* $\gamma$ *constantem, erit aequationum* (1.) *integrale*:

$$T - U + \gamma \Sigma m_i \left\{ y_i \frac{dx_i}{dt} - x_i \frac{dy_i}{dt} \right\} = \text{Const.}\ \text{“}$$

Integrale inventum etiam sic repraesentare licet:

$$(7.) \qquad T - U - \gamma \Sigma m_i r_i^2 \frac{dv_i}{dt} = \text{Const.},$$

sive etiam:

$$(8.)\quad \tfrac{1}{2}\Sigma m_i\left\{\left(\frac{dz_i}{dt}\right)^2+\left(\frac{dr_i}{dt}\right)^2+r_i^2\left(\frac{dv_i}{dt}-\gamma\right)^2\right\} = \tfrac{1}{2}\gamma^2\Sigma m_i r_i^2+U+\text{Const.}.$$

Pars laeva aequationis praecedentis (8.) est semissis summae virium vivarum systematis, siquidem refertur systema ad axes mobiles coordinatarum $x$ et $y$, in ipsarum plano circa initium coordinatarum uniformiter rotantes.

Aequationes differentiales (1.) notum est sic etiam repraesentari posse:

$$(9.)\quad \begin{cases} m_i\left(\dfrac{d^2r_i}{dt^2}-r_i\left(\dfrac{dv_i}{dt}\right)^2\right) = \dfrac{\partial U}{\partial r_i}+\lambda_1\dfrac{\partial F}{\partial r_i}+\lambda_2\dfrac{\partial \Phi}{\partial r_i}+\cdots, \\ m_i\dfrac{d.r_i^2\dfrac{dv_i}{dt}}{dt} = \dfrac{\partial U}{\partial v_i}+\lambda_1\dfrac{\partial F}{\partial v_i}+\lambda_2\dfrac{\partial \Phi}{\partial v_i}+\cdots, \\ m_i\dfrac{d^2z_i}{dt^2} = \dfrac{\partial U}{\partial z_i}+\lambda_1\dfrac{\partial F}{\partial z_i}+\lambda_2\dfrac{\partial \Phi}{\partial z_i}+\cdots. \end{cases}$$

Si statuitur

$$w_i = v_i-\gamma t,$$

erit $U$ functio ipsarum $r_i$, $w_i$, $z_i$, quae praeter has quantitates ipsam $t$ non continet; porro aequationes (9.) evadunt:

$$(10.)\quad \begin{cases} m_i\left(\dfrac{d^2r_i}{dt^2}-r_i\left(\dfrac{dw_i}{dt}\right)^2\right) = \gamma m_i r_i\left\{2\dfrac{dw_i}{dt}+\gamma\right\}+\dfrac{\partial U}{\partial r_i}+\lambda_1\dfrac{\partial F}{\partial r_i}+\lambda_2\dfrac{\partial \Phi}{\partial r_i}+\cdots, \\ m_i\dfrac{d.r_i^2\dfrac{dw_i}{dt}}{dt} = -\gamma m_i\dfrac{d.r_i^2}{dt}+\dfrac{\partial U}{\partial w_i}+\lambda_1\dfrac{\partial F}{\partial v_i}+\lambda_2\dfrac{\partial \Phi}{\partial v_i}+\cdots, \\ m_i\dfrac{d^2z_i}{dt^2} = \dfrac{\partial U}{\partial z_i}+\lambda_1\dfrac{\partial F}{\partial z_i}+\lambda_2\dfrac{\partial \Phi}{\partial z_i}+\cdots. \end{cases}$$

Si tres aequationes praecedentes per $dr_i$, $dw_i$, $dz_i$ multiplicantur atque in productis ipsi $i$ valores 1, 2, ... $n$ tribuuntur, omnium summa facile suppeditat integrale inventum (8.). Termini enim bini in $r_i$, $\frac{dr_i}{dt}$, $\frac{dw_i}{dt}$ ducti se mutuo destruunt, eritque

$$\Sigma_i\frac{\partial F}{\partial w_i}dw_i = \Sigma_i\frac{\partial F}{\partial v_i}dv_i,\qquad \Sigma_i\frac{\partial \Phi}{\partial w_i}dw_i = \Sigma_i\frac{\partial \Phi}{\partial v_i}dv_i,$$

quum e suppositione supra circa functiones $F$, $\Phi$, ... facta identice sit

$$\Sigma_i\frac{\partial F}{\partial v_i} = 0,\qquad \Sigma_i\frac{\partial \Phi}{\partial v_i} = 0,\qquad \ldots.$$

Aequationes differentiales ipsis (10.) similes ill. *Laplace* in Opere de *Machina Coelesti* tradidit, quaerens motum planetae verum circa *ipsius* medium, dum formulae praecedentes accommodatae sunt quaesitioni, qua duorum planetarum alterius motus verus circa *alterius* medium consideratur.

Functio $U$ forma antecedentibus praescripta gaudet, *quoties puncta* $m_i$ *quum a se invicem tum a centris quotcunque trahuntur, quae circa axem coordinatarum* $z$ *uniformiter rotantur, communi rotationis velocitate, et in qua neque ipsa neque puncta* $m_i$ *reagunt.* Pro quibus centris etiam substitui possunt corpora solida cuiuslibet formae exterioris ac constitutionis interioris, quae circa axem coordinatarum $z$ eadem ac constanti velocitate rotantur atque insuper neque a se invicem neque a punctis $m_i$ sollicitantur. His omnibus casibus integrale unum propositum locum habebit. Qui obveniunt casus in problemate trium corporum, siquidem statuatur, quod proxime licet, corpus principale et corpus perturbans in plano fixo uniformiter rotari circa commune eorum gravitatis centrum. Unde integrale propositum iustum erit in problemate trium corporum respectu omnium potestatum excentricitatis et inclinationis corporis perturbati atque massae corporis perturbati, reiectis terminis ab excentricitate et inclinatione corporis perturbantis atque massa ipsius corporis perturbati pendentibus.

*Ostenditur, quomodo et aequatione conservationis virium vivarum et aequatione una conservationem arearum concernente ordo integrationum binis unitatibus minuatur. Quod haudquaquam pro quolibet aequationum dynamicarum integrali contigit.*

56.

Quoties functio $U$ ipsam $t$ non explicite involvit ideoque principium conservationis virium vivarum locum habet, integrale *unum*, quo principium illud continetur, ordinem differentiationum minuit *duabus* unitatibus. Sint enim rursus aequationes differentiales numero $2m$ sequentes:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i};$$

si $U$ ideoque etiam $H = T - U$ ipsam $t$ non continet, aequationes illae sic exhiberi possunt:

$$dq_1 : dq_2 : \ldots : dq_m : dp_1 : dp_2 : \ldots : dp_m =$$
$$\frac{\partial H}{\partial p_1} : \frac{\partial H}{\partial p_2} : \ldots : \frac{\partial H}{\partial p_m} : -\frac{\partial H}{\partial q_1} : -\frac{\partial H}{\partial q_2} : \ldots : -\frac{\partial H}{\partial q_m},$$

quae sunt aequationes differentiales $2m-1$ primi ordinis, ideoque unius aequationis differentialis $(2m-1)^{ti}$ ordinis locum tenent, quae per integrale a principio conservationis virium vivarum suppeditatum ad ordinem $(2m-2)^{tum}$ reducitur. Contra si $U$ ideoque etiam $H$ ipsam $t$ continet, aequationes differentiales propositae unius aequationis ordinis $2m^{ti}$ locum tenent.

Si insuper principium conservationis arearum respectu plani cuiusdam dati locum habet, ordo differentiationum rursus duabus unitatibus minuitur, siquidem semper statuitur ordo differentiationum systematis aequationum differentialium vulgarium idem atque unius aequationis inter duas variabiles, ad quam per regulas notas eliminationis systema aequationum differentialium revocari potest, sive etiam idem atque numerus constantium arbitrariarum, quas poscit integratio completa. Sumatur enim planum datum ut planum coordinatarum $x$ et $y$, ac statuatur rursus

$$x_i = r_i \cos v_i, \qquad y_i = r_i \sin v_i;$$

casu quem consideramus continebunt aequationes propositae differentialia quidem prima et secunda singulorum angulorum $v_i$, sed ipsorum $v_i$ tantum differentias. Iam per integrale, quod casu proposito principium arearum concernit, fit, designante $\alpha$ constantem arbitrariam,

$$\alpha = \Sigma m_i r_i^2 \frac{dv_i}{dt},$$

unde posito

$$u_i = v_i - v_n, \qquad R = \Sigma m_i r_i^2, \qquad N = \Sigma m_i r_i^2 \frac{du_i}{dt},$$

fit

$$(1.) \qquad \alpha = R\frac{dv_n}{dt} + \Sigma m_i r_i^2 \frac{du_i}{dt} = R\frac{dv_n}{dt} + N.$$

Si in aequatione, qua principium conservationis virium vivarum continetur,

$$\Sigma m_i \left\{ \frac{d^2 z_i}{dt^2} + \left(\frac{dr_i}{dt}\right)^2 + r_i^2 \left(\frac{dv_i}{dt}\right)^2 \right\} = U + h,$$

in qua $h$ est constans arbitraria, substituimus valores $v_i = u_i + v_n$, fit illa:

$$\Sigma m_i \left\{ \left(\frac{dz_i}{dt}\right)^2 + \left(\frac{dr_i}{dt}\right)^2 + r_i^2 \left(\frac{du_i}{dt}\right)^2 \right\} + 2\frac{dv_n}{dt} \Sigma m_i r_i^2 \frac{du_i}{dt} + R\left(\frac{dv_n}{dt}\right)^2 = U + h,$$

sive e (1.)

$$(2.) \qquad \Sigma m_i \left\{ \left(\frac{dz_i}{dt}\right)^2 + \left(\frac{dr_i}{dt}\right)^2 + r_i^2 \left(\frac{du_i}{dt}\right)^2 \right\} + \frac{\alpha^2 - N^2}{R} = U + h.$$

Si in aequationibus differentialibus substituuntur valores

$$\frac{dv_i}{dt} = \frac{du_i}{dt} + \frac{\alpha - N}{R},$$

exulavit quantitas $v_n$ simul cum eius differentialibus, quum aequationes conditionales atque functio $U$ tantum quantitates $z_i$, $r_i$, $u_i$ contineant, unde numerus variabilium unitate, ideoque ordo differentiationum duabus unitatibus minuitur. Generaliter, quoties in aequationibus differentialibus propositis variabiles ita eligere licet, ut in iis una ex earum numero non ipsa sed tantum eius bina differentialia prima obveniant, quum novum integrale generaliter invenire licet, tum uno integrali novo invento ordo differentiationum *duabus unitatibus* diminuitur. Sit enim in aequationibus differentialibus supra propositis $q_i$ variabilis, quae in ipsa $H$ non invenitur, habetur

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} = 0$$

ideoque novum integrale

$$p_i = \text{Const.}.$$

Reiecta deinde aequatione

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

atque considerata $p_i$ in reliquis aequationibus differentialibus ut constante, sicuti inventum est, numerus variabilium $q$ et $p$ ideoque *ordo* differentiationum duabus unitatibus diminutus est.

Etiam in casu, quem §. antec. tractavi, si integrale (8.) §. 55 locum habet, ordo differentiationum duabus unitatibus diminuitur. Nam si statuuntur ut variabiles $r_i$, $w_i$, $z_i$, $r_i' = \frac{dr_i}{dt}$, $w_i' = \frac{dw_i}{dt}$, $z_i' = \frac{dz_i}{dt}$, atque aequationes (10.) §. 55 omnes per unam ex earum numero dividuntur, ipsa $t$ atque elementum $dt$ in aequationibus differentialibus non inveniuntur, ordoque differentiationum unitate diminuitur, qui deinde per integrale (8.) §. 55 altera adhuc unitate diminuitur, prorsus simili ratione atque vidimus, quoties principium conservationis virium vivarum locum habeat, ordinem differentiationum per principium illud atque eliminationem elementi temporis duabus unitatibus diminui.

At non omnibus casibus, quoties habetur integrale novum, simili ratione atque in praecedentibus exemplis ordinem differentiationum scita variabilium electione duabus unitatibus deprimere licet. Ita non fit, ut altero et tertio integrali, quod principium arearum concernit, duas variabiles cum earum differentialibus eliminare liceat ideoque *quatuor* unitatibus iste ordo deprimatur. Sunt tantum praecedentia exempla simplicissima, in quibus iam absque theoria supra condita illa depressio ordinis differentiationum sponte se offert. Theoria autem supra condita docet, semper variabilium systema investigari posse, pro quibus ordo differentiationum duabus unitatibus inferior evadat; sed generaliter illa investigatio secundum praecepta tradita postulat, ut alia condantur systemata aequationum differentialium, quae inferiorum ordinum sunt, atque singulorum istorum systematum auxiliarium integrale unum quodcunque indagetur.

*Systema propositum aequationum differentialium vulgarium vocatur canonicum. Cuiusmodi systema in aliud canonicum transformatur. Quod una cum transformatione aequationis differentialis partialis valde generali peragitur. Canonicum elementorum systema.*

57.

Revertor ad formulas perturbationum §. 52 traditas. Videmus, aequationes §. 52 (7.), in quibus elementa perturbata ut variabiles introducta sunt, prorsus eadem forma gaudere atque ipsas aequationes differentiales §. 52 (5.). Formam autem illam memorabilem, qua utrumque systema aequationum differentialium gaudet, quia frequenter in his aequationibus obvenit, dicam aequationum differentialium formam *canonicam*. Sunt in eiusmodi systemate canonico aequationum differentialium vulgarium variabiles numero pari, atque altera pars semissis variabilium alteri semissi singulae singulis ita respondent, ut differentialia illarum variabilium aequalia sint differentialibus partialibus certae cuiusdam functionis secundum has variabiles sumtis, et harum variabilium differentialia aequalia sint eiusdem functionis differentialibus partialibus secundum illas variabiles sumtis atque insuper signo negativo affectis.

His positis transformatio illa, qua vidimus §§. 52, 53 aequationes

$$\frac{dq_i}{dt} = \frac{\partial(f+\Omega)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial(f+\Omega)}{dq_i}$$

mutari in has:

$$\frac{db_i}{dt} = \frac{\partial\Omega}{\partial a_i}, \quad \frac{da_i}{dt} = -\frac{\partial\Omega}{\partial b_i},$$

continetur sub problemate generali, *quodcunque systema aequationum differentialium, quod canonica forma gaudeat, in aliud eiusdem formae per introductionem novarum variabilium transformare.* Quod problema etiam ratione prorsus diversa proponere licet.

Comprobavi enim antecedentibus integrationem systematis aequationum differentialium:

$$\frac{dq_i}{dt} = \frac{\partial f}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial f}{\partial q_i},$$

in qua maioris generalitatis gratia supponam, praeter quantitates $q_i$, $p_i$ etiam ipsam $t$ functionem $f$ explicite ingredi, pendere ab integratione aequationis differentialis partialis, quae provenit ex aequatione

$$0 = \frac{\partial V}{\partial t} + f\text{*)},$$

substituendo in functione $f$ in locum quantitatum $p_i$ differentialia functionis $V$ partialia respectu quantitatum $q_i$ sumta, sive statuendo

$$p_i = \frac{\partial V}{\partial q_i}.$$

Inventa enim functione $V$, aequationi illi differentiali partiali satisfaciente atque involvente $m$ constantes arbitrarias $a_i$, praeter unam ipsi $V$ mera additione adiungendam, erant integralia completa aequationum:

$$\frac{dq_i}{dt} = \frac{\partial f}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial f}{\partial q_i}$$

sequentia:

$$\frac{\partial V}{\partial q_i} = p_i, \quad \frac{\partial V}{\partial a_i} = b_i,$$

designantibus $b_i$ novas constantes $m$ arbitrarias. Videmus igitur, inter systematis canonici aequationum differentialium vulgarium et aequationis differentialis partialis integrationem arctissimum nexum intercedere. *Unde alterius transformatio statim etiam alterius transformationem suppeditabit.*

In promptu est aequationis differentialis partialis transformatio, si tantum in locum variabilium *independentium* aliae variabiles independentes introducuntur. *Neque alias transformationes hactenus considerasse videntur Analystae* **). Sed dantur etiam transformationes aequationis differentialis partialis primi ordinis alius in aliam primi ordinis per substitutiones, in quibus expressiones variabilium independentium alterius aequationis continent quum variabiles independentes alterius tum differentialia secundum eas sumta partialia. Methodus generalis eiusmodi efficiendi transformationem haec est:

Proposita sit aequatio:

$$(1.) \quad dV_1 = -f_1 dt + p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m,$$

in qua

$$(2.) \quad f_1 = -\frac{\partial V_1}{\partial t}$$

sit data functio ipsarum $q_1, q_2, \ldots q_m, t$ atque ipsarum

$$p_1 = \frac{\partial V_1}{\partial q_1}, \quad p_2 = \frac{\partial V_1}{\partial q_2}, \quad \ldots \quad p_m = \frac{\partial V_1}{\partial q_m}.$$

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*) Constantem $a$ §. 35 (3.) additam hic quod licet $= 0$ posui.

**) quarum tamen specimen quoddam offert *Euleriana* illa methodus qua variabiles independentes cum differentialibus secundum illas sumtis commutantur. *C.*

Aequatio (1.) locum tenet aequationis differentialis partialis (2.), atque in locum aequationis (2.) licet aequationem (1.) transformare. Ad quam efficiendam transformationem assumo functionem *prorsus arbitrariam* $V$ ipsarum $t, q_1, q_2, \dots q_m$, atque novarum variabilium $a_1, a_2, \dots a_m$. Quae determinentur novae variabiles per ipsas $t, q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ ope aequationum:

$$(3.) \quad \frac{\partial V}{\partial q_1} = p_1, \quad \frac{\partial V}{\partial q_2} = p_2, \quad \dots \quad \frac{\partial V}{\partial q_m} = p_m,$$

ac praeterea statuatur

$$(4.) \quad -\frac{\partial V}{\partial a_1} = b_1, \quad -\frac{\partial V}{\partial a_2} = b_2, \quad \dots \quad -\frac{\partial V}{\partial a_m} = b_m.$$

His positis fit:

$$(5.) \quad \begin{cases} dV = \frac{\partial V}{\partial t} dt + p_1 dq_1 + p_2 dq_2 + \dots + p_m dq_m \\ \qquad - b_1 da_1 - b_2 da_2 - \dots - b_m da_m. \end{cases}$$

Qua aequatione subducta de (1.) positoque

$$(6.) \quad V_1 - V = W,$$

nanciscimur:

$$(7.) \quad dW = -\left\{ f_1 + \frac{\partial V}{\partial t} \right\} dt + b_1 da_1 + b_2 da_2 + \dots + b_m da_m.$$

Transformationem generalem aequationis differentialis partialis primi ordinis, quae antecedentibus continetur, theoremate particulari proponere convenit.

### Theorema VIII.

*Sint $t, q_1, q_2, \dots q_m$ variabiles independentes, inter quas et functionem earum $V_1$ proposita sit aequatio differentialis partialis:*

$$\frac{\partial V_1}{\partial t} + f_1\left(t, q_1, q_2, \dots q_m, \frac{\partial V_1}{\partial q_1}, \frac{\partial V_1}{\partial q_2}, \dots \frac{\partial V_1}{\partial q_m}\right) = 0;$$

*assumatur functio prorsus arbitraria $V$ ipsarum $t, q_1, q_2, \dots q_m$ et novarum variabilium $a_1, a_2, \dots a_m$; atque in locum quantitatum $q_1, q_2, \dots q_m$, introducendo quantitates*

$$\frac{\partial V}{\partial a_1}, \quad \frac{\partial V}{\partial a_2}, \quad \dots \quad \frac{\partial V}{\partial a_m},$$

*exprimamus quantitates sequentes:*

$$\frac{\partial V}{\partial q_1}, \quad \frac{\partial V}{\partial q_2}, \quad \dots \quad \frac{\partial V}{\partial q_m},$$

*atque si in* $f_1$ *loco* $\frac{\partial V_1}{\partial q_i}$ *scribimus* $\frac{\partial V}{\partial q_i}$, functionem

$$\frac{\partial V}{\partial t}+f_1\left(t,\, q_1,\, q_2,\, \dots\, q_m,\, \frac{\partial V}{\partial q_1},\, \frac{\partial V}{\partial q_2},\, \dots\, \frac{\partial V}{\partial q_m}\right)$$

*per quantitates*

$$t,\quad a_1,\quad a_2,\quad \dots\quad a_m,\quad -\frac{\partial V}{\partial a_1},\quad -\frac{\partial V}{\partial a_2},\quad \dots\quad -\frac{\partial V}{\partial a_m};$$

*quo facto fiat:*

$$\begin{aligned}\frac{\partial V}{\partial t}&+f_1\left(t,\, q_1,\, q_2,\, \dots\, q_m,\, \frac{\partial V}{\partial q_1},\, \frac{\partial V}{\partial q_2},\, \dots\, \frac{\partial V}{\partial q_m}\right)\\ &= \varphi\left(t,\, a_1,\, a_2,\, \dots\, a_m,\, -\frac{\partial V}{\partial a_1},\, -\frac{\partial V}{\partial a_2},\, \dots\, -\frac{\partial V}{\partial a_m}\right);\end{aligned}$$

*his omnibus transactis, si in functione* $\varphi$ *loco* $-\frac{\partial V}{\partial a_i}$ *scribitur* $\frac{\partial W}{\partial a_i}$, *erit aequatio differentialis partialis proposita transformata in sequentem:*

$$\frac{\partial W}{\partial t}+\varphi\left(t,\, a_1,\, a_2,\, \dots\, a_m,\, \frac{\partial W}{\partial a_1},\, \frac{\partial W}{\partial a_2},\, \dots\, \frac{\partial W}{\partial a_m}\right) = 0,$$

*atque alterius solutio ex alterius invenitur solutione ope aequationis:*

$$V_1 = V+W,$$

*siquidem aut cognita solutione* $V_1$ *variabiles* $q_1,\, q_2,\, \dots\, q_m$ *exprimuntur per* $a_1,\, a_2,\, \dots\, a_m,\, t$ *ope aequationum*

$$\frac{\partial V_1}{\partial q_1}=\frac{\partial V}{\partial q_1},\quad \frac{\partial V_1}{\partial q_2}=\frac{\partial V}{\partial q_2},\quad \dots\quad \frac{\partial V_1}{\partial q_m}=\frac{\partial V}{\partial q_m},$$

*aut cognita solutione* $W$ *variabiles* $a_1,\, a_2,\, \dots\, a_m$ *exprimuntur per* $q_1,\, q_2,\, \dots\, q_m,\, t$ *ope aequationum:*

$$\frac{\partial W}{\partial a_1}=-\frac{\partial V}{\partial a_1},\quad \frac{\partial W}{\partial a_2}=-\frac{\partial V}{\partial a_2},\quad \dots\quad \frac{\partial W}{\partial a_m}=-\frac{\partial V}{\partial a_m}.$$

Demonstratio theorematis antecedentibus tradita eo nititur, quod positis aequationibus

$$\frac{\partial V_1}{\partial q_1}=\frac{\partial V}{\partial q_1},\quad \frac{\partial V_1}{\partial q_2}=\frac{\partial V}{\partial q_2},\quad \dots\quad \frac{\partial V_1}{\partial q_m}=\frac{\partial V}{\partial q_m}$$

inde sponte sequuntur hae aequationes:

$$\frac{\partial V}{\partial a_1}=-\frac{\partial W}{\partial a_1},\quad \frac{\partial V}{\partial a_2}=-\frac{\partial W}{\partial a_2},\quad \dots\quad \frac{\partial V}{\partial a_m}=-\frac{\partial W}{\partial a_m},$$

quod etiam inverti potest.

Transformatio generalis systematis canonici aequationum differentialium vulgarium theoremati praecedenti respondens hoc theoremate continetur:

### Theorema IX.

*Proposito systemate aequationum differentialium vulgarium canonico:*

$$\frac{dq_1}{dt} = \frac{\partial f_1}{\partial p_1}, \quad \frac{dq_2}{dt} = \frac{\partial f_1}{\partial p_2}, \quad \ldots \quad \frac{\partial q_m}{\partial t} = \frac{\partial f_1}{\partial p_m},$$

$$\frac{dp_1}{dt} = -\frac{\partial f_1}{\partial q_1}, \quad \frac{dp_2}{dt} = -\frac{\partial f_1}{\partial q_2}, \quad \ldots \quad \frac{\partial p_m}{\partial t} = -\frac{\partial f_1}{\partial q_m},$$

*in qua $f_1$ est functio ipsarum $t$, $q_1$, $q_2$, ... $q_m$, $p_1$, $p_2$, ... $p_m$ quaecunque, assumatur functio arbitraria $V$ quantitatum $t$, $q_1$, $q_2$, ... $q_m$ atque novarum variabilium $a_1$, $a_2$, ... $a_m$; quo facto condantur aequationes:*

$$\frac{\partial V}{\partial q_1} = p_1, \quad \frac{\partial V}{\partial q_2} = p_2, \quad \ldots \quad \frac{\partial V}{\partial q_m} = p_m,$$

$$\frac{\partial V}{\partial a_1} = -b_1, \quad \frac{\partial V}{\partial a_2} = -b_2, \quad \ldots \quad \frac{\partial V}{\partial a_m} = -b_m,$$

*quarum ope exprimantur et variabiles $q_1$, $q_2$, ... $q_m$, $p_1$, $p_2$, ... $p_m$ et functio*

$$f_1 + \frac{\partial V}{\partial t}$$

*per $t$ et novas variabiles $a_1$, $a_2$, ... $a_m$, $b_1$, $b_2$, ... $b_m$; inventa expressione*

$$f_1 + \frac{\partial V}{\partial t} = \varphi(t, a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m),$$

*systema aequationum differentialium vulgarium canonicum in aliud canonicum hoc per substitutiones propositas transformatur:*

$$\frac{da_1}{dt} = \frac{\partial \varphi}{\partial b_1}, \quad \frac{da_2}{dt} = \frac{\partial \varphi}{\partial b_2}, \quad \ldots \quad \frac{da_m}{dt} = \frac{\partial \varphi}{\partial b_m},$$

$$\frac{db_1}{dt} = -\frac{\partial \varphi}{\partial a_1}, \quad \frac{db_2}{dt} = -\frac{\partial \varphi}{\partial a_2}, \quad \ldots \quad \frac{db_m}{dt} = -\frac{\partial \varphi}{\partial a_m}.$$

Theorematis praecedentis gravissimi demonstratio, quamquam iam in quaestionibus supra traditis continetur, si breviter denuo adstruere placet, haec habetur.

E systemate canonico proposito fluunt aequationes symbolicae sequentes, siquidem formularum notarum

$$\delta \cdot \frac{dq_i}{dt} = \frac{d.\delta q_i}{dt}, \quad \delta \cdot \frac{dp_i}{dt} = \frac{d.\delta p_i}{dt}, \quad \delta \cdot \frac{dV}{dt} = \frac{d.\delta V}{dt}$$

recorderis:

$$\delta f_1 = \frac{\partial f_1}{\partial t}\delta t + \Sigma_i\left(\frac{dq_i}{dt}\delta p_i - \frac{dp_i}{dt}\delta q_i\right)$$

$$= \frac{\partial f_1}{\partial t}\delta t + \Sigma_i\left(\frac{dq_i}{dt}\,\delta\frac{\partial V}{\partial q_i} - \frac{d\frac{\partial V}{\partial q_i}}{dt}\delta q_i\right)$$

$$= \frac{\partial f_1}{\partial t}\delta t + \Sigma_i\left(\delta\left(\frac{\partial V}{\partial q_i}\frac{dq_i}{dt}\right) - \frac{d\cdot\frac{\partial V}{\partial q_i}\delta q_i}{dt}\right)$$

$$= \frac{\partial f_1}{\partial t}\delta t + \delta\cdot\frac{dV}{dt} - \frac{d.\delta V}{dt} - \delta\cdot\frac{\partial V}{\partial t} + \frac{d\cdot\frac{\partial V}{\partial t}\delta t}{dt}$$

$$-\Sigma_i\left(\delta\left(\frac{\partial V}{\partial a_i}\frac{da_i}{dt}\right) - \frac{d\cdot\frac{\partial V}{\partial a_i}\delta a_i}{dt}\right)$$

$$= \frac{\partial f_1}{\partial t}\delta t - \delta\cdot\frac{\partial V}{\partial t} + \frac{d\cdot\frac{\partial V}{\partial t}\delta t}{dt}$$

$$-\Sigma_i\left(\frac{da_i}{dt}\,\delta\cdot\frac{\partial V}{\partial a_i} - \delta a_i\frac{d\cdot\frac{\partial V}{\partial a_i}}{dt}\right)$$

$$= \left(\frac{\partial f_1}{\partial t} + \frac{d.(\varphi - f_1)}{dt}\right)\delta t - \delta(\varphi - f_1)$$

$$+\Sigma_i\left(\frac{da_i}{dt}\delta b_i - \frac{db_i}{dt}\delta a_i\right).$$

Unde quum e systemate canonico proposito sequatur:

$$\frac{df_1}{dt} = \frac{\partial f_1}{\partial t},$$

invenimus:

$$\delta\varphi = \frac{d\varphi}{dt}\delta t + \Sigma_i\left(\frac{da_i}{dt}\delta b_i - \frac{db_i}{dt}\delta a_i\right).$$

Quae aequatio symbolica systema canonicum transformatum suppeditat, atque insuper aequationem

$$\frac{\partial\varphi}{\partial t} = \frac{d\varphi}{dt},$$

quae ex illo sequitur.

Unde functione $W$ per ipsas $t$, $a_1$, $a_2$, ... $a_m$ expressa, fit:

$$(8.)\quad \frac{\partial W}{\partial t} = -\left\{f_1 + \frac{\partial V}{\partial t}\right\},\quad \frac{\partial W}{\partial a_1} = b_1,\quad \frac{\partial W}{\partial a_2} = b_2,\quad \dots\quad \frac{\partial W}{\partial a_m} = b_m.$$

Eliminatis quantitatibus $q_i$, $p_i$ ex expressione $f_1+\frac{\partial V}{\partial t}$ ope aequationum (3.) et (4.), evadit illa functio ipsarum $t, a_1, a_2, \dots a_m, b_1, b_2, \dots b_m$, quam statuamus:

$$f_1+\frac{\partial V}{\partial t} = \varphi(t, a_1, a_2, \dots a_m, b_1, b_2, \dots b_m),$$

eritque substitutis ipsarum $b_i$ expressionibus $\frac{\partial W}{\partial a_i}$, e (8.):

$$(9.) \qquad \frac{\partial W}{\partial t} = -\varphi\left(t, a_1, a_2, \dots a_m, \frac{\partial W}{\partial a_1}, \frac{\partial W}{\partial a_2}, \dots \frac{\partial W}{\partial a_m}\right).$$

Quae est aequatio differentialis partialis transformata, quae locum tenet aequationis differentialis partialis propositae (2.). Eritque prorsus eadem substitutione (3.) et (4.) systema canonicum aequationum differentialium vulgarium propositum in aliud canonicum transformatum.

Transformatione generali theoremate antecedente proposita, continetur illa in qua ut variabiles novae statuuntur elementa problematis approximati. Sit enim in theoremate illa

$$(10.) \qquad f_1 = f+\Omega,$$

sintque functiones $f$ et $V$ ita comparatae, ut substitutis in $f$ loco ipsarum $p_i$ expressionibus $\frac{\partial V}{\partial q_i}$, prodeat

$$(11.) \qquad \frac{\partial V}{\partial t} = -f;$$

considerari possunt quantitates $a_1, a_2, \dots a_m$ tamquam *constantes arbitrariae*, quae afficiunt solutionem $V$ aequationis differentialis partialis

$$\frac{\partial V}{\partial t}+f = 0,$$

ideoque ex iis, quae supra probata sunt, considerari possunt $a_1, a_2, \dots a_m$, $b_1, b_2, \dots b_m$ ut *elementa constantia,* quae afficiunt integralia completa

$$(12.) \qquad \begin{cases} \frac{\partial V}{\partial q_1} = p_1, & \frac{\partial V}{\partial q_2} = p_2, & \dots & \frac{\partial V}{\partial q_m} = p_m, \\ \frac{\partial V}{\partial a_1} = -b_1, & \frac{\partial V}{\partial a_2} = -b_2, & \dots & \frac{\partial V}{\partial a_m} = -b_m \end{cases}$$

aequationum differentialium

$$(13.) \qquad \begin{cases} \frac{dq_1}{dt} = \frac{\partial f}{\partial p_1}, & \frac{dq_2}{dt} = \frac{\partial f}{\partial p_2}, & \dots & \frac{dq_m}{dt} = \frac{\partial f}{\partial p_m}, \\ \frac{dp_1}{dt} = -\frac{\partial f}{\partial q_1}, & \frac{dp_2}{dt} = -\frac{\partial f}{\partial q_2}, & \dots & \frac{dp_m}{dt} = -\frac{\partial f}{\partial q_m}. \end{cases}$$

Quoties igitur vice versa in aequationibus (12.) ipsae $a_i$, $b_i$ ut constantes considerantur, sunt aequationes illae (12.) integralia completa aequationum (13.). Quoties vero in aequationibus (12.) ipsae $a_i$, $b_i$ ut variabiles considerantur, quae in locum ipsarum $q_i$, $p_i$ ope aequationum illarum substituendae sunt, docet theorema propositum, transformari ea substitutione aequationes

$$\frac{dq_i}{dt} = \frac{\partial(f+\Omega)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial(f+\Omega)}{\partial q_i}$$

in aequationes sequentes:

$$\frac{da_i}{dt} = \frac{\partial \Omega}{\partial b_i}, \quad \frac{db_i}{dt} = -\frac{\partial \Omega}{\partial a_i}.$$

Habemus enim casu proposito:

$$\varphi = f_1 + \frac{\partial V}{\partial t} = f_1 - f = \Omega.$$

Quae sunt formulae differentiales elementorum perturbatorum, quae a supra propositis tantum eo discrepant, quod in iis $-b_i$ loco $b_i$ scriptum sit. *Systema elementorum, quae in modum praecedentium per aequationes differentiales canonicas determinantur, et ipsum dicere convenit canonicum elementorum systema.*

*De transformatione systematis elementorum canonici in aliud canonicum.*

58.

In antecedentibus duae functiones, quarum differentialia partialia sumenda sunt in formandis systematibus canonicis proposito et transformato, inter se differunt. Quoties vero functio $V$, quae in praecedentibus ex arbitrio assumi poterat ipsam $t$ non continet, erit

$$\frac{\partial V}{\partial t} = 0,$$

ideoque $f_1 = \varphi$, sive in utroque systemate canonico functio illa eadem erit. Eo casu etiam relationes, quibus novae variabiles e variabilibus primordialibus determinantur, ipsam $t$ non involvunt. Unde si variabiles sunt elementa problematis approximati, etiam novae variabiles non nisi aliud systema elementorum eiusdem problematis approximati erunt. Quodsi igitur formulas differentiales elementorum perturbatorum hac ratione iterum transformamus, *nanciscimur modum maxime generalem, quo systema elementorum canonicum in alterum systema elementorum canonicum transformetur.* Habemus enim e theoremate, permutando ipsas $q_i$, $p_i$, $a_i$, $b_i$, resp. cum $a_i$, $b_i$, $\alpha_i$, $\beta_i$ propositionem sequentem:

### Theorema IX$^a$.

*Sint formulae differentiales elementorum perturbatorum, designante $\Omega$ functionem perturbatricem:*

$$\frac{da_1}{dt} = \frac{\partial \Omega}{\partial b_1}, \quad \frac{da_2}{dt} = \frac{\partial \Omega}{\partial b_2}, \quad \dots \quad \frac{da_m}{dt} = \frac{\partial \Omega}{\partial b_m},$$

$$\frac{db_1}{dt} = -\frac{\partial \Omega}{\partial a_1}, \quad \frac{db_2}{dt} = -\frac{\partial \Omega}{\partial a_2}, \quad \dots \quad \frac{db_m}{dt} = -\frac{\partial \Omega}{\partial a_m};$$

*sint $\alpha_1$, $\alpha_2$, ... $\alpha_m$, $\beta_1$, $\beta_2$, ... $\beta_m$ functiones elementorum praecedentium $a_1$, $a_2$, ... $a_m$, $b_1$, $b_2$, ... $b_m$, quae, designante $\varphi$ functionem quamlibet ipsarum $a_1$, $a_2$, ... $a_m$, $\alpha_1$, $\alpha_2$, ... $\alpha_m$, ab illis pendeant per aequationes:*

$$\frac{\partial \varphi}{\partial a_1} = b_1, \quad \frac{\partial \varphi}{\partial a_2} = b_2, \quad \dots \quad \frac{\partial \varphi}{\partial a_m} = b_m,$$

$$\frac{\partial \varphi}{\partial \alpha_1} = -\beta_1, \quad \frac{\partial \varphi}{\partial \alpha_2} = -\beta_2, \quad \dots \quad \frac{\partial \varphi}{\partial \alpha_m} = -\beta_m,$$

*determinantur elementa nova per systema aequationum differentialium prorsus simile:*

$$\frac{d\alpha_1}{dt} = \frac{\partial \Omega}{\partial \beta_1}, \quad \frac{d\alpha_2}{dt} = \frac{\partial \Omega}{\partial \beta_2}, \quad \dots \quad \frac{d\alpha_m}{dt} = \frac{\partial \Omega}{\partial \beta_m},$$

$$\frac{d\beta_1}{dt} = -\frac{\partial \Omega}{\partial \alpha_1}, \quad \frac{d\beta_2}{dt} = -\frac{\partial \Omega}{\partial \alpha_2}, \quad \dots \quad \frac{d\beta_m}{dt} = -\frac{\partial \Omega}{\partial \alpha_m}.$$

Transformatio generalis elementorum canonicorum, quae ope functionis arbitrariae theoremate praecedenti efficitur, redit in methodam notam, qua e solutione *completa* aequationis differentialis partialis primi ordinis deducitur solutio functionem arbitrariam involvens, quae dicitur *generalis*. Sit enim $V$ solutio aequationis differentialis partialis, ad cuius integrationem e theoria hic exposita reducatur problema approximatum, atque involvat $V$ constantes arbitrarias $a_1$, $a_2$, ... $a_m$, ita ut sint

$$\frac{\partial V}{\partial a_1} = -b_1, \quad \frac{\partial V}{\partial a_2} = -b_2, \quad \dots \quad \frac{\partial V}{\partial a_m} = -b_m$$

aequationes finitae problematis approximati et quantitates $a_1$, $a_2$, ... $a_m$, $b_1$, $b_2$, ... $b_m$ eius elementa canonica. In locum solutionis $V$ etiam scribere licet $V+\varphi$, designante $\varphi$ constantem; de qua solutione completa deducitur generalis, si constans $\varphi$ statuitur functio arbitraria constantium $a_1$, $a_2$, ... $a_m$, atque differentialia partialia expressionis $V+\varphi$ respectu ipsarum $a_1$, $a_2$, ... $a_m$ sumta nihilo aequiparantur. Statuamus functionem ipsarum $a_1$, $a_2$, ... $a_m$ arbitrariam involvere praeter has quantitates alias constantes arbitrarias $\alpha_1$, $\alpha_2$, ... $\alpha_m$;

eliminatis e $V+\varphi$ ope aequationum

$$-\frac{\partial V}{\partial a_1}=\frac{\partial \varphi}{\partial a_1}, \quad -\frac{\partial V}{\partial a_2}=\frac{\partial \varphi}{\partial a_2}, \quad \ldots \quad -\frac{\partial V}{\partial a_m}=\frac{\partial \varphi}{\partial a_m}$$

quantitatibus $a_1$, $a_2$, ... $a_m$, habebitur nova solutio $V+\varphi$, alias $m$ constantes arbitrarias involvens. Etiam de hac deduci possunt aequationes finitae problematis approximati, quippe quae erunt:

$$-\frac{\partial(V+\varphi)}{\partial \alpha_1}=\beta_1, \quad -\frac{\partial(V+\varphi)}{\partial \alpha_2}=\beta_2, \quad \ldots \quad -\frac{\partial(V+\varphi)}{\partial \alpha_m}=\beta_m.$$

Sed quantitates $\alpha_i$ functionem $V+\varphi$ tantum afficiunt, quatenus in ipsis $a_i$ continentur ac praeter has explicite in functione $\varphi$; differentialia autem functionis $V+\varphi$ respectu ipsarum $a_i$ sumta supposuimus evanescere; unde erit

$$\frac{\partial(V+\varphi)}{\partial \alpha_i}=\frac{\partial \varphi}{\partial \alpha_i},$$

sive aequationes novae finitae problematis habentur:

$$\frac{\partial \varphi}{\partial \alpha_1}=-\beta_1, \quad \frac{\partial \varphi}{\partial \alpha_2}=-\beta_2, \quad \ldots \quad \frac{\partial \varphi}{\partial \alpha_m}=-\beta_m;$$

eruntque $\alpha_1$, $\alpha_2$, ... $\alpha_m$, $\beta_1$, $\beta_2$, ... $\beta_m$ nova elementa canonica determinata e primordialibus per has aequationes et illas supra assumtas:

$$\frac{\partial \varphi}{\partial a_i}=-\frac{\partial V}{\partial a_i}=b_i.$$

Elementa autem nova canonica inventa si in problemate perturbato tamquam variabiles spectantur, eorum differentialia expressionibus similibus aequalia evadere debent atque elementorum primordialium. Q. D. E.

*Transformatio ea, quae in §§. antecedentibus tradita est, etiam generalior proponitur.*

59.

Sed quanta generalitate gaudeat transformatio aequationis differentialis partialis primi ordinis et quae inde pendeat systematis canonici aequationum differentialium vulgarium supra §. 57 proposita, sunt tamen aliae, quas illa transformatio non amplectatur. Scilicet adhuc sequentes addendae sunt.

Sit rursus aequatio transformanda:

$$dV_1=-f_1dt+p_1dq_1+p_2dq_2+\cdots+p_mdq_m,$$

in qua $f_1$ est data functio ipsarum $t$, $q_1$, $q_2$, ... $q_m$, $p_1$, $p_2$, ... $p_m$; sit etiam $V$ rursus functio ex arbitrio assumta ipsarum $t$, $q_1$, $q_2$, ... $q_m$, $a_1$, $a_2$, ... $a_m$, inter quas vero quantitates iam statuamus insuper locum habere aequationes:

$$F=0, \quad \Phi=0, \quad \ldots .$$

Tum, posito rursus $V_1 - V = W$, erit aequatio transformata:

$$dW = -\varphi dt + b_1 da_1 + b_2 da_2 + \cdots + b_m da_m,$$

siquidem ponitur:

$$\begin{aligned}
\varphi &= f_1 + \frac{\partial V}{\partial t} - \lambda_1 \frac{\partial F}{\partial t} - \lambda_2 \frac{\partial \Phi}{\partial t} - \cdots \\
0 &= p_1 - \frac{\partial V}{\partial q_1} + \lambda_1 \frac{\partial F}{\partial q_1} + \lambda_2 \frac{\partial \Phi}{\partial q_1} + \cdots \\
0 &= p_2 - \frac{\partial V}{\partial q_2} + \lambda_1 \frac{\partial F}{\partial q_2} + \lambda_2 \frac{\partial \Phi}{\partial q_2} + \cdots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
0 &= p_m - \frac{\partial V}{\partial q_m} + \lambda_1 \frac{\partial F}{\partial q_m} + \lambda_2 \frac{\partial \Phi}{\partial q_m} + \cdots \\
b_1 &= \quad - \frac{\partial V}{\partial a_1} + \lambda_1 \frac{\partial F}{\partial a_1} + \lambda_2 \frac{\partial \Phi}{\partial a_1} + \cdots \\
b_2 &= \quad - \frac{\partial V}{\partial a_2} + \lambda_1 \frac{\partial F}{\partial a_2} + \lambda_2 \frac{\partial \Phi}{\partial a_2} + \cdots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
b_m &= \quad - \frac{\partial V}{\partial a_m} + \lambda_1 \frac{\partial F}{\partial a_m} + \lambda_2 \frac{\partial \Phi}{\partial a_m} + \cdots.
\end{aligned}$$

Ex his aequationibus, quibus ipsae addantur $F = 0$, $\Phi = 0$, ..., eliminatis $\lambda_1$, $\lambda_2$, ... determinandae sunt $q_1$, $q_2$, ... $q_m$, $p_1$, $p_2$, ... $p_m$ per $a_1$, $a_2$, ... $a_m$, $b_1$, $b_2$, ... $b_m$, earumque valores in expressione ipsius $\varphi$ substituendae.

Si multiplicatores $\lambda_1$, $\lambda_2$, ... evitare placet neque tamen symmetriae formularum derogare, transformationem etiam sic proponere licet. Aequationibus enim $F = 0$, $\Phi = 0$, ... inter quantitates $q_i$, $a_i$, $t$ assumtis, quarum numerus sit $m-k$, sint $r_1$, $r_2$, ... $r_k$ functiones quaelibet ipsarum $t$, $q_1$, $q_2$, ... $q_m$, $a_1$, $a_2$, ... $a_m$; licet omnes $q_1$, $q_2$, ... $q_m$ per ipsas $t$, $a_1$, $a_2$, ... $a_m$ atque novas quantitates $r_1$, $r_2$, ... $r_k$ exprimere, quae expressiones eliminatis $r_1$, $r_2$, ... $r_k$ suppeditabunt $m-k$ aequationes inter ipsas $q_1$, $q_2$, ... $q_m$, $a_1$, $a_2$, ... $a_m$, $t$.

Aequationes $F = 0$, $\Phi = 0$, ..., quum ex arbitrio assumi possint, iam earum in locum statuere licet, ipsas $q_1$, $q_2$, ... $q_m$ esse functiones arbitrarias quantitatum $t$, $a_1$, $a_2$, ... $a_m$, $r_1$, $r_2$, ... $r_k$. Quibus electis, sit

$$\begin{aligned}
p_1 \frac{\partial q_1}{\partial r_i} + p_2 \frac{\partial q_2}{\partial r_i} + \cdots + p_m \frac{\partial q_m}{\partial r_i} &= R_i, \\
p_1 \frac{\partial q_1}{\partial a_i} + p_2 \frac{\partial q_1}{\partial a_i} + \cdots + p_m \frac{\partial q_m}{\partial a_i} &= A_i, \\
p_1 \frac{\partial q_1}{\partial t} + p_2 \frac{\partial q_2}{\partial t} + \cdots + p_m \frac{\partial q_m}{\partial t} &= T.
\end{aligned}$$

Unde erit:

$$\begin{aligned} dV_1 &= \quad -f_1\,dt+p_1\,dq_1+p_2\,dq_2+\cdots+p_m\,dq_m \\ &= -(f-T)\,dt+R_1\,dr_1+R_2\,dr_2+\cdots+R_k\,dr_k \\ &\qquad +A_1\,da_1+A_2\,da_2+\cdots+A_m\,da_m. \end{aligned}$$

Assumta iam ex arbitrio ipsarum $a_1, a_2, \ldots a_m, r_1, r_2, \ldots r_k, t$ functione $V$, statuatur:

$$f_1-T+\frac{\partial V}{\partial t} = \varphi$$

$$\frac{\partial V}{\partial r_1} = R_1, \qquad \frac{\partial V}{\partial r_2} = R_2, \quad \ldots \quad \frac{\partial V}{\partial r_k} = R_k$$

$$A_1-\frac{\partial V}{\partial a_1} = b_1, \quad A_2-\frac{\partial V}{\partial a_2} = b_2, \quad \ldots \quad A_m-\frac{\partial V}{\partial a_m} = b_m;$$

erit

$$d(V_1-V) = dW = -\varphi\,dt+b_1\,da_1+b_2\,da_2+\cdots+b_m\,da_m.$$

Ex aequationibus $m+k$:

$$p_1\frac{\partial q_1}{\partial r_i}+p_2\frac{\partial q_2}{\partial r_i}+\cdots+p_m\frac{\partial q_m}{\partial r_i} = R_i = \frac{\partial V}{\partial r_i}$$

$$p_1\frac{\partial q_1}{\partial a_i}+p_2\frac{\partial q_2}{\partial a_i}+\cdots+p_m\frac{\partial q_m}{\partial a_i} = A_i = \frac{\partial V}{\partial a_i}+b_i$$

et per resolutionem aequationum linearium determinantur $p_1, p_2, \ldots p_m$ per $r_1, r_2, \ldots r_k, a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m, t$, et eliminatis $p_1, p_2, \ldots p_m$ habentur $k$ aequationes inter ipsas $r_1, r_2, \ldots r_k, a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m, t$, quarum ope ipsae $r_1, r_2, \ldots r_k$ ideoque etiam $p_1, p_2, \ldots p_m, q_1, q_2, \ldots q_m$ per $a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m, t$ determinari possunt. Qui deinde valores in expressione $\varphi$ substituendi sunt, ut illa solarum $a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m, t$ functio evadat. Transformata autem aequatione

$$dV_1 = -f_1\,dt+p_1\,dq_1+p_2\,dq_2+\cdots+p_m\,dq_m$$

in hanc

$$dW = -\varphi\,dt+b_1\,da_1+b_2\,da_2+\cdots+b_m\,da_m,$$

in quibus $f_1$ est ipsarum $t, q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m$, atque $\varphi$ ipsarum $t, a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m$ functio, simul aequationes

$$\frac{\partial V}{\partial t}+f_1 = 0, \quad \frac{\partial W}{\partial t}+\varphi = 0$$

altera in alteram transformatae sunt. Substitutis in $f_1$ loco ipsarum $p_i$ expressionibus $\frac{\partial V_1}{\partial q_i}$, in $\varphi$ loco ipsarum $b_i$ expressionibus $\frac{\partial V}{\partial a_i}$, evadunt aequationes illae aequationes differentiales partiales, quarum igitur iam per methodum propositam transformationes novas nacti sumus.

Si $k=m$, transformatio eadem obtinetur atque §. 57. Transformationes, quas ex antecedentibus pro $k<m$ obtinemus, etiam novas transformationes systematum canonicorum aequationum differentialium vulgarium suppeditant. Et similiter atque in §. antec. etiam systematum elementorum canonicorum novas inde eruimus transformationes. Quas transformationes, singulas binis modis, sequentibus theorematibus proponam.

### Theorema X.

*Designante i numeros 1, 2, ... m propositum sit systema aequationum differentialium vulgarium:*

$$\frac{dq_i}{dt}=\frac{\partial f_1}{\partial p_i},\quad \frac{dp_i}{dt}=-\frac{\partial f_1}{\partial q_i};$$

*assumatur functio arbitraria $V$ ipsarum $t$, $q_1$, $q_2$, ... $q_m$ novarumque variabilium $a_1$, $a_2$, ... $a_m$, inter quas quantitates $2m+1$ statuatur locum habere aequationes quascunque*

$$F=0,\quad \Phi=0,\quad \dots;$$

*condantur porro aequationes $2m+1$ sequentes:*

$$\begin{aligned}
\varphi &= f_1+\frac{\partial V}{\partial t}-\lambda_1\frac{\partial F}{\partial t}-\lambda_2\frac{\partial \Phi}{\partial t}-\cdots,\\
p_i &= \frac{\partial V}{\partial q_i}-\lambda_1\frac{\partial F}{\partial q_i}-\lambda_2\frac{\partial \Phi}{\partial q_i}-\cdots,\\
-b_i &= \frac{\partial V}{\partial a_i}-\lambda_1\frac{\partial F}{\partial a_i}-\lambda_2\frac{\partial \Phi}{\partial a_i}-\cdots,
\end{aligned}$$

*quarum aequationum ope, advocatis ipsis $F=0$, $\Phi=0$, ..., et eliminari possunt multiplicatores $\lambda_1$, $\lambda_2$, ..., et determinantur $2m$ quantitates $q_i$ et $p_i$ nec non functio $\varphi$ per ipsam $t$ atque $2m$ novas quantitates $a_i$, $b_i$; qui ipsorum $q_i$, $p_i$ valores si substituuntur in systemate proposito aequationum differentialium vulgarium, transformatur illud in sequens:*

$$\frac{da_i}{dt}=\frac{\partial \varphi}{\partial b_i},\quad \frac{db_i}{dt}=-\frac{\partial \varphi}{\partial a_i}.$$

*Porro, in locum ipsorum $p_i$, $b_i$ scribendo $\frac{\partial V_1}{\partial q_i}$, $\frac{\partial W}{\partial a_i}$, ubi conduntur aequationes differentiales partiales*

$$\frac{\partial V_1}{\partial t}+f_1=0, \qquad \frac{\partial W}{\partial t}+\varphi=0,$$

*alterius solutio ex alterius obtinetur solutione per aequationem:*

$$V_1 = V+W.$$

Theorema XI.

*Designante i numeros* 1, 2, ... *m, data sint differentialia elementorum problematis alicuius perturbati per aequationes:*

$$\frac{da_i}{dt}=\frac{\partial \Omega}{\partial b_i}, \qquad \frac{db_i}{dt}=-\frac{\partial \Omega}{\partial a_i},$$

*designante $\Omega$ functionem perturbatricem; assumatur functio arbitraria $V$ elementorum $a_1$, $a_2$, ... $a_m$ novarumque quantitatum $\alpha_1$, $\alpha_2$, ... $\alpha_m$; condantur porro aequationes 2m sequentes:*

$$b_i=\frac{\partial V}{\partial a_i}-\lambda_1\frac{\partial F}{\partial a_i}-\lambda_2\frac{\partial \Phi}{\partial a_i}-\cdots,$$

$$-\beta_i=\frac{\partial V}{\partial \alpha_i}-\lambda_1\frac{\partial F}{\partial \alpha_i}-\lambda_2\frac{\partial \Phi}{\partial \alpha_i}-\cdots,$$

*quarum aequationum ope, advocatis ipsis $F=0$, $\Phi=0$, ..., et eliminari possunt multiplicatores $\lambda_1$, $\lambda_2$, ... et determinantur 2m elementa $a_i$, $b_i$ per novum systema elementorum $\alpha_i$, $\beta_i$; quae elementa nova si etiam in functionem perturbatricem $\Omega$ loco ipsorum $a_i$, $b_i$ introducuntur, inveniuntur differentialia elementorum novi systematis $\alpha_i$, $\beta_i$ per formulas:*

$$\frac{d\alpha_i}{dt}=\frac{\partial \Omega}{\partial \beta_i}, \qquad \frac{d\beta_i}{dt}=-\frac{\partial \Omega}{\partial \alpha_i}.$$

Obtinetur theorema praecedens e theoremate X. statuendo, functiones $V$, $F$, $\Phi$, ... ipsam $t$ non involvere, et permutando $p_i$, $q_i$ cum $b_i$, $a_i$, atque $b_i$, $a_i$ cum $\beta_i$, $\alpha_i$. Theoremata praecedentia etiam hanc alteram formam induere possunt:

Theorema X$^a$.

*Designante i numeros* 1, 2, ... *m, propositum sit systema aequationum differentialium vulgarium:*

$$\frac{dq_i}{dt}=\frac{\partial f_1}{\partial p_i}, \qquad \frac{dp_i}{dt}=-\frac{\partial f_1}{\partial q_i};$$

*statuantur quantitates $q_1$, $q_2$, ... $q_m$ aequales expressionibus quibuscunque ipsius $t$ novarumque quantitatum $a_1$, $a_2$, ... $a_m$, $r_1$, $r_2$, ... $r_k$, designante k numerum aut ipsi m aequalem aut ipso m minorem; assumatur deinde* *functio arbitraria* $V$ *earundem quantitatum* $a_1, a_2, \dots a_m, r_1, r_2, \dots r_k,$ $t$, *atque condantur aequationes* $m+k$:

$$\frac{\partial V}{\partial r_i} = p_1\frac{\partial q_1}{\partial r_i}+p_2\frac{\partial q_2}{\partial r_i}+\dots+p_m\frac{\partial q_m}{\partial r_i},$$

$$b_i+\frac{\partial V}{\partial a_i} = p_1\frac{\partial q_1}{\partial a_i}+p_2\frac{\partial q_2}{\partial a_i}+\dots+p_m\frac{\partial q_m}{\partial a_i};$$

*per quas determinentur quantitates* $r_1, r_2, \dots r_k, p_1, p_2, \dots p_m$ *per quantitates* $a_1, a_2, \dots a_m, b_1, b_2, \dots b_m, t$, *unde etiam quantitates* $q_1, q_2, \dots q_m$ *per easdem quantitates determinatae erunt; quibus substitutis quantitatum* $r_i, q_i, p_i$ *expressionibus per easdem quantitates* $a_1, a_2, \dots a_m,$ $b_1, b_2, \dots b_m, t$ *etiam exhibeatur functio*

$$\varphi = f_1+\frac{\partial V}{\partial t}-\left\{p_1\frac{\partial q_1}{\partial t}+p_2\frac{\partial q_2}{\partial t}+\dots+p_m\frac{\partial q_m}{\partial t}\right\};$$

*quibus factis si expressiones ipsarum* $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ *per* $a_1, a_2, \dots a_m, b_1, b_2, \dots b_m, t$ *inventae substituuntur in systemate aequationum differentialium propositarum, obtinebitur hoc similis formae:*

$$\frac{da_i}{dt}=\frac{\partial\varphi}{\partial b_i},\qquad \frac{db_i}{dt}=-\frac{\partial\varphi}{\partial a_i};$$

*simul aequationes differentiales partiales*

$$\frac{\partial V_1}{\partial t}+f_1=0,\qquad \frac{\partial W}{\partial t}+\varphi=0,$$

*quae obtinentur ponendo in altera* $\frac{\partial V_1}{\partial q_i}$ *loco* $p_i$, *in altera* $\frac{\partial W}{\partial a_i}$ *loco* $b_i$, *per easdem aequationes altera in alteram transformantur, et alterius solutio ex alterius habetur per aequationem*

$$V_1 = W+V.$$

### Theorema XI^a.

*Designante i numeros* 1, 2, ... *m, data sint differentialia elementorum problematis alicuius perturbati per aequationes*

$$\frac{da_i}{dt}=\frac{\partial\Omega}{\partial b_i},\qquad \frac{db_i}{dt}=-\frac{\partial\Omega}{\partial a_i},$$

*designante* $\Omega$ *functionem perturbatricem; statuantur elementa* $a_1, a_2, \dots a_m$ *aequalia expressionibus quibuscunque novarum quantitatum* $m+k$,

$$\alpha_1,\ \alpha_2,\ \dots\ \alpha_k,\ \varepsilon_1,\ \varepsilon_2,\ \dots\ \varepsilon_k,$$

*designante k numerum aut ipsi m aequalem aut ipso m minorem; assumta* *deinde functione arbitraria $V$ earundem $m+k$ quantitatum, formentur aequationes $m+k$*

$$\frac{\partial V}{\partial \varepsilon_i} = b_1\frac{\partial a_1}{\partial \varepsilon_i}+b_2\frac{\partial a_2}{\partial \varepsilon_i}+\cdots+b_m\frac{\partial a_m}{\partial \varepsilon_i},$$

$$\beta_i+\frac{\partial V}{\partial \alpha_i} = b_1\frac{\partial a_1}{\partial \alpha_i}+b_2\frac{\partial a_2}{\partial \alpha_i}+\cdots+b_m\frac{\partial a_m}{\partial \alpha_i};$$

*per quas determinentur $m+k$ quantitates $\varepsilon_1, \varepsilon_2, \dots \varepsilon_k, b_1, b_2, \dots b_m$ per quantitates novas $2m$:*

$$\alpha_1, \quad \alpha_2, \quad \dots \quad \alpha_m, \quad \beta_1, \quad \beta_2, \quad \dots \quad \beta_m,$$

*unde etiam $a_1, a_2, \dots a_m$ per easdem quantitates determinatae erunt; quae expressiones elementorum $a_1, a_2, \dots a_m, b_1, b_2, \dots b_m$ per nova elementa $\alpha_1, \alpha_2, \dots \alpha_m, \beta_1, \beta_2, \dots \beta_m$ si substituuntur in functione perturbatrice $\Omega$, habentur differentialia novi elementorum systematis per formulas similes:*

$$\frac{d\alpha_i}{dt} = \frac{\partial \Omega}{\partial \beta_i}, \qquad \frac{d\beta_i}{dt} = -\frac{\partial \Omega}{\partial \alpha_i}.$$

Haec altera theorematum forma commodior est, quoties in priore forma numerus aequationum $F=0$, $\Phi=0$, ... valde magnus habetur.

*De casu quodam simplicissimo, quo e systemate elementorum canonico aliud ejusmodi systema eruatur.*

60.

Paucis agam de transformationibus simplicissimis, quarum tamen frequentissimus usus erit, alius systematis canonici elementorum in aliud canonicum. Sunt elementa canonica, quorum differentialia huiusmodi aequationibus exprimuntur:

$$\frac{da_i}{dt} = \frac{\partial \Omega}{\partial b_i}, \qquad \frac{db_i}{dt} = -\frac{\partial \Omega}{\partial a_i},$$

quarum aequationum integratio redit in integrationem aequationis differentialis partialis:

$$0 = \frac{\partial V}{\partial t}+\Omega,$$

siquidem in functione perturbatrice $\Omega$ loco ipsorum $b_1, b_2, \dots b_m$ ponitur $\frac{\partial V}{\partial a_1}, \frac{\partial V}{\partial a_2}, \dots \frac{\partial V}{\partial a_m}$, sive quod idem est, redit integratio aequationum differentialium vulgarium praecedentium in integrationem aequationis:

$$dV = -\Omega\, dt+b_1 da_1+b_2 da_2+\cdots+b_m da_m.$$

Vocemus duas classes systematis canonici elementorum, alteram, ipsas $a_1, a_2, \dots a_m$

amplectentem, *positivam classem*, alteram, ipsas $b_1, b_2, \dots b_m$ amplectentem *negativam classem*. Elementa bina $a_i$ et $b_i$ vocemus coniugata. Bina elementa coniugata systematis simul ex altera classi in alteram transeunt, si *alterius elementi signum mutatur*. Si functiones quascunque elementorum, quae ad alteram tantum classem pertinent, ut nova elementa illius classis introducere placet, facillime alterius classis elementa, quae novis illis coniugata sunt, determinantur. *Sint ex. gr. classis positivae elementa $a_1, a_2, \dots a_m$ expressa per alias quantitates $\alpha_1, \alpha_2, \dots \alpha_m$, quae ut nova elementa classis positivae spectentur; posito*

$$\beta_i = b_1\frac{\partial a_1}{\partial \alpha_i}+b_2\frac{\partial a_2}{\partial \alpha_i}+\dots+b_m\frac{\partial a_m}{\partial \alpha_i},$$

*erit*

$$dV = -\Omega dt+\beta_1 d\alpha_1+\beta_2 d\alpha_2+\dots+\beta_m d\alpha_m.$$

*Unde considerari debent quantitates $\beta_i$ ut nova elementa alterius classis, eritque elementum $\beta_i$ ipsi $\alpha_i$ conjugatum, sive erit:*

$$\frac{\partial \Omega}{\partial \beta_i}=\frac{d\alpha_i}{dt},\quad \frac{\partial \Omega}{\partial \alpha_i}=-\frac{d\beta_i}{dt}.$$

In propositione praecedente loco $a_i$, $\alpha_i$ ponere licet $-b_i$, $-\beta_i$ simulque loco $b_i$, $\beta_i$ ponere $a_i$, $\alpha_i$. Unde fluit altera propositio, *expressis $b_i$ per alia elementa $\beta_i$, elementa classis positivae, ipsis $\beta_i$ coniugata, fieri*

$$\alpha_i = a_1\frac{\partial b_1}{\partial \beta_i}+a_2\frac{\partial b_2}{\partial \beta_i}+\dots+a_m\frac{\partial b_m}{\partial \beta_i}.$$

Si hunc transformationis modum una cum illo qui sola mutatione signi unius seu plurium elementorum efficitur, vicissim iterum iterumque adhibes, iam hac via simplicissima ex uno systemate elementorum canonico diversissima alia deducere licet.

Ut per methodum generalem supra traditam eruatur transformatio illa simplicissima, qua bina elementa coniugata, ex. gr. $a_1$ et $b_1$, alterum in alterius classem transeunt, statuatur

$$V_0 = a_1\alpha_1,\quad b_1=\frac{\partial V_0}{\partial a_1}=\alpha_1;$$

erit

$$d(V-V_0) = -\Omega dt - a_1 d\alpha_1 + b_2 da_2+\dots+b_m da_m,$$

quae docet aequatio, si loco ipsius $a_1$ introducatur $\alpha_1 = b_1$, elementum coniugatum quod erat $b_1$ fieri $-a_1$, q. d. e.

*Transformationes antecedentes, variabili t et ipsa mutata, ad summam generalitatem perducuntur.*

## 61.

In §§. 57., 59. transformationum aequationum differentialium partialium generalitatem eo restrinximus, quod unam variabilium independentium immutatam reliquimus. Quam restrictionem si missam facimus, ita agendum est.

Sit rursus

$$dV_1 = -f_1 dt + p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m$$

atque $V$ functio arbitraria ipsorum $q_1, q_2, \ldots q_m, t$ atque novarum $a_1, a_2, \ldots a_m, \tau$: fit aequatio transformata

$$d(V_1 - V) = -\frac{\partial V}{\partial \tau} d\tau + b_1 da_1 + b_2 da_2 + \cdots + b_m da_m,$$

siquidem statuitur:

$$\frac{\partial V}{\partial t} = -f_1, \quad \frac{\partial V}{\partial q_1} = p_1, \quad \frac{\partial V}{\partial q_2} = p_2, \quad \ldots \quad \frac{\partial V}{\partial q_m} = p_m,$$

$$\frac{\partial V}{\partial a_1} = -b_1, \quad \frac{\partial V}{\partial a_2} = -b_2, \quad \ldots \quad \frac{\partial V}{\partial a_m} = -b_m.$$

Ex his aequationibus determinandae sunt $t, q_1, q_2, \ldots q_m$ per $a_1, a_2, \ldots a_m$, $b_1, b_2, \ldots b_m, \tau$ atque valores eruti substituendi in expressione $\frac{\partial V}{\partial \tau}$, quo facto erit

$$d(V_1 - V) = -\frac{\partial V}{\partial \tau} d\tau + b_1 da_1 + b_2 da_2 + \cdots + b_m da_m$$

aequatio transformata. Si in modum §. 59. introducuntur aequationes $F = 0$, $\Phi = 0, \ldots$ inter ipsas $q_1, q_2, \ldots q_m, t, a_1, a_2, \ldots a_m, \tau$, nil in praecedentibus mutabitur, nisi quod loco ipsius $V$ ponendum sit $V - \lambda_1 F - \lambda_2 \Phi - \cdots$, cuius expressionis differentialia partialia multiplicatorum $\lambda_1, \lambda_2, \ldots$ non continent differentialia, ut quae in expressiones $F, \Phi, \ldots$ evanescentes ducta sunt.

Ut e propositione praecedente generaliori deducatur casus, quo variabilis independens $t$ immutata manet, scribatur in locum ipsius $V$ expressio

$$V + (\tau - t) f_1,$$

designante $V$ functionem ab ipsa $t$ vacuam. Quo facto aequatio

$$\frac{\partial V}{\partial t} = -f_1$$

abit in

$$(\tau - t)\frac{\partial f_1}{\partial t} = 0,$$

unde deducitur

$$\tau = t.$$

Reiectis igitur post differentiationes factas terminis in $\tau - t$ ductis ut evanescentibus, mutato $V$ in $V + (\tau - t) f_1$, abit $\frac{\partial V}{\partial \tau}$ in $\frac{\partial V}{\partial \tau} + f_1$, differentialia $\frac{\partial V}{\partial q_i}$, $\frac{\partial V}{\partial a_i}$ non mutantur. Unde si postremo $t$ loco $\tau$ scribitur, facile patet, quomodo e praecedente propositione generaliori deducantur eae, quae §§. antecedentibus traditae sunt.

Si aequatio differentialis partialis proposita ipsam functionem quaesitam $V_1$ continet, ita agendum est. Sit

$$dV = -f dt + p_1 dq_1 + p_2 dq_2 + \cdots + p_m dq_m.$$

atque contineat $f$ praeter variabiles $t$, $q_1$, $q_2$, ... $q_m$, $p_1$, $p_2$, ... $p_m$ adhuc ipsam $V$; assumatur ipsarum $V$, $t$, $q_1$, $q_2$, ... $q_m$, $a_1$, $a_2$, ... $a_m$, $\tau$ functio quaelibet $W$; erit

$$dW = \frac{\partial W}{\partial \tau} d\tau + b_1 da_1 + b_2 da_2 + \cdots + b_m da_m,$$

siquidem statuitur:

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial V} f, \quad \frac{\partial W}{\partial q_1} = -\frac{\partial W}{\partial V} p_1, \quad \frac{\partial W}{\partial q_2} = -\frac{\partial W}{\partial V} p_2, \quad \ldots \quad \frac{\partial W}{\partial q_m} = -\frac{\partial W}{\partial V} p_m,$$

$$\frac{\partial W}{\partial a_1} = b_1, \qquad \frac{\partial W}{\partial a_2} = b_2, \qquad \ldots \quad \frac{\partial W}{\partial a_m} = b_m.$$

Ex his $2m+1$ aequationibus, advocata ipsa functionis $W$ expressione ex arbitrio assumta, determinandae sunt $V$, $t$, $q_1$, $q_2$, ... $q_m$, $p_1$, $p_2$, ... $p_m$ per $W$, $\tau$, $a_1$, $a_2$, ... $a_m$, $b_1$, $b_2$, ... $b_m$, ac valores inventi in expressione ipsius $\frac{\partial W}{\partial \tau}$ substituendi, quo facto aequatio differentialis praecedens erit aequatio transformata. Si inter ipsas $V$, $t$, $q_1$, $q_2$, ... $q_m$, $\tau$, $a_1$, $a_2$, ... $a_m$ aequationes $F = 0$, $\Phi = 0$, ... locum habere statuis, tantum necesse est, ut in praecedentibus loco ipsius $W$ ponatur $W + \lambda_1 F + \lambda_2 \Phi + \cdots$.

Haec sunt maxime generalia, quae de transformatione aequationum differentialium partialium primi ordinis inter numerum quemcunque variabilium praecipere licet.

*Agitur de usu integralis cuiuslibet systematis aequationum differentialium vulgarium, quod non eam in seriem integralium redeat, quam methodus supra proposita sibi poscat.*

62.

Adnotavi supra §. 56., si principium conservationis arearum respectu certi cuiusdam plani locum habeat, unam variabilem cum eius differentiali prorsus ex aequationibus differentialibus abire, ideoque ordinem differentiationum unitatibus duabus diminui; idem vero, quod pro uno integrali illo fit, non evenire pro secundo et tertio integrali, quod datur, si principium conservationis arearum respectu *cuiuslibet* plani locum habet. Quaeramus iam, quemnam alium usum e tribus illis integralibus percipere liceat in ea integrationis via, quam supra proposuimus. Eum in finem antemitto has considerationes generales. Antea autem processum generalem integrationum, qualis e supra traditis habetur, paucis repetam.

Proposita integratione aequationum differentialium

$$dq_1 : dq_2 : \cdots : dq_m : \quad dp_1 : \quad dp_2 : \cdots : \quad dp_m =$$
$$\frac{\partial f}{\partial p_1} : \frac{\partial f}{\partial p_2} : \cdots : \frac{\partial f}{\partial p_m} : -\frac{\partial f}{\partial q_1} : -\frac{\partial f}{\partial q_2} : \cdots : -\frac{\partial f}{\partial q_m},$$

in quibus $f$ est data functio ipsarum $q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m$, e praeceptis supra traditis ita agendum erat.

Integrale *primum* sponte habetur per aequationem

$$f = a,$$

in qua $a$ est constans arbitraria. Integrale *secundum* habetur $H_1 = a_1$, si datur functio $H_1$, quae identice satisfacit aequationi

$$(1.) \qquad 0 = [H_1, f],$$

siquidem semper statuitur

$$[\varphi, \psi] = \frac{\partial\varphi}{\partial q_1}\frac{\partial\psi}{\partial p_1} + \frac{\partial\varphi}{\partial q_2}\frac{\partial\psi}{\partial p_2} + \cdots + \frac{\partial\varphi}{\partial q_m}\frac{\partial\psi}{\partial p_m}$$
$$-\frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial q_1} - \frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_2} - \cdots - \frac{\partial\varphi}{\partial p_m}\frac{\partial\psi}{\partial q_m}.$$

Iam e praeceptis traditis non integrale tertium quodcunque investigandum est seu functio $H_2$ quaecunque satisfaciat aequationi

$$[H_2, f] = 0,$$

sed eiusmodi functio $H_2$, quae duabus simul satisfaciat aequationibus,

$$(2.) \qquad [H_2, f] = 0, \quad [H_2, H_1] = 0.$$

Deinde investiganda erit functio $H_3$, quae tribus aequationibus

$$[H_3, f] = 0, \quad [H_3, H_1] = 0, \quad [H_3, H_2] = 0,$$

functio $H_4$, quae quatuor aequationibus

$$[H_4, f] = 0, \quad [H_4, H_1] = 0, \quad [H_4, H_2] = 0, \quad [H_4, H_3] = 0,$$

etc. etc., denique functio $H_{m-1}$, quae $m-1$ aequationibus

$$[H_{m-1}, f] = 0, \quad [H_{m-1}, H_1] = 0, \quad \ldots \quad [H_{m-1}, H_{m-2}] = 0$$

satisfaciat. His inventis functionibus, integratio completa aequationum differentialium propositarum ad meras quadraturas revocata erat. Scilicet sunt $m$ integralia aequationum differentialium propositarum ipsae aequationes,

$$f = a, \quad H_1 = a_1, \quad H_2 = a_2, \quad \ldots \quad H_{m-1} = a_{m-1},$$

e quibus deinde si determinantur $p_1$, $p_2$, $\ldots$ $p_m$ per $q_1$, $q_2$, $\ldots$ $q_m$, habentur $m-1$ reliqua integralia per formulas:

$$\int \left\{ \frac{\partial p_1}{\partial a_1} dq_1 + \frac{\partial p_2}{\partial a_1} dq_2 + \cdots + \frac{\partial p_m}{\partial a_1} dq_m \right\} + b_1 \quad = 0,$$

$$\int \left\{ \frac{\partial p_1}{\partial a_2} dq_1 + \frac{\partial p_2}{\partial a_2} dq_2 + \cdots + \frac{\partial p_m}{\partial a_2} dq_m \right\} + b_2 \quad = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int \left\{ \frac{\partial p_1}{\partial a_{m-1}} dq_1 + \frac{\partial p_2}{\partial a_{m-1}} dq_2 + \cdots + \frac{\partial p_m}{\partial a_{m-1}} dq_m \right\} + b_{m-1} = 0,$$

in quibus expressiones sub signo sunt differentialia completa, quorum igitur integratio nonnisi quadraturas poscit. Quantitates $a$, $a_1$, $\ldots$ $a_{m-1}$, $b_1$, $b_2$, $\ldots$ $b_{m-1}$ sunt constantes arbitrariae.

Aequationes, quibus functio $H_i$ definitur, ex iis quae supra §. 32 demonstravi, variis modis transformare licet. Scilicet in aequationibus

$$[H_i, f] = 0, \quad [H_i, H_1] = 0, \quad \ldots \quad [H_i, H_{i-1}] = 0$$

loco functionum $f$, $H_1$, $H_2$, $\ldots$ $H_{i-1}$ alias quascunque ponere licet, in quas illae ope aequationum $f = a$, $H_1 = a_1$, $H_2 = a_2$, $\ldots$ $H_{i-1} = a_{i-1}$ transformari possunt; nec non supponere licet, ope harum aequationum e functione quaesita $H_i$ eliminatas esse quantitates $p_1$, $p_2$, $\ldots$ $p_i$. Statuantur ope earundem aequationum eliminatas esse $p_1$, $p_2$, $\ldots$ $p_i$ ex ipsa $f$ omnes praeter $p_1$, ex ipsa $H_1$ omnes praeter $p_2$, $\ldots$ ex ipsa $H_{i-1}$ omnes praeter $p_i$, singulis aequationibus $f = a$, $H_1 = a_1$, $\ldots$ $H_{i-1} = a_{i-1}$ in singulis eliminationibus efficiendis respective reiectis. Tum aequationes, quibus $H_i$ satisfacere debet, hae fiunt:

$$0 = [H_i, f] = \frac{\partial H_i}{\partial q_1}\frac{\partial f}{\partial p_1} + \frac{\partial H_i}{\partial q_{i+1}}\frac{\partial f}{\partial p_{i+1}} + \cdots + \frac{\partial H_i}{\partial q_m}\frac{\partial f}{\partial p_m}$$
$$- \frac{\partial H_i}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}} - \cdots - \frac{\partial H_i}{\partial p_m}\frac{\partial f}{\partial q_m},$$

$$0 = [H_i, H_1] = \frac{\partial H_i}{\partial q_2}\frac{\partial H_1}{\partial p_2} + \frac{\partial H_i}{\partial q_{i+1}}\frac{\partial H_1}{\partial p_{i+1}} + \cdots + \frac{\partial H_i}{\partial q_m}\frac{\partial H_1}{\partial p_m}$$
$$- \frac{\partial H_i}{\partial p_{i+1}}\frac{\partial H_1}{\partial q_{i+1}} - \cdots - \frac{\partial H_i}{\partial p_m}\frac{\partial H_1}{\partial q_m},$$

. . . . . . . . . . . . . . . . . . . . . .

$$0 = [H_i, H_{i-1}] = \frac{\partial H_i}{\partial q_i}\frac{\partial H_{i-1}}{\partial p_i} + \frac{\partial H_i}{\partial q_{i+1}}\frac{\partial H_{i-1}}{\partial p_{i+1}} + \cdots + \frac{\partial H_i}{\partial q_m}\frac{\partial H_{i-1}}{\partial p_m}$$
$$- \frac{\partial H_i}{\partial p_{i+1}}\frac{\partial H_{i-1}}{\partial q_{i+1}} - \cdots - \frac{\partial H_i}{\partial p_m}\frac{\partial H_{i-1}}{\partial q_m}.$$

Sit e $f = a$ ipsa $p_1$, e $H_1 = a_1$ ipsa $p_2$, ... e $H_{i-1} = a_{i-1}$ ipsa $p_i$ expressa per $p_{i+1}, p_{i+2}, \ldots p_m, q_1, q_2, \ldots q_m$; possunt aequationes praecedentes sic quoque exhiberi:

$$\frac{\partial H_i}{\partial q_1} = \frac{\partial p_1}{\partial p_{i+1}}\frac{\partial H_i}{\partial q_{i+1}} + \cdots + \frac{\partial p_1}{\partial p_m}\frac{\partial H_i}{\partial q_m}$$
$$- \frac{\partial p_1}{\partial q_{i+1}}\frac{\partial H_i}{\partial p_{i+1}} - \cdots - \frac{\partial p_1}{\partial q_m}\frac{\partial H_i}{\partial p_m},$$

$$\frac{\partial H_i}{\partial q_2} = \frac{\partial p_2}{\partial p_{i+1}}\frac{\partial H_i}{\partial q_{i+1}} + \cdots + \frac{\partial p_2}{\partial p_m}\frac{\partial H_i}{\partial q_m}$$
$$- \frac{\partial p_2}{\partial q_{i+1}}\frac{\partial H_i}{\partial p_{i+1}} - \cdots - \frac{\partial p_2}{\partial q_m}\frac{\partial H_i}{\partial p_m},$$

. . . . . . . . . . . . . . . .

$$\frac{\partial H_i}{\partial q_i} = \frac{\partial p_i}{\partial p_{i+1}}\frac{\partial H_i}{\partial q_{i+1}} + \cdots + \frac{\partial p_i}{\partial p_m}\frac{\partial H_i}{\partial q_m}$$
$$- \frac{\partial p_i}{\partial q_{i+1}}\frac{\partial H_i}{\partial p_{i+1}} - \cdots - \frac{\partial p_i}{\partial q_m}\frac{\partial H_i}{\partial p_m}.$$

Hae sunt aequationes (*d.*) §. 17, quarum docui supra integrationem simultaneam §§. 19, 20, primum quaerens functionem, quae aequationi primae, deinde quae duabus primis, deinde quae tribus primis etc. satisfaciat. Sed interdum fieri potest, ut alia via determinandi functionem $H_i$ commodius ineatur, cuius rei exemplum simplicissimum prodam.

63.

Integratio simultanea binarum aequationum in theoria praecedente primum obvenit in investigatione functionis $H_2$, quippe quae duabus aequationibus

$$[H_2, f] = 0, \qquad [H_2, H_1] = 0$$

simul satisfacere debet. E §. antec. hae duae aequationes ope integralium iam inventorum $f = a$, $H_1 = a_1$ in duas alias transformandae sunt. Sed statuamus, aequationum differentialium propositarum praeter duo integralia illa $f = a$, $H_1 = a_1$ haberi tertium:

$$\varphi = \text{Const.},$$

ita ut identice sit

$$[\varphi, f] = 0,$$

transformatio illa non adhibenda, sed haec investigandae ipsius $H_2$ ineunda via est.

Si habetur $[\varphi, H_1] = 0$, statui potest $H_2 = \varphi$ neque igitur ulteriore disquisitione opus est. Casum quo $[\varphi, H_1]$ in valorem numericum, ex. gr. $\pm 1$ redit, sive generalius in functionem ipsarum $f$, $H$, in sequentibus excludemus, quippe quo methodum supra traditam retinere convenit neque tertii integralis inventi usus erit. Quibus suppositis poterit sequente methodo e functione $\varphi$ alia $H_2$ derivari, pro qua, sicuti pro ipsa $\varphi$, sit $[H_2, f] = 0$, simul vero etiam $[H_2, H_1] = 0$. Ponamus:

$$[\varphi, H_1] = A_1, \qquad [A_1, H_1] = A_2, \qquad [A_2, H_1] = A_3, \qquad \ldots,$$

quae eo usque continuanda est novarum functionum formatio, donec perveniatur ad functionem $[A_{k-1}, H_1] = A_k$, quae antecedentium $f$, $H_1$, $\varphi$, $A_1$, $A_2$, ... $A_{k-1}$ functio est

$$A_k = \Psi(f, H_1, \varphi, A_1, A_2, \ldots A_{k-1}),$$

quae functio praeterea e variabilibus $q_i$, $p_i$ nullam involvat. Sit $F$ functio alia quaecunque earundem quantitatum $f$, $H_1$, $\varphi$, $A_1$, $A_2$, ... $A_{k-1}$, dico *primum*, haberi identice

$$[F, f] = 0.$$

Etenim ex aequatione identica generali (Theor. V, §. 26):

$$[[\psi, f], H_1] + [[f, H_1], \psi] + [[H_1, \psi], f] = 0$$

sequitur casu nostro, quo $[f, H_1] = 0$, aequatio:

$$[[\psi, f], H_1] + [[H_1, \psi], f] = 0.$$

In qua si loco $\psi$ successive ponitur $\varphi$, $A_1$, $A_2$, $A_3$, ..., sequuntur, quum etiam sit ex hypothesi $[\varphi, f] = 0$, alia ex alia aequationes

$$[A_1, f] = 0, \qquad [A_2, f] = 0, \qquad \ldots \qquad [A_{k-1}, f] = 0.$$

Porro fit

$$[F,f] = \frac{\partial F}{\partial f}[f,f]+\frac{\partial F}{\partial H_1}[H_1,f]+\frac{\partial F}{\partial \varphi}[\varphi,f]+\frac{\partial F}{\partial A_1}[A_1,f]+\cdots+\frac{\partial F}{\partial A_{k-1}}[A_{k-1},f].$$

Unde, quum expressiones in singula differentialia partialia ipsius $F$ multiplicatae identice evanescant, fit $[F,f]=0$, q. d. e.

Habetur secundo loco:

$$[F, H_1] = \frac{\partial F}{\partial f}[f, H_1]+\frac{\partial F}{\partial H_1}[H_1, H_1]+\frac{\partial F}{\partial \varphi}[\varphi, H_1]$$
$$+\frac{\partial F}{\partial A_1}[A_1, H_1]+\cdots+\frac{\partial F}{\partial A_{k-1}}[A_{k-1}, H_1],$$

sive quum sit:

$$[f, H_1] = [H_1, H_1] = 0,$$
$$[\varphi, H_1] = A_1, \quad [A_1, H_1] = A_2, \quad \ldots \quad [A_{k-1}, H_1] = A_k = \Psi,$$

habetur

$$[F, H_1] = \frac{\partial F}{\partial \varphi}A_1+\frac{\partial F}{\partial A_1}A_2+\cdots+\frac{\partial F}{\partial A_{k-2}}A_{k-1}+\frac{\partial F}{\partial A_{k-1}}\Psi.$$

Unde eruitur functio $F$, quae duabus aequationibus

$$[F,f]=0, \quad [F,H_1]=0$$

simul satisfacit, si ea indagatur ipsarum $\varphi$, $A_1$, $A_2$, ... $A_{k-1}$ functio $F$, quae identice efficiat

$$\frac{\partial F}{\partial \varphi}A_1+\frac{\partial F}{\partial A_1}A_2+\cdots+\frac{\partial F}{\partial A_{k-2}}A_{k-1}+\frac{\partial F}{\partial A_{k-1}}\Psi = 0,$$

qua in aequatione $\Psi$ est data ipsarum $\varphi$, $A_1$, $A_2$, ... $A_{k-1}$ functio. Ipsae $f$, $H_1$ in hac investigatione tamquam constantes considerari possunt, quippe secundum quas functio quaesita $F$ non differentiatur. Qua de re in $\Psi$ loco $f$, $H_1$ etiam earum valores constantes $a$, $a_1$ ponere licet.

Per regulas notas habetur $F$, si $F=$Constans est integrale aequationum:

$$d\varphi : dA_1 : dA_2 : \ldots : dA_{k-2} : dA_{k-1} =$$
$$A_1 : A_2 : A_3 : \ldots : A_{k-1} : \Psi,$$

quod systema locum tenet unius aequationis differentialis ordinis $(k-1)^{ti}$ inter duas variabiles. Quam, si introducitur elementum $d\tau$, etiam per hanc aequationem $k^{ti}$ ordinis repraesentare licet:

$$\frac{d^k\varphi}{d\tau^k} = \Psi,$$

siquidem in expressione ipsius $\Psi$ loco ipsarum $A_1$, $A_2$, ... $A_{k-1}$ ponitur

$\frac{d\varphi}{d\tau}, \frac{d^2\varphi}{d\tau^2}, \ldots \frac{d^{k-1}\varphi}{d\tau^{k-1}}$, cuius aequationis si $F = \text{Const.}$ est integrale, invenitur functio quaesita $H_2$, si in $F$ loco ipsarum $\frac{d\varphi}{d\tau}, \frac{d^2\varphi}{d\tau^2}, \ldots \frac{d^{k-1}\varphi}{d\tau^{k-1}}$ restituuntur valores $A_1, A_2, \ldots A_{k-1}$. In hac quaestione usui est, quod *quantus sit ordo aequationis differentialis, cuius integrale unum inveniri debet ad determinandam functionem $H_2 = F$, totidem e duobus integralibus $H_1 = a_1$, $\varphi = \text{Const.}$ integralia nova aequationum differentialium propositarum derivare liceat.* Vidimus enim, si $k-1$ iste ordo sit, haberi integralia nova,

$$A_1 = \text{Const.}, \quad A_2 = \text{Const.}, \quad \ldots \quad A_{k-1} = \text{Const.},$$

quae a se et a tribus integralibus datis independentia sunt. Qua re altioris integrationis incommodum quodammodo compensatur.

Patet antecedentibus, etsi integrale $\varphi = \text{Const.}$ non id sit, quod in serie integralium secundum methodum a me propositam successive investigandorum ut integrale tertium adhiberi possit, eius integralis tamen cognitionem investigationem tertii integralis $H_2 = \text{Const.}$ plurimum adjuvare, siquidem expressio $[H_1, \varphi]$ non in numerum redit alium atque zero, neque in ipsarum $f$, $H_1$ functionem.

*Praecedentia applicantur ad investigandum usum trium integralium quae conservationem arearum concernunt in aequationibus dynamicis secundum methodum supra propositam integrandis.*

### 64.

Praemissis considerationibus praecedentibus generalibus, revertor ad propositum; quaerimus enim usum, quem in integratione nostra percipere liceat e tribus integralibus, quae principium conservationis arearum concernunt. In applicationibus ad Dynamicam est $f = a$ aequatio, qua principium conservationis virium vivarum continetur. Sint $H_1 = a_1$, $\varphi = \text{Const.}$ aequationes binae e tribus, quae principium arearum constituunt, sive tribus expressionibus

$$\Sigma m_i\left(y_i\frac{dz_i}{dt} - z_i\frac{dy_i}{dt}\right), \quad \Sigma m_i\left(z_i\frac{dx_i}{dt} - x_i\frac{dz_i}{dt}\right), \quad \Sigma m_i\left(x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt}\right)$$

exhibitis per quantitates $q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m$, sit

$$\Sigma m_i\left(y_i\frac{dz_i}{dt} - z_i\frac{dy_i}{dt}\right) = \varphi,$$

$$\Sigma m_i\left(z_i\frac{dx_i}{dt} - x_i\frac{dz_i}{dt}\right) = \psi,$$

$$\Sigma m_i\left(x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt}\right) = H_1.$$

Demonstravi supra §. 51 (2.), haberi

$$[\varphi, \psi] = H_1, \qquad [\psi, H_1] = \varphi, \qquad [H_1, \varphi] = \psi.$$

Hinc secundum praecepta §. antec. tradita, si statuitur $[\varphi, H_1] = A_1$, $[A_1, H_1] = A_2$, fit $A_1 = -\psi$, $A_2 = -\varphi$, ideoque $k = 2$, $A_k = \Psi = -\varphi$.

Ponendum jam est:

$$d\varphi : dA_1 = A_1 : A_2$$

sive

$$d\varphi : -d\psi = -\psi : -\varphi,$$

sive

$$\varphi d\varphi + \psi d\psi = 0,$$

cuius aequationis habetur integrale:

$$H_2 = \varphi\varphi + \psi\psi = \text{Const.}.$$

Casu igitur proposito aequatio differentialis, cuius integrale inveniri debebat ad determinandam $H_2$, tantum *primum* ordinem ascendebat, eratque aequationum differentialium propositarum *unum* tantum integrale novum $\psi = \text{Const.}$, quod e dato $\varphi = \text{Const.}$ derivari poterat. Aequatio illa primi ordinis quum sine negotio integrata sit, habemus, si tria integralia principii arearum locum habent, duas functiones $H_1$, $H_2$, quae identice satisfaciunt aequationibus:

$$[f, H_1] = 0, \qquad [f, H_2] = 0, \qquad [H_1, H_2] = 0.$$

Loco $H_2$ etiam aliam quamlibet functionem ipsarum $f$, $H_1$, $H_2$ ponere licet, ex. gr. functionem

$$H_2 = \sqrt{H_1 H_1 + \varphi\varphi + \psi\psi},$$

quae plerumque commodiores formulas suppeditat. Invento quolibet integrali $H_i = \text{Const.}$ in serie integralium secundum methodum propositam investigandorum, supra vidimus §. 22 ordinem systematis aequationum differentialium, quae integrandae restant, unitatibus duabus diminui. Hinc introducto uno integrali $H_2 = \text{Const.}$ in locum duorum $\varphi = \text{Const.}$, $\psi = \text{Const.}$, nihil nos profecisse videri potest, quum etiam duobus integralibus *quibuscunque* inventis aequationum differentialium propositarum ordo duabus unitatibus diminuatur. Sed hoc interest discrimen, quod introducto uno integrali $H_2 = \text{Const.}$ methodum nostram integrationum adhibere liceat, qua quolibet novo integrali $H_i = \text{Const.}$ invento ordo differentiationum duabus unitatibus diminuitur. Sed melius adhuc methodi propositae indoles his considerationibus perspicitur.

Demonstravi supra §. 56 modo particulari, quod etiam e theoria proposita generali peti poterat, quoties unum integrale

$$H_1 = \Sigma m_i \left( x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt} \right) = \Sigma m_i r_i^2 \frac{dv_i}{dt}$$

locum habeat, ope huius integralis unam variabilem $v_n$ una cum eius differentiali $\frac{dv_n}{dt}$ ex aequationibus differentialibus propositis prorsus abire, unde ordo differentiationum duabus unitatibus minuitur. Scilicet posito $v_i = u_i + v_n$ et eliminata $\frac{dv_n}{dt}$ ope aequationis praecedentis, variabilium $v_1, v_2, \ldots v_n$ eorumque differentialium solae differentiae $u_i$, $\frac{du_i}{dt}$ in aequationibus differentialibus propositis remanent. At ne hoc, quod ea ratione lucramur, commodum rursus perdamus, necesse est, ut ad reductionem ulteriorem ordinis differentiationum ea tantum adhibeamus integralia, quae et ipsa variabilium $v_1, v_2, \ldots v_n$ non nisi differentias continent. Id quod in integralibus duobus reliquis, quae ad principium conservationis arearum pertinent,

$$\varphi = \text{Const.}, \quad \psi = \text{Const.},$$

in quibus

$$\varphi = \Sigma m_i\left(y_i\frac{dz_i}{dt} - z_i\frac{dy_i}{dt}\right) = \Sigma m_i\left\{\quad \sin v_i\left(r_i\frac{dz_i}{dt} - z_i\frac{dr_i}{dt}\right) - z_i r_i \cos v_i \frac{dv_i}{dt}\right\},$$

$$\psi = \Sigma m_i\left(z_i\frac{dx_i}{dt} - x_i\frac{dz_i}{dt}\right) = \Sigma m_i\left\{-\cos v_i\left(r_i\frac{dz_i}{dt} - z_i\frac{dr_i}{dt}\right) - z_i r_i \sin v_i \frac{dv_i}{dt}\right\},$$

locum non habet. Qua de re pro his duobus integralibus certa eorum combinatio adhibenda est, in qua angulorum $v_1, v_2, \ldots v_n$ non nisi differentiae obveniunt. Cuiusmodi combinatio est aequatio

$$\varphi\varphi + \psi\psi = \text{Const.}.$$

Habetur enim, designante $e$ basin logarithmorum naturalium,

$$\psi + \varphi\sqrt{-1} = \Sigma m_i e^{-v_i\sqrt{-1}}\left\{z_i\frac{dr_i}{dt} - r_i\frac{dz_i}{dt} - \sqrt{-1}\,.\,z_i r_i \frac{dv_i}{dt}\right\},$$

$$\psi - \varphi\sqrt{-1} = \Sigma m_k e^{+v_k\sqrt{-1}}\left\{z_k\frac{dr_k}{dt} - r_k\frac{dz_k}{dt} + \sqrt{-1}\,.\,z_k r_k \frac{dv_k}{dt}\right\},$$

sive posito

$$z_i = \varrho_i \cos\eta_i, \qquad r_i = \varrho_i \sin\eta_i,$$

habetur:

$$\psi + \varphi\sqrt{-1} = \Sigma m_i e^{-v_i\sqrt{-1}}\varrho_i^2\left\{\frac{d\eta_i}{dt} - \sqrt{-1}\,.\,\cos\eta_i \sin\eta_i \frac{dv_i}{dt}\right\},$$

$$\psi - \varphi\sqrt{-1} = \Sigma m_k e^{+v_k\sqrt{-1}}\varrho_k^2\left\{\frac{d\eta_k}{dt} + \sqrt{-1}\,.\,\cos\eta_k \sin\eta_k \frac{dv_k}{dt}\right\};$$

unde

$$\psi\psi+\varphi\varphi =$$

$$\Sigma m_i m_k e^{-(v_i-v_k)\sqrt{-1}}\varrho_i^2\varrho_k^2\left\{\frac{d\eta_i}{dt}-\sqrt{-1}\,.\cos\eta_i\sin\eta_i\frac{dv_i}{dt}\right\}\left\{\frac{d\eta_k}{dt}+\sqrt{-1}\,.\cos\eta_k\sin\eta_k\frac{dv_k}{dt}\right\},$$

ipsis $i$ et $k$ in summa duplici praecedente tributis valoribus $1, 2, \dots n$, siquidem rursus $n$ est numerus punctorum materialium, quorum motus determinandus proponitur. Quam summam patet ipsarum $v_1, v_2, \dots v_n$ solas differentias continere. Quae ut etiam ipsarum $\frac{dv_1}{dt}, \frac{dv_2}{dt}, \dots \frac{dv_n}{dt}$ solas differentias contineat, facile efficitur adiumento aequationis

$$H_1 = \Sigma m_i r_i^2\frac{dv_i}{dt} = \Sigma m_i\varrho_i^2\sin^2\eta_i\frac{dv_i}{dt} = \text{Const.}.$$

Imaginariis eiectis, aequatio praecedens in hanc abit:

$$\psi\psi+\varphi\varphi =$$

$$\Sigma m_i m_k\cos(v_i-v_k)\varrho_i^2\varrho_k^2\left\{\frac{d\eta_i}{dt}\frac{d\eta_k}{dt}+\tfrac{1}{4}\sin 2\eta_i\sin 2\eta_k\frac{dv_i}{dt}\frac{dv_k}{dt}\right\}$$

$$+\Sigma m_i m_k\sin(v_i-v_k)\varrho_i^2\varrho_k^2\left\{\tfrac{1}{2}\sin 2\eta_k\frac{d\eta_i}{dt}\frac{dv_k}{dt}-\tfrac{1}{2}\sin 2\eta_i\frac{d\eta_k}{dt}\frac{dv_i}{dt}\right\},$$

quam data occasione adnotare placet formulam.

Quum antecedentibus casu proposito pateat, in ipsa integratione aequationum differentialium propositarum in locum duorum integralium $\varphi = \text{Const.}$, $\psi = \text{Const.}$ adhibendam esse unicum $\varphi\varphi+\psi\psi = \text{Const.}$, quaeritur, quinam usus sit integralis $\varphi = \text{Const.}$ sive $\psi = \text{Const.}$, quod praeter integrale hoc $\varphi\varphi+\psi\psi = \text{Const.}$ habetur. Cuius is est usus, quod eius integralis beneficio *quadraturae* supersedeatur. Sint enim $H_1 = a_1$, $\varphi = \sqrt{a_2}\cos\beta$, $\psi = \sqrt{a_2}\sin\beta$, tria integralia, quae principium conservationis arearum constituunt, designantibus $a_1$, $a_2$, $\beta$ constantes arbitrarias. Inventis omnibus aequationibus integralibus inter quantitates $r_i$, $u_i$, $z_i$, $\frac{dr_i}{dt}, \frac{du_i}{dt}, \frac{dz_i}{dt}$, invenitur $\frac{dv_n}{dt}$ ope aequationis

$$H_1 = \Sigma m_i r_i^2\frac{dv_i}{dt} = \alpha$$

per formulam supra traditam §. 56 (1.):

$$\frac{dv_n}{dt} = \frac{\alpha-\Sigma m_i r_i^2\frac{du_i}{dt}}{\Sigma m_i r_i^2}.$$

E qua formula per quadraturam valor ipsius $v_n$ eruendus foret, cui tamen per aequationem $\varphi = \sqrt{a_2}\cos\beta$ sive $\psi = \sqrt{a_2}\sin\beta$ sive aliam ex iis conflatam supersedetur. Fit enim, advocatis ipsorum $\varphi$, $\psi$ expressionibus supra traditis:

$$\varphi\cos v_n+\psi\sin v_n = \sqrt{a_2}\cos(v_n-\beta) = \Sigma m_i\left\{\sin u_i\left(r_i\frac{dz_i}{dt}-z_i\frac{dr_i}{dt}\right)-z_i r_i\cos u_i\frac{dv_i}{dt}\right\},$$

$$\varphi\sin v_n-\psi\cos v_n = \sqrt{a_2}\sin(v_n-\beta) = \Sigma m_i\left\{\cos u_i\left(r_i\frac{dz_i}{dt}-z_i\frac{dr_i}{dt}\right)+z_i r_i\sin u_i\frac{dv_i}{dt}\right\}.$$

Quarum aequationum alterutra post substitutionem valorum $\frac{dv_i}{dt}=\frac{du_i}{dt}+\frac{dv_n}{dt}$, qui iam pro datis habentur, determinatur $v_n$ sine quadratura.

*Demonstratur, quodvis problema mechanicum, quo principia conservationis virium vivarum et arearum valeant, atque in quo positio systematis a tribus tantum quantitatibus pendeat, ad quadraturas revocari.*

65.

Considerationibus praecedentibus aestimari potest, quae methodo proposita lucramur, si in problemate mechanico praeter principium conservationis virium vivarum tria integralia locum habent, quae principium conservationis arearum concernunt. Systematis aequationum differentialium propositi

$$dq_1 : dq_2 : \ldots : dq_m : \quad dp_1 : \quad dp_2 : \ldots : \quad dp_m =$$
$$\frac{\partial f}{\partial p_1} : \frac{\partial f}{\partial p_2} : \ldots : \frac{\partial f}{\partial p_m} : -\frac{\partial f}{\partial q_1} : -\frac{\partial f}{\partial q_2} : \ldots : -\frac{\partial f}{\partial q_m}$$

est ordo $2m-1$, qui per integrale $f=a$ revocatur ad ordinem $2m-2$, ac per tria integralia principii arearum $H_1=a_1$, $\varphi=\sqrt{a_2}\cos\beta$, $\psi=\sqrt{a_2}\sin\beta$, ad ordinem $2m-5$. Nam licet iam per duo integralia $f=a$, $H_1=a_1$ aequationes propositae ad ordinem $2m-4$ revocentur, ponendo $v_i=v_n+u_i$ et eliminando differentiale $\frac{dv_n}{dt}$, ipsa $v_n$ ex aequationibus differentialibus sponte abeunte, adnotavi tamen, ne ipsa $V_n$ in aequationes differentiales redeat, loco integralium $\varphi=\sqrt{a_2}\cos\beta$, $\psi=\sqrt{a_2}\sin\beta$ unicum tantum $\varphi\varphi+\psi\psi=a_2$ adhiberi posse, ideoque tantum unitate ordo $2m-4$ adhuc diminuetur. At methodo nostra, qua integrale quodlibet datis considerationibus satisfaciens ordinem differentiationum unitatibus duabus diminuit, fit ut duobus integralibus $H_1=a_1$, $H_2=a_2$ adhibitis, quippe pro quibus identice habetur

$$[H_1, f]=0, \qquad [H_2, f]=0, \qquad [H_1, H_2]=0,$$

*ordo* $2m-2$ *revocetur ad ordinem* $2m-6$. Fluit igitur casu speciali, quo $m=3$, e methodo tradita propositio memorabilis:

*Quodlibet problema mechanicum, pro quo principia conservationis virium vivarum et arearum locum habent, et in quo positio geometrica systematis tribus quantitatibus determinatur, ad quadraturas revocari potest.*

Propositio praecedens paullum a gravitate sua eo amittit, quod, ni vehementer fallor, nullum extat problema mechanicum, pro quo dicta principia generalia locum habeant, et in quo positio systematis a tribus quantitatibus pendeat, praeter duo illa motus puncti versus centrum fixum attracti et rotationis corporis solidi nullis viribus sollicitati circa punctum eius fixum. Horum autem problematum solutio completa iam ex longo temporis intervallo inter Analystas constat. At posterioris certe problematis reductio ad quadraturas extolli solet ut pulcherrimus gloriae titulus, quem adepti sint Analystae decimi octavi saeculi, qui tamen aequationum differentialium integrationem perbene calluerunt. Et postea magnas laudes iustamque admirationem meruit ill. *Lagrange,* qui istam ad quadraturas problematis reductionem vel sine advocatis axium principalium corporum proprietatibus, quibus *Euleri* analysis nitebatur, suscipere ausus fuerit; id quod pro splendida ac paene luxuriante artis manifestatione habeatur. Qua de re fortasse non displicebit Analystis, quod hic non tantum sine auxilio proprietatum axium principalium, sed adeo sine certa variabilium electione — quid? quod sine formatione aequationum differentialium problemati illi particularium, reductio ad quadraturas efficiatur, nulla re in subsidium vocata, nisi quod in problemate assignato principia generalia mechanica valeant.

Operae pretium videtur, *simultaneam* duorum problematum mechanicorum, de quibus dixi, reductionem ad quadraturas, secundum methodum traditam generalem efficiendam, accuratius exponere. Si motus propositi perturbantur, eadem analysis sine ulteriore calculo differentialia elementorum perturbatorum suppeditat (§. 52).

*Solutio simultanea problematis de motu puncti versus centrum fixum attracti atque problematis de rotatione corporis solidi nullis viribus sollicitati circa punctum fixum, una cum expressionibus differentialibus elementorum perturbatorum utriusque problematis.*

66.

Eligatur modus quicunque, in altero problemate positionem puncti in spatio, in altero positionem corporis solidi circa punctum eius fixum determinandi, quod fit in utroque per quantitates tres, quas voco $q_1$, $q_2$, $q_3$. Sit $\frac{dq_1}{dt} = q_1'$, $\frac{dq_2}{dt} = q_2'$, $\frac{dq_3}{dt} = q_3'$, ac expressa semisumma virium vivarum $T$ per $q_1$, $q_2$, $q_3$, $q_1'$, $q_2'$, $q_3'$, sit

$$\frac{\partial T}{\partial q_1'} = p_1, \quad \frac{\partial T}{\partial q_2'} = p_2, \quad \frac{\partial T}{\partial q_3'} = p_3.$$

Ponamus, $H = a$ esse aequationem principium conservationis virium vivarum constituentem, atque $H_1 = a_1$, $\varphi = a_1'$, $\psi = a_1''$ esse tres aequationes, quae principium conservationis arearum repraesentant respectu trium planorum inter se orthogonalium, in altero problemate per centrum attractionis, in altero per punctum fixum corporis ductorum.

Quantitates $a, a_1, a_1', a_1''$, sunt constantes arbitrariae, quae ipsas functiones $H, H_1, \varphi, \psi$ non afficiunt. Expressis $H, H_1, H_2 = \sqrt{H_1 H_1 + \varphi\varphi + \psi\psi}$ per quantitates $q_1, q_2, q_3, p_1, p_2, p_3$, atque ex aequationibus

$$H = a, \quad H_1 = a_1, \quad H_2 = a_2,$$

in quibus $a_2 = \sqrt{a_1 a_1 + a_1' a_1' + a_1'' a_1''}$ determinatis quantitatibus $p_1, p_2, p_3$ per $q_1, q_2, q_3$, est $p_1 dq_1 + p_2 dq_2 + p_3 dq_3$ differentiale completum, positoque

$$\int \{p_1 dq_1 + p_2 dq_2 + p_3 dq_3\} = V,$$

ac designantibus $a, a_1, a_2, b, b_1, b_2$ constantes arbitrarias, evadunt e §§. 33., 34. aequationes:

$$H = a, \quad H_1 = a_1, \quad H_2 = a_2,$$

$$\frac{\partial V}{\partial a} = \int \left\{ \frac{\partial p_1}{\partial a} dq_1 + \frac{\partial p_2}{\partial a} dq_2 + \frac{\partial p_3}{\partial a} dq_3 \right\} = t + b,$$

$$\frac{\partial V}{\partial a_1} = \int \left\{ \frac{\partial p_1}{\partial a_1} dq_1 + \frac{\partial p_2}{\partial a_1} dq_2 + \frac{\partial p_3}{\partial a_1} dq_3 \right\} = b_1,$$

$$\frac{\partial V}{\partial a_2} = \int \left\{ \frac{\partial p_1}{\partial a_2} dq_1 + \frac{\partial p_2}{\partial a_2} dq_2 + \frac{\partial p_3}{\partial a_2} dq_3 \right\} = b_2$$

integralia completa utriusque problematis propositi, eruntque tres aequationes postremae aequationes finitae problematum.

Si motus propositi perturbantur atque in problematibus perturbatis aequatio concernens principium conservationis virium vivarum fit

$$H + \Omega = \text{Const.},$$

sunt e §. 52. differentialia elementorum perturbatorum data per formulas:

$$\frac{da}{dt} = -\frac{\partial \Omega}{\partial b}, \quad \frac{da_1}{dt} = -\frac{\partial \Omega}{\partial b_1}, \quad \frac{da_2}{dt} = -\frac{\partial \Omega}{\partial b_2},$$

$$\frac{db}{dt} = \frac{\partial \Omega}{\partial a}, \quad \frac{db_1}{dt} = \frac{\partial \Omega}{\partial a_1}, \quad \frac{db_2}{dt} = \frac{\partial \Omega}{\partial a_2}.$$

Iam olim ill. *Poisson* in Commentatione praeclara Actis Academiae Parisiensis anni 1816 inserta expressiones differentiales elementorum perturbatorum pro utroque problemate communi analysi investigari posse demonstravit. Sed ipsa problemata duo imperturbata eadem analysi hic primum, quantum credo, amplexus sum.

Iam certa variabilium electione facta formulas inventas pro altero problemate seorsim evolvam.

*De motu puncti versus centrum fixum attracti secundum legem Neutonianam; formulae differentiales elementorum perturbatorum.*

67.

Sint $\varrho\cos\eta$, $\varrho\sin\eta\cos v$, $\varrho\sin\eta\sin v$ coordinatae orthogonales puncti attracti, spectato centro fixo ut initio coordinatarum; posito $\frac{d\varrho}{dt} = \varrho'$, $\frac{dv}{dt} = v'$, $\frac{d\eta}{dt} = \eta'$, massaque corporis $= 1$, fit, designante $\varkappa^2$ vim attractivam pro unitate distantiae:

$$(1.)\quad \begin{cases} H = \frac{1}{2}\{\varrho'\varrho' + \varrho^2\eta'\eta' + \varrho^2\sin^2\eta . v'v'\} - \frac{\varkappa^2}{\varrho} = a, \\ H_1 = \varrho^2\sin^2\eta . v' = a_1, \\ H_2 = \{H_1 H_1 + \varphi\varphi + \psi\psi\}^{\frac{1}{2}} = \varrho^2\{\eta'\eta' + \sin^2\eta . v'v'\}^{\frac{1}{2}} = a_2. \end{cases}$$

Quae notae sunt formulae et facillime probantur. Quantitates $\varrho$, $\eta$, $v$ hic sunt eaedem, quas §. antec. per $q_1$, $q_2$, $q_3$ denotavi. Fit porro

$$T = \tfrac{1}{2}\{\varrho'\varrho' + \varrho^2\eta'\eta' + \varrho^2\sin^2\eta . v'v'\},$$

unde

$$\frac{\partial T}{\partial \varrho'} = \varrho', \qquad \frac{\partial T}{\partial \eta'} = \eta_1 = \varrho^2\eta', \qquad \frac{\partial T}{\partial v'} = v_1 = \varrho^2\sin^2\eta . v'.$$

Quantitates $\varrho'$, $\eta_1$, $v_1$ hic eaedem sunt atque §. antec. per $p_1$, $p_2$, $p_3$ denotatae.

Eliminata $v'$, fit e (1.):

$$a + \frac{\varkappa^2}{\varrho} = \tfrac{1}{2}\left\{\varrho'\varrho' + \varrho^2\eta'\eta' + \frac{a_1 a_1}{\varrho^2\sin^2\eta}\right\},$$

$$a_2^2 = \varrho^4\eta'\eta' + \frac{a_1 a_1}{\sin^2\eta},$$

unde

$$(2.)\quad \begin{cases} \varrho' = \left\{2\left(a + \frac{\varkappa^2}{\varrho}\right) - \frac{a_2^2}{\varrho^2}\right\}^{\frac{1}{2}}, \\ \eta_1 = \varrho^2\eta' = \left\{a_2^2 - \frac{a_1 a_1}{\sin^2\eta}\right\}^{\frac{1}{2}}, \\ v_1 = \varrho^2\sin^2\eta . v' = a_1. \end{cases}$$

His substitutis valoribus fit

$$(3.)\quad \begin{cases} V = \int\left\{\varrho'\,d\varrho + \eta_1\,d\eta + v_1\,dv\right\} \\ \quad = \int\left\{2\left(a + \frac{\varkappa^2}{\varrho}\right) - \frac{a_2^2}{\varrho^2}\right\}^{\frac{1}{2}} d\varrho + \int\left\{a_2^2 - \frac{a_1^2}{\sin^2\eta}\right\}^{\frac{1}{2}} d\eta + a_1 v. \end{cases}$$

Hic non tantum patet, quod generaliter probavimus, expressionem

$$p_1 dq_1 + p_2 dq_2 + \cdots$$

differentiale exactum esse, sed ea, qua usi sumus, variabilium electione id effectum esse videmus, ut in expressione illa differentiali adeo variabiles separatae sint. Idem evenit pro lege attractionis quacunque, quae si exprimitur per functionem $-\frac{\partial f(\varrho)}{\partial \varrho}$, tantum opus est, ut in expressione ipsius $V$ antecedente loco ipsius $\frac{\varkappa^2}{\varrho}$ ponatur $f(\varrho)$.

Ex aequatione (3.) fluunt secundum praecepta §. antec. tradita integralia finita problematis:

$$(4.)\quad t+b = \frac{\partial V}{\partial a} = \int \frac{d\varrho}{\left\{2\left(a+\frac{\varkappa^2}{\varrho}\right)-\frac{a_2^2}{\varrho^2}\right\}^{\frac{1}{2}}},$$

$$(5.)\quad b_1 = \frac{\partial V}{\partial a_1} = -\int \frac{a_1\, d\eta}{\left\{a_2^2-\frac{a_1^2}{\sin^2\eta}\right\}^{\frac{1}{2}}\sin^2\eta} + v,$$

$$(6.)\quad b_2 = \frac{\partial V}{\partial a_2} = a_2 \int \frac{d\eta}{\left\{a_2^2-\frac{a_1^2}{\sin^2\eta}\right\}^{\frac{1}{2}}} - a_2 \int \frac{1}{\left\{2\left(a+\frac{\varkappa^2}{\varrho}\right)-\frac{a_2^2}{\varrho^2}\right\}^{\frac{1}{2}}}\cdot\frac{d\varrho}{\varrho^2}.$$

Fit primum

$$(7.)\quad \int \frac{d\eta}{\left\{a_2^2-\frac{a_1^2}{\sin^2\eta}\right\}^{\frac{1}{2}}} = \int \frac{\sin\eta\, d\eta}{\{a_2^2-a_1^2-a_2^2\cos^2\eta\}^{\frac{1}{2}}} = \frac{1}{a_2}\operatorname{Arccos}\left(\sqrt{\frac{a_2^2}{a_2^2-a_1^2}}\cdot\cos\eta\right),$$

porro e (5.) habetur:

$$(8.)\quad v-b_1 = -\int \frac{a_1\, d\operatorname{ctg}\eta}{\{a_2^2-a_1^2-a_1^2\operatorname{ctg}^2\eta\}^{\frac{1}{2}}} = \operatorname{Arccos}\left(\sqrt{\frac{a_1^2}{a_2^2-a_1^2}}\cdot\operatorname{ctg}\eta\right),$$

unde

$$(9.)\quad \operatorname{ctg}\eta = \sqrt{\frac{a_2^2-a_1^2}{a_1^2}}\cdot\cos(v-b_1).$$

Deinde erit:

$$(10.)\quad \int \frac{a_2}{\left\{2\left(a+\frac{\varkappa^2}{\varrho}\right)-\frac{a_2^2}{\varrho^2}\right\}^{\frac{1}{2}}}\cdot\frac{d\varrho}{\varrho^2} = \int \frac{a_2^2}{\left\{\varkappa^4+2aa_2^2-\left(\varkappa^2-\frac{a_2^2}{\varrho}\right)^2\right\}^{\frac{1}{2}}}\cdot\frac{d\varrho}{\varrho^2} = u,$$

siquidem statuitur

$$(11.)\quad \frac{a_2^2}{\varrho}-\varkappa^2 = \sqrt{\varkappa^4+2aa_2^2}\cdot\cos u.$$

Substitutis (7.) et (10.), aequatio (6.) in hanc abit:

$$(12.)\quad b_2 = \operatorname{Arccos}\left(\sqrt{\frac{a_2^2}{a_2^2-a_1^2}}\cdot\cos\eta\right)-u,$$

unde

$$(13.)\qquad \cos\eta = \sqrt{\frac{a_2^2-a_1^2}{a_2^2}}\cdot\cos(u+b_2).$$

Denique habetur e (4.):

$$t+b = \int\frac{\varrho\, d\varrho}{\sqrt{2a\varrho^2+2\varkappa^2\varrho-a_2^2}} = \int\frac{\sqrt{-2a}.\varrho\, d\varrho}{\{\varkappa^4+2aa_2^2-(2a\varrho+\varkappa^2)^2\}^{\frac{1}{2}}},$$

sive posito

$$(14.)\qquad \varkappa^2+2a\varrho = \sqrt{\varkappa^4+2aa_2^2}.\cos E,$$

fit

$$(15.)\qquad t+b = \frac{1}{\sqrt{-2a}}\int\varrho\, dE = \frac{\varkappa^2 E}{(-2a)^{\frac{3}{2}}} - \frac{\sqrt{\varkappa^4+2aa_2^2}}{(-2a)^{\frac{3}{2}}}\cdot\sin E.$$

Ut aequationes inventae induant formam simpliciorem, pro constantibus arbitrariis adhibitis alias introducam. Sit

$$(16.)\qquad \sqrt{1+\frac{2aa_2^2}{\varkappa^4}} = e,\quad a = -\frac{\varkappa^2}{2A},\quad \mu = \frac{(-2a)^{\frac{3}{2}}}{\varkappa^2} = \frac{\varkappa}{A^{\frac{3}{2}}},$$

fiunt (14.), (11.), (15.):

$$(17.)\quad \varrho = A(1-e.\cos E),\quad \frac{A(1-ee)}{\varrho} = 1+e.\cos u,\quad \mu(t+b) = E-e.\sin E.$$

E quibus aequationibus patet, quantitates $E$, $u$, $\mu(t+b)$, $A$, $e$, $-b$, $\frac{a_2}{\varkappa}$ esse *anomaliam excentricam, anomaliam veram, anomaliam mediam, semiaxem maiorem, excentricitatem, tempus perihelii, radicem quadraticam semiparametri.*

Ponamus porro:

$$(18.)\qquad \sqrt{\frac{a_2^2-a_1^2}{a_1^2}} = \operatorname{tg} i,\quad\text{unde}\quad \frac{a_1}{a_2} = \cos i,$$

fit e (9.)

$$(19.)\qquad \cos i\cos\eta = \sin i\sin\eta\cos(v-b_1),$$

quae docet aequatio orbitam puncti attracti esse planam atque designare $i$ *inclinationem orbitae* atque $b_1$ *longitudinem nodi ascendentis orbitae* ideoque erit $\frac{a_1}{\varkappa}$ aequale *radici quadraticae semiparametri multiplicatae per cosinum inclinationis orbitae.* Unde iam quinque constantes arbitrariae $a$, $a_1$, $a_2$, $b$, $b_1$ significationem geometricam invenerunt. Restat aequatio (13.) quae e (18.) in hanc abit:

$$(20.)\qquad \cos\eta = \sin i\cos(u+b_2) = \sin i.\sin\left(u+\frac{\pi}{2}+b_2\right).$$

Haec formula docet, esse $\frac{\pi}{2}+b_2$ ***distantiam perihelii a nodo ascendente.***

E theoremate §. 66 proposito fiunt differentialia elementorum perturbatorum:

$$\frac{da}{dt} = -\frac{\partial \Omega}{\partial b}, \quad \frac{da_1}{dt} = -\frac{\partial \Omega}{\partial b_1}, \quad \frac{da_2}{dt} = -\frac{\partial \Omega}{\partial b_2},$$

$$\frac{db}{dt} = \frac{\partial \Omega}{\partial a}, \quad \frac{db_1}{dt} = \frac{\partial \Omega}{\partial a_1}, \quad \frac{db_2}{dt} = \frac{\partial \Omega}{\partial a_2},$$

quibus in formulis est

$\frac{-\varkappa^2}{2a}$ ... semiaxis major,

$\frac{a_1}{\varkappa}$ ... radix quadratica semiparametri multiplicata per cosinum inclinationis,

$\frac{a_2}{\varkappa}$ ... radix quadratica semiparametri,

$-b$ ... tempus perihelii,

$b_1$ ... longitudo nodi ascendentis,

$\frac{\pi}{2}+b_2$ ... distantia perihelii a nodo ascendente.

Designante igitur, ut notationem usitatiorem adhibeam, $A$ semiaxem maiorem, $h$ radicem quadraticam semiparametri, $i$ inclinationem, $\tau$ tempus perihelii, ☊ longitudinem nodi ascendentis, $\varpi$ distantiam eius a perihelio, fiunt formulae differentiales elementorum perturbatorum:

$$(21.) \quad \begin{cases} \varkappa^2 \frac{dA}{dt} = 2A^2 \frac{\partial \Omega}{\partial \tau}, & \varkappa^2 \frac{d\tau}{dt} = -2A^2 \frac{\partial \Omega}{\partial A}, \\ \varkappa \frac{d\varpi}{dt} = \frac{\partial \Omega}{\partial h}, & \varkappa \frac{dh}{dt} = -\frac{\partial \Omega}{\partial \varpi}, \\ \varkappa \frac{d☊}{dt} = \frac{\partial \Omega}{\partial . h\cos i}, & \varkappa \frac{d.h\cos i}{dt} = -\frac{\partial \Omega}{\partial ☊}. \end{cases}$$

quae formulae aequivalent aequationibus differentialibus sequentibus, in quibus $\varrho = \sqrt{xx+yy+zz}$:

$$(22.) \quad \begin{cases} \frac{d^2x}{dt^2} = -\frac{\varkappa^2 x}{\varrho^3} - \frac{\partial \Omega}{\partial x}, \\ \frac{d^2y}{dt^2} = -\frac{\varkappa^2 y}{\varrho^3} - \frac{\partial \Omega}{\partial y}, \\ \frac{d^2z}{dt^2} = -\frac{\varkappa^2 z}{\varrho^3} - \frac{\partial \Omega}{\partial z}, \end{cases}$$

in quibus $\Omega$ data est functio ipsarum $x, y, z, t$, vel adeo aequationibus differentialibus generalioribus sequentibus:

$$(23.)\quad \begin{cases} \dfrac{dx}{dt} = x'+\dfrac{\partial\Omega}{\partial x'}, & \dfrac{dx'}{dt} = -\dfrac{\varkappa^2 x}{\varrho^3}-\dfrac{\partial\Omega}{\partial x}, \\ \dfrac{dy}{dt} = y'+\dfrac{\partial\Omega}{\partial y'}, & \dfrac{dy'}{dt} = -\dfrac{\varkappa^2 y}{\varrho^3}-\dfrac{\partial\Omega}{\partial y}, \\ \dfrac{dz}{dt} = z'+\dfrac{\partial\Omega}{\partial z'}, & \dfrac{dz'}{dt} = -\dfrac{\varkappa^2 z}{\varrho^3}-\dfrac{\partial\Omega}{\partial z}, \end{cases}$$

in quibus $\Omega$ est data functio ipsarum $t$, $x$, $y$, $z$, $x'$, $y'$, $z'$. Per varias methodos supra traditas ex elementorum canonicorum systemate proposito innumera alia facillime derivantur.

Si placet, quod in calculis commodius est, in locum temporis perihelii ut elementum introducere *epocham* seu *valorem anomaliae mediae pro* $t=0$,

$$c = -\mu\tau = \mu b,$$

facile deducitur e (21.):

$$\varkappa\frac{dc}{dt} = 2A^{\frac{1}{2}}\frac{\partial\Omega}{\partial A}, \qquad \varkappa\frac{dA}{dt} = -2A^{\frac{1}{2}}\frac{\partial\Omega}{\partial c},$$

reliquis formulis (21.) immutatis manentibus.

Quaeramus adhuc ipsius functionis $V$ expressionem finitam. Fit

$$\left\{2\left(a+\frac{\varkappa^2}{\varrho}\right)-\frac{a_2^2}{\varrho^2}\right\}^{\frac{1}{2}} = \frac{\varkappa e}{\sqrt{A}}\,\frac{\sin E}{1-e\cos E},$$
$$d\varrho = Ae\sin E\,dE,$$

unde

$$\int\left\{2\left(a+\frac{\varkappa^2}{\varrho}\right)-\frac{a_2^2}{\varrho^2}\right\}^{\frac{1}{2}}d\varrho = \varkappa e^2\sqrt{A}\int\frac{\sin^2 E\,dE}{1-e\cos E}$$
$$= \varkappa\sqrt{A}\left\{E+e\sin E-2\sqrt{1-e^2}\,\mathrm{Arctg}\left(\frac{\sqrt{1+e}}{\sqrt{1-e}}\,\mathrm{tg}\frac{E}{2}\right)\right\}.$$

Fit porro, posito $a_2 = \varkappa h = \varkappa\sqrt{A(1-e^2)}$,

$$\int\left\{a_2^2-\frac{a_1^2}{\sin^2\eta}\right\}^{\frac{1}{2}}d\eta = \varkappa h\int\left\{1-\frac{\cos^2 i}{\sin^2\eta}\right\}^{\frac{1}{2}}d\eta$$
$$= \varkappa h\,\mathrm{Arccos}\left(\frac{\cos\eta}{\sin i}\right)-\varkappa h\cos i\,\mathrm{Arccos}(\mathrm{ctg}\,i\,\mathrm{ctg}\,\eta).$$

Unde

$$V = \varkappa\sqrt{A}\left\{E+e\sin E-2\sqrt{1-e^2}\,\mathrm{Arctg}\left(\frac{\sqrt{1+e}}{\sqrt{1-e}}\,\mathrm{tg}\frac{E}{2}\right)\right\}+\varkappa h\,\mathrm{Arccos}\left(\frac{\cos\eta}{\sin i}\right)$$
$$+\varkappa h\cos i\,\{v-\mathrm{Arccos}(\mathrm{ctg}\,i\,\mathrm{ctg}\,\eta)\}.$$

In differentianda hac expressione secundum constantes arbitrarias $A$, $e$, $i$, quae in locum ipsarum $a$, $a_1$, $a_2$ introduci possunt, adhibendae sunt aequationes:

$$0 = 1-e\cos E+Ae\sin E\frac{\partial E}{\partial A},$$
$$0 = \cos E-e\sin E\frac{\partial E}{\partial e},$$
$$0 = \frac{\partial E}{\partial i}.$$

Integrationes antecedentibus factae sunt respective inde a limitibus $\varrho = A(1-e)$, $\eta = \frac{\pi}{2} - i$. Quorum limitum in differentianda $V$ secundum constantes arbitrarias respectum non habuimus. Scilicet quia in expressione $V$ termini sub signo integrationis pro limitibus illis evanescunt, unde facile patet, terminos e limitum variatione prodeuntes evanescere ideoque negligi posse.

*De methodo proposita in varia problemata applicanda, ac praesertim in problemata isoperimetrica.*

### 68.

Methodus generalis etiam facillime applicatur problemati celeberrimo de puncto versus duo puncta fixa, quorum datae sunt massae, secundum legem Neutonianam attracto. In cuius solutione occupatus invenerat *Eulerus* praeter integralia duo a principiis conservationis virium vivarum et arearum suppeditata integrale tertium, quo problema ad aequationem differentialem primi ordinis inter duas variabiles revocabatur. At summo viri egregii acumine et intrepido animo indigebat, ut per varia tentamina aequationis differentialis complicatissimae reductio ad quadraturas succederet. Nostra methodo per regulam generalem absque omni calculo instituendo aequatio differentialis revocari potuisset ad quadraturas. Determinatis enim e tribus illis integralibus $x'$, $y'$, $z'$ per $x$, $y$, $z$ et tres constantes arbitrarias $a$, $a_1$, $a_2$, quas principia conservationis virium vivarum et arearum et integrale ab *Eulero* inventum involvunt, erunt aequationes tres:

$$\int \left\{ \frac{\partial x'}{\partial a} dx + \frac{\partial y'}{\partial a} dy + \frac{\partial z'}{\partial a} dz \right\} = t + b,$$

$$\int \left\{ \frac{\partial x'}{\partial a_1} dx + \frac{\partial y'}{\partial a_1} dy + \frac{\partial z'}{\partial a_1} dz \right\} = b_1,$$

$$\int \left\{ \frac{\partial x'}{\partial a_2} dx + \frac{\partial y'}{\partial a_2} dy + \frac{\partial z'}{\partial a_2} dz \right\} = b_2,$$

in quibus expressiones sub signis integrationum differentialia exacta sunt, problematis propositi aequationes finitae. Nec non etiam huius problematis sine ullo calculo per propositiones nostras generales habentur formulae perturbatoriae.

Quoties in problemate mechanico, in quo principium conservationis virium vivarum valet, positio systematis duabus quantitatibus a se independentibus $q_1$, $q_2$ determinatur — quod ex. gr. evenit, puncto supra datam superficiem *in linea brevissima* moto — nova methodo integrandi aequationes differentiales partiales primi ordinis non egebat, sed sufficit *Lagrangiana,* quae de tribus variabilibus pro perfecta haberi potest. Problemata eiusmodi pendent ab integratione aequationis differentialis secundi ordinis inter duas variabiles, quae, si praeter dictum principium alterum innotescit integrale

$$f_1(q_1, q_2, p_1, p_2) = a_1,$$

revocatur ad ordinem primum. Sed secundum methodum nostram generalem vel etiam secundum ipsam methodum *Lagrangianam* integrandi aequationes differentiales partiales primi ordinis inter tres variabiles, *haec aequatio differentialis primi ordinis inter duas variabiles semper ad quadraturas revocari potest.* Sit enim $f = a$ aequatio pro viribus vivis, eruantur ipsarum $p_1$, $p_2$ valores ex aequationibus $f = a$, $f_1 = a_1$, determinabit aequatio

$$\int \left\{ \frac{\partial p_1}{\partial a_1} dq_1 + \frac{\partial p_2}{\partial a_1} dq_2 \right\} = b_1$$

positionem systematis, sive pro puncto singulo, in data superficie moto, eius *orbitam,* atque altera aequatio

$$\int \left\{ \frac{\partial p_1}{\partial a} dq_1 + \frac{\partial p_2}{\partial a} dq_2 \right\} = b + t$$

*positionis tempus.* In hunc casum, qui tantum integrationem aequationis differentialis partialis primi ordinis inter variabiles *tres* requirit, in qua una variabilium, functio quaesita, ipsa non obvenit, exempla praecedentibus allegata redeunt. Nam introducendo coordinatas polares per integrale a principio conservationis arearum petitum unam variabilem simul atque differentiale partiale secundum eam sumtum ex aequatione differentiali partiali eliminare licet, unde tres variabiles independentes ad duas revocantur atque aequatio differentialis partialis, a cuius integratione problema pendet, ad aliam iam apud ill. *Lagrange* tractatam.

Notum est, aequationes differentiales vulgares *lineares* secundi ordinis inter duas variabiles ita comparatas esse, ut post alterum integrale inventum alterum tantum a quadraturis pendeat. Problemata mechanica videmus ducere ad alias aequationes differentiales vulgares secundi ordinis inter duas variabiles ita comparatas, ut post alterum integrale inventum alterum a solis quadraturis pendeat, et quae neutiquam sunt lineares.

Cuiusmodi ex. gr. est aequatio differentialis notissima, quae lineam brevissimam in data superficie concernit, quippe quam describit punctum in superficie data moveri coactum et a viribus nullis acceleratricibus sollicitatum; unde habetur propositio:

*aequationis differentialis secundi ordinis, a qua linea brevissima pendet, si integrale unum inventum est, lineae determinatio ad solas quadraturas revocatur.*

Cuius propositionis exempla suggerunt lineae brevissimae in superficiebus rotundis, conicis, cylindricis, in quibus integrale unum sponte se offert. Ac generalius, quod diximus, valebit de aequationibus differentialibus secundi ordinis, a quibus pendent problemata, integralia huiusmodi

$$\int \varphi\left(x, y, \frac{dy}{dx}\right) dx$$

*maxima* vel *minima* reddere. Facile autem patet e supra traditis, exhibendo problemata mechanica hic tractata sub forma

$$\delta \int (T+U)\, dt = 0,$$

methodum propositam omnibus problematibus isoperimetricis adhiberi posse, in quibus expressio sub signo integrationis quemlibet functionum incognitarum numerum earumque differentialia prima involvat. Posito enim $q_i' = \frac{dq_i}{dt}$, si proponitur aequatio:

$$\delta \int \varphi(t, q_1, q_2, \ldots q_m, q_1', q_2', \ldots q_m')\, dt = 0,$$

ponatur

$$H = q_1' \frac{\partial \varphi}{\partial q_1'} + q_2' \frac{\partial \varphi}{\partial q_2'} + \cdots + q_m' \frac{\partial \varphi}{\partial q_m'} - \varphi,$$

atque eliminentur ex hac expressione ipsae $q_1'$, $q_2'$, ... $q_m'$ per aequationes

$$\frac{\partial \varphi}{\partial q_1'} = p_1, \quad \frac{\partial \varphi}{\partial q_2'} = p_2, \quad \ldots \quad \frac{\partial \varphi}{\partial q_m'} = p_m.$$

Quo facto pendebit problema, quod facile e regulis notis calculi variationum comprobatur, atque ex ipsa analysi elucet, quam apud ill. *Hamilton* invenis, ab integratione completa systematis aequationum differentialium vulgarium sequentium:

$$\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1}, \quad \frac{dq_2}{dt} = \frac{\partial H}{\partial p_2}, \quad \ldots \quad \frac{dq_m}{dt} = \frac{\partial H}{\partial p_m},$$

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial q_1}, \quad \frac{dp_2}{dt} = -\frac{\partial H}{\partial q_2}, \quad \ldots \quad \frac{dp_m}{dt} = -\frac{\partial H}{\partial q_m}.$$

Quam integrationem completam demonstravi obtineri per integrationem completam aequationis differentialis partialis

$$\frac{\partial V}{\partial t} + H = 0,$$

in qua ponendum est

$$p_1 = \frac{\partial V}{\partial q_1}, \quad p_2 = \frac{\partial V}{\partial q_2}, \quad \dots \quad p_m = \frac{\partial V}{\partial q_m}.$$

Haec autem integratio per methodum novam hic propositam absolvitur.

Etiam generaliorem casum problematum isoperimetricorum, quo expressio sub signo integrationis praeter differentialia prima functionum incognitarum differentialia altiora ordinis cuiuslibet involvit, contingit ad integrationem aequationis differentialis partialis primi ordinis revocare. Quae aequationes differentiales partiales omnes eo commodo gaudent, quod functionem quaesitam sive variabilem dependentem ipsam non involvunt. Sunt tamen problemata isoperimetrica, quae ad aequationes differentiales partiales conducant ipsam etiam variabilem dependentem involventes, ea dico, in quibus expressio, cuius variationem evanescentem reddi oportet, non immediate ut integrale proponitur, sed et ipsa ab integratione aequationis differentialis primi ordinis pendet, quae praeterea functiones incognitas earumque differentialia involvit. Quin adeo, si expressio, cuius variationem evanescentem reddere proponitur, per aequationem differentialem cuiuslibet ordinis datur, quae etiam functiones incognitas earumque differentialia involvat, quaestionem ad integrationem aequationis differentialis partialis primi ordinis revocare contigit. Unde et illis quaestionibus valde generalibus methodos nostras applicare licet.

Quaestiones isoperimetricas, quae ad aequationes differentiales partiales primi ordinis revocari possunt, antecedentibus eas esse supposuimus, in quibus *functiones unius variabilis* seu *curvae* indagantur proprietati maximi minimive satisfacientes. Quaenam analoga extent circa problemata isoperimetrica, in quibus functiones duarum variabilium seu superficies quaeruntur, integrale duplex propositum maximum minimumve reddentes, felicioribus conatibus relinquo investiganda.

*De relationibus simplicissimis, quibus differentialia partialia variabilium secundum elementa canonica sumta differentialibus elementorum secundum variabiles sumtis vel nude vel mutato signo singula singulis aequiparantur.*

69.

Systema elementorum, quae afficiunt solutiones problematum mechanicorum secundum methodum a me propositam inventas, praeterea quod formulas perturbatorias simplicissimas suppeditant, aliis adhuc gravissimis proprietatibus gaudent. Quas sequentibus exponam.

Sit $V$ functio quantitatum

$$q_1,\ q_2,\ \dots\ q_m,\ a_1,\ a_2,\ \dots\ a_m,\ t_1,\ t_2,\ \dots\ t_\mu$$

ac ponamus

$$(1.)\quad \frac{\partial V}{\partial q_1} = p_1,\quad \frac{\partial V}{\partial q_2} = p_2,\quad \dots\quad \frac{\partial V}{\partial q_m} = p_m,$$

$$(2.)\quad \frac{\partial V}{\partial a_1} = b_1,\quad \frac{\partial V}{\partial a_2} = b_2,\quad \dots\quad \frac{\partial V}{\partial a_m} = b_m,$$

$$(3.)\quad \frac{\partial V}{\partial t_1} = T_1,\quad \frac{\partial V}{\partial t_2} = T_2,\quad \dots\quad \frac{\partial V}{\partial t_\mu} = T_\mu.$$

Ex aequationibus (1.) et (2.) sint $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ expressae per $a_1, a_2, \dots a_m, b_1, b_2, \dots b_m, t_1, t_2, \dots t_\mu$ ac vice versa $a_1, a_2, \dots a_m, b_1, b_2, \dots b_m$ expressae per $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m, t_1, t_2, \dots t_\mu$. Quae sunt expressiones, quas in sequentibus subintelligam, si quantitates $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ secundum $a_i, b_i, t_i$ vel vice versa quantitates $a_1, a_2, \dots a_m, b_1, b_2, \dots b_m$ secundum $q_i, p_i, t_i$ differentiantur. Suppositionem, quantitates $q_i, p_i$ per $a_i, b_i, t_i$ ita expressas esse, ut (1.), (2.) identicae evadant, vocabo *suppositionem primam;* suppositionem, quantitates $a_i, b_i$ per $q_i, p_i, t_i$ ita expressas esse, ut (1.), (2.) identicae evadant, vocabo *suppositionem secundam.*

Suppositione *prima* facta, differentiemus aequationes

$$\frac{\partial V}{\partial a_k} = b_k$$

secundum $a_i, b_i, t_i$, prodit:

$$(4.)\quad \frac{\partial^2 V}{\partial a_k \partial a_i} + \frac{\partial^2 V}{\partial a_k \partial q_1}\frac{\partial q_1}{\partial a_i} + \frac{\partial^2 V}{\partial a_k \partial q_2}\frac{\partial q_2}{\partial a_i} + \dots + \frac{\partial^2 V}{\partial a_k \partial q_m}\frac{\partial q_m}{\partial a_i} = 0,$$

$$(5.)\quad \frac{\partial^2 V}{\partial a_k \partial q_1}\frac{\partial q_1}{\partial b_i} + \frac{\partial^2 V}{\partial a_k \partial q_2}\frac{\partial q_2}{\partial b_i} + \dots + \frac{\partial^2 V}{\partial a_k \partial q_m}\frac{\partial q_m}{\partial b_i} = \frac{\partial b_k}{\partial b_i},$$

$$(6.)\quad \frac{\partial^2 V}{\partial a_k \partial t_i} + \frac{\partial^2 V}{\partial a_k \partial q_1}\frac{\partial q_1}{\partial t_i} + \frac{\partial^2 V}{\partial a_k \partial q_2}\frac{\partial q_2}{\partial t_i} + \dots + \frac{\partial^2 V}{\partial a_k \partial q_m}\frac{\partial q_m}{\partial t_i} = 0.$$

Suppositione *secunda* facta differentiemus aequationes

$$\frac{\partial V}{\partial q_\lambda} = p_\lambda$$

secundum $q_{i'}, p_{i'}, t_{i'}$, prodit:

$$(7.)\quad \frac{\partial^2 V}{\partial q_\lambda \partial q_{i'}} + \frac{\partial^2 V}{\partial q_\lambda \partial a_1}\frac{\partial a_1}{\partial q_{i'}} + \frac{\partial^2 V}{\partial q_\lambda \partial a_2}\frac{\partial a_2}{\partial q_{i'}} + \dots + \frac{\partial^2 V}{\partial q_\lambda \partial a_m}\frac{\partial a_m}{\partial q_{i'}} = 0,$$

$$(8.)\quad \frac{\partial^2 V}{\partial q_\lambda \partial a_1}\frac{\partial a_1}{\partial p_{i'}} + \frac{\partial^2 V}{\partial q_\lambda \partial a_2}\frac{\partial a_2}{\partial p_{i'}} + \dots + \frac{\partial^2 V}{\partial q_\lambda \partial a_m}\frac{\partial a_m}{\partial p_{i'}} = \frac{\partial p_\lambda}{\partial p_{i'}},$$

$$(9.)\quad \frac{\partial^2 V}{\partial q_\lambda \partial t_{i'}} + \frac{\partial^2 V}{\partial q_\lambda \partial a_1}\frac{\partial a_1}{\partial t_{i'}} + \frac{\partial^2 V}{\partial q_\lambda \partial a_2}\frac{\partial a_2}{\partial t_{i'}} + \dots + \frac{\partial^2 V}{\partial q_\lambda \partial a_m}\frac{\partial a_m}{\partial t_{i'}} = 0.$$

In his formulis indicibus $i$, $i'$, $k$, $\lambda$ tribui possunt valores $1, 2, \dots m$, excepto casu, quo $i$, $i'$ ipsam $t$ afficiunt, quo casu iis valores $1, 2, \dots \mu$ conveniunt. Expressiones $\frac{\partial b_k}{\partial b_i}$, $\frac{\partial p_\lambda}{\partial p_{i'}}$ sunt aut $=0$, si $k$, $\lambda$ ab $i$, $i'$ diversi sunt, aut $=1$, si $k=i$, $\lambda=i'$.

Multiplicentur aequationes (4.), (5.), (6.) per

$$\frac{\partial a_k}{\partial q_{i'}},\quad \frac{\partial a_k}{\partial p_{i'}},\quad \frac{\partial a_k}{\partial t_{i'}},$$

ac post multiplicationes factas instituatur summatio secundum indicem $k$, hoc est, ponatur successive $1, 2, \dots m$ loco ipsius $k$ et expressiones, quae inde prodeunt, addantur. Quo facto per (7.), (8.), (9.) nanciscimur e (4.):

$$(10.)\quad \begin{cases} \frac{\partial^2 V}{\partial a_1 \partial a_i}\frac{\partial a_1}{\partial q_{i'}} + \frac{\partial^2 V}{\partial a_2 \partial a_i}\frac{\partial a_2}{\partial q_{i'}} + \dots + \frac{\partial^2 V}{\partial a_m \partial a_i}\frac{\partial a_m}{\partial q_{i'}} = \\ \frac{\partial^2 V}{\partial q_1 \partial q_{i'}}\frac{\partial q_1}{\partial a_i} + \frac{\partial^2 V}{\partial q_2 \partial q_{i'}}\frac{\partial q_2}{\partial a_i} + \dots + \frac{\partial^2 V}{\partial q_m \partial q_{i'}}\frac{\partial q_m}{\partial a_i}, \end{cases}$$

$$(11.)\quad \frac{\partial^2 V}{\partial a_1 \partial a_i}\frac{\partial a_1}{\partial p_{i'}} + \frac{\partial^2 V}{\partial a_2 \partial a_i}\frac{\partial a_2}{\partial p_{i'}} + \dots + \frac{\partial^2 V}{\partial a_m \partial a_i}\frac{\partial a_m}{\partial p_{i'}} = -\frac{\partial q_{i'}}{\partial a_i},$$

$$(12.)\quad \begin{cases} \frac{\partial^2 V}{\partial a_1 \partial a_i}\frac{\partial a_1}{\partial t_{i'}} + \frac{\partial^2 V}{\partial a_2 \partial a_i}\frac{\partial a_2}{\partial t_{i'}} + \dots + \frac{\partial^2 V}{\partial a_m \partial a_i}\frac{\partial a_m}{\partial t_{i'}} = \\ \frac{\partial^2 V}{\partial q_1 \partial t_{i'}}\frac{\partial q_1}{\partial a_i} + \frac{\partial^2 V}{\partial q_2 \partial t_{i'}}\frac{\partial q_2}{\partial a_i} + \dots + \frac{\partial^2 V}{\partial q_m \partial t_{i'}}\frac{\partial q_m}{\partial a_i}; \end{cases}$$

e (5.):

$$(13.)\quad -\frac{\partial a_i}{\partial q_{i'}} = \frac{\partial^2 V}{\partial q_1 \partial q_{i'}}\frac{\partial q_1}{\partial b_i} + \frac{\partial^2 V}{\partial q_2 \partial q_{i'}}\frac{\partial q_2}{\partial b_i} + \dots + \frac{\partial^2 V}{\partial q_m \partial q_{i'}}\frac{\partial q_m}{\partial b_i},$$

$$(14.)\quad \frac{\partial a_i}{\partial p_{i'}} = \frac{\partial q_{i'}}{\partial b_i},$$

$$(15.)\quad -\frac{\partial a_i}{\partial t_{i'}} = \frac{\partial^2 V}{\partial q_1 \partial t_{i'}}\frac{\partial q_1}{\partial b_i} + \frac{\partial^2 V}{\partial q_2 \partial t_{i'}}\frac{\partial q_2}{\partial b_i} + \dots + \frac{\partial^2 V}{\partial q_m \partial t_{i'}}\frac{\partial q_m}{\partial b_i};$$

e (6.):

$$(16.)\quad \begin{cases} \frac{\partial^2 V}{\partial a_1 \partial t_i}\frac{\partial a_1}{\partial q_{i'}} + \frac{\partial^2 V}{\partial a_2 \partial t_i}\frac{\partial a_2}{\partial q_{i'}} + \dots + \frac{\partial^2 V}{\partial a_m \partial t_i}\frac{\partial a_m}{\partial q_{i'}} = \\ \frac{\partial^2 V}{\partial q_1 \partial q_{i'}}\frac{\partial q_1}{\partial t_i} + \frac{\partial^2 V}{\partial q_2 \partial q_{i'}}\frac{\partial q_2}{\partial t_i} + \dots + \frac{\partial^2 V}{\partial q_m \partial q_{i'}}\frac{\partial q_m}{\partial t_i}, \end{cases}$$

$$(17.)\quad \frac{\partial^2 V}{\partial a_1 \partial t_i}\frac{\partial a_1}{\partial p_{i'}} + \frac{\partial^2 V}{\partial a_2 \partial t_i}\frac{\partial a_2}{\partial p_{i'}} + \dots + \frac{\partial^2 V}{\partial a_m \partial t_i}\frac{\partial a_m}{\partial p_{i'}} = -\frac{\partial q_{i'}}{\partial t_i},$$

$$(18.)\quad \begin{cases} \frac{\partial^2 V}{\partial a_1 \partial t_i}\frac{\partial a_1}{\partial t_{i'}} + \frac{\partial^2 V}{\partial a_2 \partial t_i}\frac{\partial a_2}{\partial t_{i'}} + \dots + \frac{\partial^2 V}{\partial a_m \partial t_i}\frac{\partial a_m}{\partial t_{i'}} = \\ \frac{\partial^2 V}{\partial q_1 \partial t_{i'}}\frac{\partial q_1}{\partial t_i} + \frac{\partial^2 V}{\partial q_2 \partial t_{i'}}\frac{\partial q_2}{\partial t_i} + \dots + \frac{\partial^2 V}{\partial q_m \partial t_{i'}}\frac{\partial q_m}{\partial t_i}. \end{cases}$$

Si utrique parti aequationum (10.), (12.), (16.), (18.) respective additur:

$$\frac{\partial^2 V}{\partial q_{i'} \partial a_i}, \quad \frac{\partial^2 V}{\partial t_{i'} \partial a_i}, \quad \frac{\partial^2 V}{\partial q_{i'} \partial t_i}, \quad \frac{\partial^2 V}{\partial t_{i'} \partial t_i}$$

aequationes inventae (10.) — (18.) sic exhiberi possunt:

$$(10.^*) \quad \frac{\partial b_i}{\partial q_{i'}} = \frac{\partial p_{i'}}{\partial a_i},$$

$$(11.^*) \quad \frac{\partial b_i}{\partial p_{i'}} = -\frac{\partial q_{i'}}{\partial a_i},$$

$$(12.^*) \quad \frac{\partial b_i}{\partial t_{i'}} = \frac{\partial T_{i'}}{\partial a_i},$$

$$(13.^*) \quad \frac{\partial a_i}{\partial q_{i'}} = -\frac{\partial p_{i'}}{\partial b_i},$$

$$(14.^*) \quad \frac{\partial a_i}{\partial p_{i'}} = \frac{\partial q_{i'}}{\partial b_i},$$

$$(15.^*) \quad \frac{\partial a_i}{\partial t_{i'}} = -\frac{\partial T_{i'}}{\partial b_i},$$

$$(16.^*) \quad \frac{\partial T_i}{\partial q_{i'}} = \frac{\partial p_{i'}}{\partial t_i},$$

$$(17.^*) \quad \frac{\partial T_i}{\partial p_{i'}} = -\frac{\partial q_{i'}}{\partial t_i},$$

$$(18.^*) \quad \frac{\partial T_i}{\partial t_{i'}} = \frac{\partial T_{i'}}{\partial t_i}.$$

Ut aequationes novem praecedentes, quae sunt gravia et elegantia theoremata, recte intelligantur, teneri oportet, expressiones ad laevam omnes referri ad suppositionem secundam, qua considerantur $2m$ quantitates $a_i$ et $b_i$ ut functiones ipsarum $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m, t_1, t_2, \dots t_\mu$ aequationibus (1.) et (2.) identice satisfacientes atque illi ipsarum $a_i$ et $b_i$ valores in expressionibus $T_i$ substituti supponuntur, antequam secundum $t_{i'}$ differentiantur; contra expressiones ad dextram omnes ad suppositionem primam pertinent, qua considerantur $2m$ quantitates $q_i$ et $p_i$ ut functiones ipsarum $a_1, a_2, \dots a_m, b_1, b_2, \dots b_m, t_1, t_2, \dots t_\mu$ aequationibus (1.) et (2.) identice satisfacientes, atque illi ipsarum $q_i$ et $p_i$ valores in expressionibus $T_{i'}$ substituti supponuntur, antequam secundum $t_i$ differentiantur.

## 70.

Aequationes integrales systematis aequationum differentialium vulgarium propositi sub duabus maxime formis considerantur; exprimuntur enim aut incognitae omnes per unam ex earum numero (ex. gr. tempus in quaestionibus mechanicis) atque constantes arbitrarias, quas integratio completa secum fert, aut exprimuntur constantes arbitrariae per incognitas. Qua in re incognitas etiam dicimus earum differentialia, quae inferioris ordinis sunt atque summi, ad quem in aequationibus differentialibus propositis ascendunt. Aequationes posterioris formae ita comparatae sunt, ut semel differentiando constantes omnes arbitrariae sponte abeant, ideoque aequationibus differentialibus, quae inde proveniunt, per ipsas aequationes differentiales propositas sponte satisfiat, cuiusmodi aequationes integrales prae ceteris vocavi integralia aequationum differentialium propositarum. Aequationes integrales, quae motum ellipticum concernunt puncti secundum legem Neutonianam ad punctum fixum attracti, saepius sub utraque forma propositae sunt, variosque ad usus indagatae sunt quotientes differentiales partiales provenientes, si in altera forma incognitae secundum singulas constantes arbitrarias, sive in altera functiones constantibus arbitrariis aequivalentes secundum singulas incognitas differentiantur. Qua de re memoratu dignum mihi videtur, quod e formulis praecedentibus patet, proposito systemate aequationum differentialium vulgarium, quale in mechanicis integrandum est:

$$\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1}, \quad \frac{dq_2}{dt} = \frac{\partial H}{\partial p_2}, \quad \dots \quad \frac{dq_m}{dt} = \frac{\partial H}{\partial p_m},$$

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial q_1}, \quad \frac{dp_2}{dt} = -\frac{\partial H}{\partial q_2}, \quad \dots \quad \frac{dp_m}{dt} = -\frac{\partial H}{\partial q_m},$$

*si eligatur constantium arbitrarium sive elementorum systema canonicum, fore ut quotientes differentiales incognitarum secundum elementa aut elementorum secundum incognitas singulae singulis aequales evadant aut solo signo inter se differant.*

Sit enim in formulis praecedentibus $\mu = 1$, sive una tantum adsit quantitatum $t_1, t_2, \dots t_\mu$, quam vocabo $t$. Ponendo in expressione

$$\frac{\partial V}{\partial t} = T$$

loco $a_1, a_2, \dots a_m$ earum valores per $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m, t$ expressos, quales obtinetur ex $m$ aequationibus

$$\frac{\partial V}{\partial q_1} = p_1, \quad \frac{\partial V}{\partial q_2} = p_2, \quad \dots \quad \frac{\partial V}{\partial q_m} = p_m,$$

abeat $T$ in $-H$, ideo ut sit $-H$ expressio ipsius $T$ in suppositione secunda. Unde erit $V$ integrale aequationis differentialis partialis

$$(1.)\qquad \frac{\partial V}{\partial t}+H = 0,$$

quod integrale continebit $m$ constantes arbitrarias $a_1, a_2, \dots a_m$. Consideremus in aequationibus

$$(2.)\qquad \begin{cases} \dfrac{\partial V}{\partial q_1}=p_1, & \dfrac{\partial V}{\partial q_2}=p_2, & \dots & \dfrac{\partial V}{\partial q_m}=p_m, \\ \dfrac{\partial V}{\partial a_1}=b_1, & \dfrac{\partial V}{\partial a_2}=b_2, & \dots & \dfrac{\partial V}{\partial a_m}=b_m, \end{cases}$$

e quibus sequebantur aequationes §$^i$. antec. (10.*) — (18.*), ipsas $a_1, a_2, \dots a_m$, $b_1, b_2, \dots b_m$ ut constantes, unde ex aequationibus illis fiunt $q_1, q_2, \dots q_m$, $p_1, p_2, \dots p_m$ solius $t$ functiones. Ac scribendo $-H$ loco $T_i$ in parte laeva aequationum (16.*), (17.*) §$^i$. antec. obtinemus $2m$ aequationes differentiales, quibus aequationes (2.) satisfaciunt:

$$(3.)\qquad -\frac{\partial H}{\partial q_{i'}}=\frac{dp_{i'}}{dt},\quad \frac{\partial H}{\partial p_{i'}}=\frac{dq_{i'}}{dt},$$

quibus in formulis indici $i'$ valores $1, 2, \dots m$ tribuendi sunt. Aequationes vero (10.*) — (15.*) suppeditant theorema propositum, videlicet: differentialia partialia variabilium secundum elementa differentialibus partialibus elementorum secundum variabiles singula singulis aequivalere. Casu, quo functio $H$ ipsam $t$ non implicat, qui est frequentissimus in problematis mechanicis, ita agere licet. Statuamus, in §$^o$. antec. functionem $V$ quantitates $t_1, t_2, \dots t_\mu$ omnino non implicare; porro loco $a_m$ scribamus $h$, loco $b_m$ vero $t+\tau$. Unde aequationes (1.), (2.) §$^i$. antec. fiunt:

$$(4.)\qquad \frac{\partial V}{\partial q_1}=p_1,\quad \frac{\partial V}{\partial q_2}=p_2,\quad \dots\quad \frac{\partial V}{\partial q_m}=p_m,$$

$$(5.)\qquad \frac{\partial V}{\partial a_1}=b_1,\quad \frac{\partial V}{\partial a_2}=b_2,\quad \dots\quad \frac{\partial V}{\partial a_{m-1}}=b_{m-1},\quad \frac{\partial V}{\partial h}=t+\tau.$$

Eliminatis e (4.) quantitatibus $a_1, a_2, \dots a_{m-1}$, prodeat

$$H = h,$$

designante $H$ functionem ipsarum $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$, quae ipsam $h$ non implicat. Unde vice versa considerari potest $V$ ut solutio completa aequationis differentialis partialis:

$$H = h,$$

in qua $a_1, a_2, \dots a_{m-1}$ sunt constantes arbitrariae (constantem arbitrariam, quae functioni $V$ sola additione iungi potest, ut plerumque, non respicimus) atque $h$ constans data, quae ipsam iam aequationem differentialem afficit. Consideremus porro in aequationibus (4.), (5.) ipsas $a_1$, $a_2$, ... $a_{m-1}$, $h$, $b_1$, $b_2$, ... $b_m$, $\tau$ ut constantes, erunt per aequationes illas ipsae $q_1$, $q_2$, ... $q_m$, $p_1$, $p_2$, ... $p_m$ datae functiones quantitatis $t$. Ponamus in aequationibus (13.*), (14.*) §[i]. antec. $i=m$, atque, sicuti convenimus, loco $a_m$, $b_m$ scribamus $h$ atque $t+\tau$, in parte laeva aequationum illarum exprimendum erit $a_m$ sive $h$ per ipsas $q_1$, $q_2$, ... $q_m$, $p_1$, $p_2$, ... $p_m$ ope aequationum (4.), (5.), sive loco $a_m$ ponendum est $H$. Quo facto, si insuper animadvertimus, loco expressionum $\frac{\partial q_{i'}}{\partial b_m}$, $\frac{\partial p_{i'}}{\partial b_m}$ sive $\frac{\partial q_{i'}}{\partial(t+\tau)}$, $\frac{\partial p_{i'}}{\partial(t+\tau)}$ scribendum esse, si $t$ ut variabilis independens spectetur,

$$\frac{\partial q_{i'}}{\partial b_m}=\frac{dq_{i'}}{dt},\quad \frac{\partial p_{i'}}{\partial b_m}=\frac{dp_{i'}}{dt},$$

abeunt aequationes (13.*), (14.*) in systema aequationum differentialium vulgarium, quae aequationibus (4.), (5.) satisfaciunt:

$$\frac{\partial H}{\partial q_{i'}}=-\frac{dp_{i'}}{dt},\quad \frac{\partial H}{\partial p_{i'}}=\frac{dq_{i'}}{dt}.$$

Ac vice versa aequationibus (4.), (5.) hae aequationes differentiales complete integrantur. Porro aequationes §[i] antec. (10.*), (11.*), (13.*), (14.*) suppeditant formulas:

$$\frac{\partial b_i}{\partial q_{i'}}=\frac{\partial p_{i'}}{\partial a_i},\quad \frac{\partial b_i}{\partial p_{i'}}=-\frac{\partial q_{i'}}{\partial a_i},$$
$$\frac{\partial a_i}{\partial q_{i'}}=-\frac{\partial p_{i'}}{\partial b_i},\quad \frac{\partial a_i}{\partial p_{i'}}=\frac{\partial q_{i'}}{\partial b_i},$$

in quibus indici $i$ valores 1, 2, ... $m-1$, indici $i'$ valores 1, 2, ... $m$ tribuendi sunt; atque fit:

$$\frac{\partial \tau}{\partial q_{i'}}=\frac{\partial p_{i'}}{\partial h},\quad \frac{\partial \tau}{\partial p_{i'}}=-\frac{\partial q_{i'}}{\partial h},$$

in quibus aequationibus indici $i'$ rursus valores 1, 2, ... $m$ tribuendi sunt*).

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*) Formulae eiusmodi, cum Academia Berolinensi scientiarum communicatae, iam inveniuntur in commentatiuncula „Neues Theorem der analytischen Mechanik", huius diarii tom. XXX, p. 117. *C.*

*Formulae antecedentes applicantur in motum liberum n punctorum materialium quo aequatio conservationis virium vivarum locum habet.*

## 71.

Operae pretium mihi videtur, nonnulla eorum, quae antecedentibus invenimus, pro casu systematis liberi per vires internas sollicitati seorsim theoremate exponere. Quo casu loco quantitatum $q_i$ ponamus coordinatas orthogonales, unde loco ipsarum $p_i$ ponendae erunt expressiones $m_i x'_i$, $m_i y'_i$, $m_i z'_i$.

### Theorema.

*Consideremus motum systematis liberi n punctorum materialium; sint $x_i$, $y_i$, $z_i$ coordinatae orthogonales puncti, cuius massa $m_i$, ac sollicitentur singula puncta $m_i$ secundum directiones axium coordinatarum viribus $m_i X_i$, $m_i Y_i$, $m_i Z_i$ talibus, ut evadat summa*

$$\Sigma m_i (X_i dx_i + Y_i dy_i + Z_i dz_i),$$

*extensa ad puncta omnia systematis, differentiale completum*

$$dU = \Sigma m_i (X_i dx_i + Y_i dy_i + Z_i dz_i),$$

*qui habetur casus, quoties systema punctorum materialium tantum viribus internis attractionis aut repulsionis sollicitatur. Ad inveniendum motum systematis, integretur aequatio differentialis partialis:*

$$\tfrac{1}{2}\Sigma \frac{1}{m_i}\left\{\left(\frac{\partial V}{\partial x_i}\right)^2 + \left(\frac{\partial V}{\partial y_i}\right)^2 + \left(\frac{\partial V}{\partial z_i}\right)^2\right\} = U + h,$$

*in qua h est constans; inventaque solutione completa V, quae praeter constantem, quae sola additione ei iungi potest, constantes arbitrarias $a_1, a_2, \dots a_{3n-1}$ implicet, erunt aequationes finitae, quibus motus punctorum materialium definiuntur:*

$$\frac{\partial V}{\partial a_1} = b_1, \quad \frac{\partial V}{\partial a_2} = b_2, \quad \dots \quad \frac{\partial V}{\partial a_{3n-1}} = b_{3n-1},$$

$$\frac{\partial V}{\partial h} = t + \tau,$$

*designantibus $b_1, b_2, \dots b_{3n-1}, \tau$ novas constantes arbitrarias; porro erit*

$$\frac{\partial V}{\partial x_1} = m_1 \frac{dx_1}{dt}, \quad \frac{\partial V}{\partial x_2} = m_2 \frac{dx_2}{dt}, \quad \dots \quad \frac{\partial V}{\partial x_n} = m_n \frac{dx_n}{dt},$$

$$\frac{\partial V}{\partial y_1} = m_1 \frac{dy_1}{dt}, \quad \frac{\partial V}{\partial y_2} = m_2 \frac{dy_2}{dt}, \quad \dots \quad \frac{\partial V}{\partial y_n} = m_n \frac{dy_n}{dt},$$

$$\frac{\partial V}{\partial z_1} = m_1 \frac{dz_1}{dt}, \quad \frac{\partial V}{\partial z_2} = m_2 \frac{dz_2}{dt}, \quad \dots \quad \frac{\partial V}{\partial z_n} = m_n \frac{dz_n}{dt}.$$

*Per aequationes propositas, statuto*

$$x_i' = \frac{dx_i}{dt}, \qquad y_i' = \frac{dy_i}{dt}, \qquad z_i' = \frac{dz_i}{dt},$$

*considerari possunt* $6n$ *quantitates* $x_i$, $y_i$, $z_i$, $x_i'$, $y_i'$, $z_i'$ *ut functiones* $6n$ *quantitatum* $a_1, a_2, \ldots a_{3n-1}, h, b_1, b_2, \ldots b_{3n-1}, t+\tau$, *vel vice versa spectari possunt* $6n$ *quantitates* $a_1, a_2, \ldots a_{3n-1}, h, b_1, b_2, \ldots b_{3n-1}, t+\tau$, *ut functiones* $6n$ *quantitatum* $x_i$, $y_i$, $z_i$, $x_i'$, $y_i'$, $z_i'$.

*Si sub utraque suppositione functionum illarum quotientes differentiales partiales sumuntur, quotientes differentiales partiales sub altera suppositione sumtae quotientibus differentialibus partialibus sub altera suppositione sumtis singulae singulis aequales fiunt aut tantum signo differunt; fit enim, designante i unum quemcunque e numeris* $1, 2, \ldots n$, *atque k unum quemcunque e numeris* $1, 2, 3, \ldots 3n-1$:

$$m_i \frac{\partial x_i}{\partial a_k} = -\frac{\partial b_k}{\partial x_i'}, \qquad m_i \frac{\partial x_i}{\partial b_k} = \frac{\partial a_k}{\partial x_i'},$$

$$m_i \frac{\partial y_i}{\partial a_k} = -\frac{\partial b_k}{\partial y_i'}, \qquad m_i \frac{\partial y_i}{\partial b_k} = \frac{\partial a_k}{\partial y_i'},$$

$$m_i \frac{\partial z_i}{\partial a_k} = -\frac{\partial b_k}{\partial z_i'}, \qquad m_i \frac{\partial z_i}{\partial b_k} = \frac{\partial a_k}{\partial z_i'},$$

$$m_i \frac{\partial x_i'}{\partial a_k} = \frac{\partial b_k}{\partial x_i}, \qquad m_i \frac{\partial x_i'}{\partial b_k} = -\frac{\partial a_k}{\partial x_i},$$

$$m_i \frac{\partial y_i'}{\partial a_k} = \frac{\partial b_k}{\partial y_i}, \qquad m_i \frac{\partial y_i'}{\partial b_k} = -\frac{\partial a_k}{\partial y_i},$$

$$m_i \frac{\partial z_i'}{\partial a_k} = \frac{\partial b_k}{\partial z_i}, \qquad m_i \frac{\partial z_i'}{\partial b_k} = -\frac{\partial a_k}{\partial z_i},$$

$$m_i \frac{\partial x_i}{\partial h} = -\frac{\partial(\tau+t)}{\partial x_i'}, \qquad m_i \frac{\partial x_i'}{\partial h} = \frac{\partial(\tau+t)}{\partial x_i},$$

$$m_i \frac{\partial y_i}{\partial h} = -\frac{\partial(\tau+t)}{\partial y_i'}, \qquad m_i \frac{\partial y_i'}{\partial h} = \frac{\partial(\tau+t)}{\partial y_i},$$

$$m_i \frac{\partial z_i}{\partial h} = -\frac{\partial(\tau+t)}{\partial z_i'}, \qquad m_i \frac{\partial z_i'}{\partial h} = \frac{\partial(\tau+t)}{\partial z_i}.$$

*Statuamus propositos motus perturbari, viribus* $m_i X_i$, $m_i Y_i$, $m_i Z_i$ *punctum* $m_i$ *sollicitantibus accedentibus novis viribus* $m_i X_i'$, $m_i Y_i'$, $m_i Z_i'$, *designantibus* $X_i'$, $Y_i'$, $Z_i'$ *functiones omnium* $3n$ *coordinatarum* $x_i$, $y_i$, $z_i$ *atque temporis* $t$, *ac sit, si solae coordinatae variantur neque simul tempus, summa*

$$\Sigma m_i\{X_i'\delta x_i+Y_i'\delta y_i+Z_i'\delta z_i\},$$

*extensa ad puncta omnia systematis, variatio completa*

$$-\delta\Omega = \Sigma m_i\{X_i'\delta x_i+Y_i'\delta y_i+Z_i'\delta z_i\};$$

*quibus statutis, aequationes problematis imperturbati*

$$\frac{\partial V}{\partial a_1}=b_1,\quad \frac{\partial V}{\partial a_2}=b_2,\quad \dots\quad \frac{\partial V}{\partial a_{3n-1}}=b_{3n-1},\quad \frac{\partial V}{\partial h}=t+\tau,$$

$$\frac{\partial V}{\partial x_1}=m_1x_1',\quad \frac{\partial V}{\partial x_2}=m_2x_2',\quad \dots\quad \frac{\partial V}{\partial x_n}=m_nx_n',$$

$$\frac{\partial V}{\partial y_1}=m_1y_1',\quad \frac{\partial V}{\partial y_2}=m_2y_2',\quad \dots\quad \frac{\partial V}{\partial y_n}=m_ny_n',$$

$$\frac{\partial V}{\partial z_1}=m_1z_1',\quad \frac{\partial V}{\partial z_2}=m_2z_2',\quad \dots\quad \frac{\partial V}{\partial z_n}=m_nz_n'$$

*etiam motus suppeditabunt perturbatos, si loco elementorum* $a_1, a_2, \dots a_{3n-1}$, $h$, $b_1$, $b_2$, $\dots$ $b_{3n-1}$, $\tau$ *sumuntur functiones temporis satisfacientes aequationibus differentialibus:*

$$\frac{da_1}{dt}=-\frac{\partial\Omega}{\partial b_1},\quad \frac{da_2}{dt}=-\frac{\partial\Omega}{\partial b_2},\quad \dots\quad \frac{da_{3n-1}}{dt}=-\frac{\partial\Omega}{\partial b_{3n-1}},$$

$$\frac{db_1}{dt}=\frac{\partial\Omega}{\partial a_1},\quad \frac{db_2}{dt}=\frac{\partial\Omega}{\partial a_2},\quad \dots\quad \frac{db_{3n-1}}{dt}=\frac{\partial\Omega}{\partial a_{3n-1}},$$

$$\frac{dh}{dt}=-\frac{\partial\Omega}{\partial\tau},\quad \frac{d\tau}{dt}=\frac{\partial\Omega}{\partial h},$$

*quibus in aequationibus supponitur, functionem* $\Omega$ *ope aequationum pro motu imperturbato inventarum*

$$\frac{\partial V}{\partial a_1}=b_1,\quad \frac{\partial V}{\partial a_2}=b_2,\quad \dots\quad \frac{\partial V}{\partial a_{3n-1}}=b_{3n-1},\quad \frac{\partial V}{\partial h}=t+\tau$$

*per sola elementa et tempus expressam esse.*

Aequatio differentialis partialis in theoremate antecedente proposita invenitur ex aequatione

$$H=T-U=h,$$

quum sit $T$ semissis summae virium vivarum

$$T = \tfrac{1}{2}\Sigma m_i\{x_i'x_i'+y_i'y_i'+z_i'z_i'\};$$

aequationes vero

$$\frac{\partial T}{\partial q_i'}=p_i=\frac{\partial V}{\partial q_i}$$

in aequatione $H=h$ substituendae, quo aequatio differentialis partialis evadat, hic sunt

$$m_i x'_i = \frac{\partial V}{\partial x_i}, \quad m_i y'_i = \frac{\partial V}{\partial y_i}, \quad m_i z'_i = \frac{\partial V}{\partial z_i}.$$

Unde aequatio

$$H = \tfrac{1}{2}\Sigma m_i(x'_i x'_i + y'_i y'_i + z'_i z'_i) - U = h,$$

abit in aequationem

$$\tfrac{1}{2}\Sigma \frac{1}{m_i}\left\{\left(\frac{\partial V}{\partial x_i}\right)^2 + \left(\frac{\partial V}{\partial y_i}\right)^2 + \left(\frac{\partial V}{\partial z_i}\right)^2\right\} - U = h,$$

quae est aequatio differentialis partialis in theoremate antecedente proposita. Formulae perturbatoriae theorematis e §. 52. petitae sunt, scriptis $h$ et $\tau$ loco $a$ et $b$. Eaedem expressiones differentialium elementorum habentur etiam pro generalioribus aequationibus differentialibus, in quibus $\Omega$ praeter ipsas $x_i$, $y_i$, $z_i$ etiam quantitates $x'_i$, $y'_i$, $z'_i$ involvere potest:

$$\frac{dx_i}{dt} = x'_i + \frac{\partial \Omega}{\partial x'_i}, \qquad \frac{dy_i}{dt} = y'_i + \frac{\partial \Omega}{\partial y'_i}, \qquad \frac{dz_i}{dt} = z'_i + \frac{\partial \Omega}{\partial z'_i},$$

$$m_i \frac{dx'_i}{dt} = \frac{\partial (U-\Omega)}{\partial x_i}, \qquad m_i \frac{dy'_i}{dt} = \frac{\partial (U-\Omega)}{\partial y_i}, \qquad m_i \frac{dz'_i}{dt} = \frac{\partial (U-\Omega)}{\partial z_i},$$

quae, quoties $\Omega$ ipsas non implicat $x'_i$, $y'_i$, $z'_i$ in aequationes differentiales perturbatas, quae vulgo habentur, redeunt:

$$m_i \frac{d^2 x_i}{dt^2} = \frac{\partial (U-\Omega)}{\partial x_i}, \qquad m_i \frac{d^2 y_i}{dt^2} = \frac{\partial (U-\Omega)}{\partial y_i}, \qquad m_i \frac{d^2 z_i}{dt^2} = \frac{\partial (U-\Omega)}{\partial z_i}.$$

Si theorema antecedens ad motum ellipticum planetarum applicare placet, ponamus, uti in formulis §. 67 factum est,

$$h = -\frac{\varkappa^2}{2A}, \qquad a_1 = \varkappa\sqrt{p}\cos i, \qquad a_2 = \varkappa\sqrt{p},$$

$$b = \tau, \qquad b_1 = \Omega, \qquad \frac{\pi}{2} + b_2 = \varpi,$$

designantibus $A$, $p$, $i$, $-\tau$, $\Omega$, $\varpi$ semiaxem maiorem, parametrum, inclinationem, tempus perihelii, longitudinem nodi ascendentis, distantiam perihelii a nodo ascendente, atque $\varkappa^2$ vim attractivam pro unitate distantiae. Si $x$, $y$, $z$, $x'$, $y'$, $z'$ per $A$, $p$, $\sqrt{p}\cos i$, $\tau$, $\Omega$, $\varpi$, $t$ vel vice versa $A$, $p$, $i$, $\tau$, $\Omega$, $\varpi$ per $x$, $y$, $z$, $x'$, $y'$, $z'$ exprimimus, et expressiones illas sub utraque suppositione differentiamus, prodeunt e theoremate antecedente formulae:

$$\frac{\partial x}{\partial.\sqrt{p}\cos i} = -\varkappa\frac{\partial\Omega}{\partial x'}, \quad \frac{\partial y}{\partial.\sqrt{p}\cos i} = -\varkappa\frac{\partial\Omega}{\partial y'}, \quad \frac{\partial z}{\partial.\sqrt{p}\cos i} = -\varkappa\frac{\partial\Omega}{\partial z'},$$

$$\frac{\partial x}{\partial\Omega} = \varkappa\frac{\partial.\sqrt{p}\cos i}{\partial x'}, \quad \frac{\partial y}{\partial\Omega} = \varkappa\frac{\partial.\sqrt{p}\cos i}{\partial y'}, \quad \frac{\partial z}{\partial\Omega} = \varkappa\frac{\partial.\sqrt{p}\cos i}{\partial z'},$$

$$\frac{\partial x}{\partial\sqrt{p}} = -\varkappa\frac{\partial\varpi}{\partial x'}, \quad \frac{\partial y}{\partial\sqrt{p}} = -\varkappa\frac{\partial\varpi}{\partial y'}, \quad \frac{\partial z}{\partial\sqrt{p}} = -\varkappa\frac{\partial\varpi}{\partial z'},$$

$$\frac{\partial x}{\partial\varpi} = \varkappa\frac{\partial\sqrt{p}}{\partial x'}, \quad \frac{\partial y}{\partial\varpi} = \varkappa\frac{\partial\sqrt{p}}{\partial y'}, \quad \frac{\partial z}{\partial\varpi} = \varkappa\frac{\partial\sqrt{p}}{\partial z'},$$

$$2A^2\frac{\partial x}{\partial A} = -\varkappa^2\frac{\partial\tau}{\partial x'}, \quad 2A^2\frac{\partial y}{\partial A} = -\varkappa^2\frac{\partial\tau}{\partial y'}, \quad 2A^2\frac{\partial z}{\partial A} = -\varkappa^2\frac{\partial\tau}{\partial z'},$$

$$\frac{\partial x'}{\partial.\sqrt{p}\cos i} = \varkappa\frac{\partial\Omega}{\partial x}, \quad \frac{\partial y'}{\partial.\sqrt{p}\cos i} = \varkappa\frac{\partial\Omega}{\partial y}, \quad \frac{\partial z'}{\partial.\sqrt{p}\cos i} = \varkappa\frac{\partial\Omega}{\partial z},$$

$$\frac{\partial x'}{\partial\Omega} = -\varkappa\frac{\partial.\sqrt{p}\cos i}{\partial x}, \quad \frac{\partial y'}{\partial\Omega} = -\varkappa\frac{\partial.\sqrt{p}\cos i}{\partial y}, \quad \frac{\partial z'}{\partial\Omega} = -\varkappa\frac{\partial.\sqrt{p}\cos i}{\partial z},$$

$$\frac{\partial x'}{\partial\sqrt{p}} = \varkappa\frac{\partial\varpi}{\partial x}, \quad \frac{\partial y'}{\partial\sqrt{p}} = \varkappa\frac{\partial\varpi}{\partial y}, \quad \frac{\partial z'}{\partial\sqrt{p}} = \varkappa\frac{\partial\varpi}{\partial z},$$

$$\frac{\partial x'}{\partial\varpi} = -\varkappa\frac{\partial\sqrt{p}}{\partial x}, \quad \frac{\partial y'}{\partial\varpi} = -\varkappa\frac{\partial\sqrt{p}}{\partial y}, \quad \frac{\partial z'}{\partial\varpi} = -\varkappa\frac{\partial\sqrt{p}}{\partial z},$$

$$2A^2\frac{\partial x'}{\partial A} = \varkappa^2\frac{\partial\tau}{\partial x}, \quad 2A^2\frac{\partial y'}{\partial A} = \varkappa^2\frac{\partial\tau}{\partial y}, \quad 2A^2\frac{\partial z'}{\partial A} = \varkappa^2\frac{\partial\tau}{\partial z}.$$

Quibus in formulis designat $i$ inclinationem plani orbitae ad unum planorum coordinatarum orthogonalium $x$, $y$, $z$, atque $\Omega$ angulum, quem intersectio utriusque plani cum altera axi coordinatarum facit, quae in plano illo coordinatarum ducta est. E formulis notis motus elliptici verificationem formularum praecedentium facile obtinere licet. Quae facile etiam in alias varias formas transfunduntur.

*De expressionibus $(\varphi, \psi)$ et $[\varphi, \psi]$, quae in modum coefficientium in ill. Lagrange et Poisson formulis perturbatoriis obvenientium conflatae sunt. Innotescente integrali quolibet $H_i = a_i$ aequationum dynamicarum, differentialia omnia functionis cuiuslibet secundum elementum $b_i$, quod ipsi $a_i$ in systemate quolibet elementorum canonico fiat coniugatum, assignari possunt.*

## 72.

Statuamus rursus:

$$[\varphi, \psi] = \frac{\partial\varphi}{\partial q_1}\frac{\partial\psi}{\partial p_1} + \frac{\partial\varphi}{\partial q_2}\frac{\partial\psi}{\partial p_2} + \cdots + \frac{\partial\varphi}{\partial q_m}\frac{\partial\psi}{\partial p_m}$$
$$-\frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial q_1} - \frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_2} - \cdots - \frac{\partial\varphi}{\partial p_m}\frac{\partial\psi}{\partial q_m}$$

porro, uncis rotundis adhibitis,

$$(\varphi, \psi) = \frac{\partial q_1}{\partial \psi}\frac{\partial p_1}{\partial \varphi} + \frac{\partial q_2}{\partial \psi}\frac{\partial p_2}{\partial \varphi} + \dots + \frac{\partial q_m}{\partial \psi}\frac{\partial p_m}{\partial \varphi}$$

$$- \frac{\partial q_1}{\partial \varphi}\frac{\partial p_1}{\partial \psi} - \frac{\partial q_2}{\partial \varphi}\frac{\partial p_2}{\partial \psi} - \dots - \frac{\partial q_m}{\partial \varphi}\frac{\partial p_m}{\partial \psi}$$

facile probatur e formulis §. 69 traditis, haberi

$$(1.)\qquad [a_i, a_k] = 0, \quad [a_i, b_k] = 0, \quad [b_i, b_k] = 0,$$

$$(2.)\qquad (a_i, a_k) = 0, \quad (a_i, b_k) = 0, \quad (b_i, b_k) = 0,$$

exceptis aequationibus:

$$(3.)\qquad [a_i, b_i] = -1, \qquad (4.)\qquad (a_i, b_i) = 1.$$

Formulae (1.) ad suppositionem secundam pertinent, qua consideravimus ipsas $a_i$, $b_i$ ut functiones ipsarum $q_i$, $p_i$, $t_i$; formulae (2.) ad suppositionem primam, qua considerantur $q_i$, $p_i$ ut functiones ipsarum $a_i$, $b_i$, $t_i$. Quae formulae e (10.*) seqq. §. 69 sic demonstrantur:

Habetur, extensa summatione ad ipsius $i'$ valores 1, 2, ... $m$:

$$0 = \frac{\partial a_k}{\partial a_i} = \Sigma_{i'}\left\{\frac{\partial a_k}{\partial q_{i'}}\frac{\partial q_{i'}}{\partial a_i} + \frac{\partial a_k}{\partial p_{i'}}\frac{\partial p_{i'}}{\partial a_i}\right\},$$

$$0 = \frac{\partial a_k}{\partial b_i} = \Sigma_{i'}\left\{\frac{\partial a_k}{\partial q_{i'}}\frac{\partial q_{i'}}{\partial b_i} + \frac{\partial a_k}{\partial p_{i'}}\frac{\partial p_{i'}}{\partial b_i}\right\},$$

$$0 = \frac{\partial b_k}{\partial a_i} = \Sigma_{i'}\left\{\frac{\partial b_k}{\partial q_{i'}}\frac{\partial q_{i'}}{\partial a_i} + \frac{\partial b_k}{\partial p_{i'}}\frac{\partial p_{i'}}{\partial a_i}\right\},$$

$$0 = \frac{\partial b_k}{\partial b_i} = \Sigma_{i'}\left\{\frac{\partial b_k}{\partial q_{i'}}\frac{\partial q_{i'}}{\partial b_i} + \frac{\partial b_k}{\partial p_{i'}}\frac{\partial p_{i'}}{\partial b_i}\right\},$$

exceptis casibus, quibus in aequatione prima et quarta fit $k = i$, quibus casibus habetur:

$$1 = \Sigma_{i'}\left\{\frac{\partial a_i}{\partial q_{i'}}\frac{\partial q_{i'}}{\partial a_i} + \frac{\partial a_i}{\partial p_{i'}}\frac{\partial p_{i'}}{\partial a_i}\right\},$$

$$1 = \Sigma_{i'}\left\{\frac{\partial b_i}{\partial q_{i'}}\frac{\partial q_{i'}}{\partial b_i} + \frac{\partial b_i}{\partial p_{i'}}\frac{\partial p_{i'}}{\partial b_i}\right\}.$$

Substituamus in formulis praecedentibus aequationes (10.*), (11.*), (13.*), (14.*) §. 69

$$\frac{\partial p_{i'}}{\partial a_i} = \frac{\partial b_i}{\partial q_{i'}}, \quad \frac{\partial q_{i'}}{\partial a_i} = -\frac{\partial b_i}{\partial p_{i'}}, \quad \frac{\partial p_{i'}}{\partial b_i} = -\frac{\partial a_i}{\partial q_{i'}}, \quad \frac{\partial q_{i'}}{\partial b_i} = \frac{\partial a_i}{\partial p_{i'}},$$

abeunt illae in sequentes:

$$-[a_k, b_i] = 0, \quad [a_k, a_i] = 0, \quad [b_i, b_k] = 0, \quad -[a_i, b_k] = 0,$$
$$1 = -[a_i, b_i] = [b_i, a_i],$$

quae conveniunt cum aequationibus demonstrandis (1.), (3.).

Porro in iisdem formulis easdem substituamus aequationes (10.*) — (14.*) §. 69, in quibus tamen loco indicis $i$ scribamus $k$, unde evadunt:

$$\frac{\partial b_k}{\partial q_{i'}} = \frac{\partial p_{i'}}{\partial a_k}, \quad \frac{\partial b_k}{\partial p_{i'}} = -\frac{\partial q_{i'}}{\partial a_k}, \quad \frac{\partial a_k}{\partial q_{i'}} = -\frac{\partial p_{i'}}{\partial b_k}, \quad \frac{\partial a_k}{\partial p_{i'}} = \frac{\partial q_{i'}}{\partial b_k}.$$

Quibus substitutis, aequationes supra traditae in sequentes abeunt:

$$0 = (a_i, b_k), \quad 0 = (b_i, b_k), \quad 0 = (a_k, a_i), \quad 0 = (a_k, b_i)$$
$$1 = (a_i, b_i),$$

quae sunt aequationes demonstrandae (2.), (4.).

Sint $\varphi$, $\psi$ datae quaecunque functiones ipsarum $a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m$, quae quantitates $t_1, t_2, \ldots t_\mu$ non contineant. Substitutis ipsarum $a_i$, $b_i$, valoribus per $q_i$, $p_i$, $t_i$ expressis, evadant $\varphi$, $\psi$ harum quantitatum functiones, eritque

$$[\varphi, \psi] = \Sigma_{i'}\left\{\frac{\partial\varphi}{\partial q_{i'}}\frac{\partial\psi}{\partial p_{i'}} - \frac{\partial\varphi}{\partial p_{i'}}\frac{\partial\psi}{\partial q_{i'}}\right\} =$$

$$\Sigma_{i'}\left\{\begin{array}{l} \Sigma_i\left(\frac{\partial\varphi}{\partial a_i}\frac{\partial a_i}{\partial q_{i'}} + \frac{\partial\varphi}{\partial b_i}\frac{\partial b_i}{\partial q_{i'}}\right)\Sigma_k\left(\frac{\partial\psi}{\partial a_k}\frac{\partial a_k}{\partial p_{i'}} + \frac{\partial\psi}{\partial b_k}\frac{\partial b_k}{\partial p_{i'}}\right) \\ -\Sigma_i\left(\frac{\partial\varphi}{\partial a_i}\frac{\partial a_i}{\partial p_{i'}} + \frac{\partial\varphi}{\partial b_i}\frac{\partial b_i}{\partial p_{i'}}\right)\Sigma_k\left(\frac{\partial\psi}{\partial a_k}\frac{\partial a_k}{\partial q_{i'}} + \frac{\partial\psi}{\partial b_k}\frac{\partial b_k}{\partial q_{i'}}\right) \end{array}\right\}.$$

Quae expressio sic repraesentari potest:

$$[\varphi, \psi] = \Sigma_{i,k}\left\{\frac{\partial\varphi}{\partial a_i}\frac{\partial\psi}{\partial a_k}[a_i, a_k] + \frac{\partial\varphi}{\partial a_i}\frac{\partial\psi}{\partial b_k}[a_i, b_k]\right\}$$
$$+\Sigma_{i,k}\left\{\frac{\partial\varphi}{\partial b_i}\frac{\partial\psi}{\partial a_k}[b_i, a_k] + \frac{\partial\varphi}{\partial b_i}\frac{\partial\psi}{\partial b_k}[b_i, b_k]\right\},$$

unde e (1.), (3.) habetur:

$$[\varphi, \psi] = \Sigma_i\left\{\frac{\partial\varphi}{\partial b_i}\frac{\partial\psi}{\partial a_i} - \frac{\partial\varphi}{\partial a_i}\frac{\partial\psi}{\partial b_i}\right\},$$

sive

$$(5.)\quad \left\{\begin{array}{l} [\varphi, \psi] = \frac{\partial\varphi}{\partial q_1}\frac{\partial\psi}{\partial p_1} + \frac{\partial\varphi}{\partial q_2}\frac{\partial\psi}{\partial p_2} + \cdots + \frac{\partial\varphi}{\partial q_m}\frac{\partial\psi}{\partial p_m} \\ \qquad - \frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial q_1} - \frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_2} - \cdots - \frac{\partial\varphi}{\partial p_m}\frac{\partial\psi}{\partial q_m} \\ \quad = \frac{\partial\varphi}{\partial b_1}\frac{\partial\psi}{\partial a_1} + \frac{\partial\varphi}{\partial b_2}\frac{\partial\psi}{\partial a_2} + \cdots + \frac{\partial\varphi}{\partial b_m}\frac{\partial\psi}{\partial a_m} \\ \qquad - \frac{\partial\varphi}{\partial a_1}\frac{\partial\psi}{\partial b_1} - \frac{\partial\varphi}{\partial a_2}\frac{\partial\psi}{\partial b_2} - \cdots - \frac{\partial\varphi}{\partial a_m}\frac{\partial\psi}{\partial b_m}. \end{array}\right.$$

Quoties igitur accidit, ut $\varphi$, $\psi$ sint eiusmodi ipsarum $q_i$, $p_i$, $t_i$ functiones, quae per solas $a_i$, $b_i$ absque quantitatibus $t_i$ exprimi queant, erit etiam

$$[\varphi, \psi] = \frac{\partial\varphi}{\partial q_1}\frac{\partial\psi}{\partial p_1}+\frac{\partial\varphi}{\partial q_2}\frac{\partial\psi}{\partial p_2}+\cdots+\frac{\partial\varphi}{\partial q_m}\frac{\partial\psi}{\partial p_m}$$
$$-\frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial q_1}-\frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_2}-\cdots-\frac{\partial\varphi}{\partial p_m}\frac{\partial\psi}{\partial q_m}$$

eiusmodi functio, quippe quae aequalis fit expressioni

$$\frac{\partial\varphi}{\partial b_1}\frac{\partial\psi}{\partial a_1}+\frac{\partial\varphi}{\partial b_2}\frac{\partial\psi}{\partial a_2}+\cdots+\frac{\partial\varphi}{\partial b_m}\frac{\partial\psi}{\partial a_m}$$
$$-\frac{\partial\varphi}{\partial a_1}\frac{\partial\psi}{\partial b_1}-\frac{\partial\varphi}{\partial a_2}\frac{\partial\psi}{\partial b_2}-\cdots-\frac{\partial\varphi}{\partial a_m}\frac{\partial\psi}{\partial b_m},$$

quae, si $\varphi$ et $\psi$ sint solarum $a_i$, $b_i$ functiones ab ipsis $t_i$ vacuae, et ipsa erit solarum $a_i$, $b_i$ functio ab ipsis $t_i$ libera. Si quantitatum $t_i$ una tantum in problemate proposito adest, quam $t$ vocemus, redit propositio antecedens in eam, quam olim ill. *Poisson* demonstravit, quoties $\varphi = \text{Const.}$, $\psi = \text{Const.}$ sint integralia systematis aequationum differentialium

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \text{ *)},$$

quantitatem $[\varphi, \psi]$ per sola elementa absque $t$ exprimi.

Casus specialis aequationis (5.) valde memorabilis is est, quo functio $\psi$ uni quantitatum $a_i$, $b_i$ aequalis existit; tum enim abit aequatio illa in has simplices:

$$(6.) \quad \begin{cases} [\varphi, a_i] = \dfrac{\partial\varphi}{\partial b_i}, \\ [\varphi, b_i] = -\dfrac{\partial\varphi}{\partial a_i}. \end{cases}$$

Docent hae aequationes sequentia: *Quoties enim habetur aequationum differentialium propositarum integrale*

$$\varphi = \text{Const.},$$

*$\varphi$ per ipsas $a_i$, $b_i$ absque $t$ exprimi potest, quam vero expressionem ipsam*

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*) Aequationes differentiales ill. *Poisson* sub alia forma exhibuit; licet enim iam ille animadverterit in commentatione prima de Variatione Constantium, expressiones ipsis $\frac{dq_i}{dt}$, $\frac{dq_k}{dt}$ aequales per $q_i$, $p_i$ exhibitas ita comparatas esse, ut prioris differentiale secundum $p_k$ alterius differentiali secundum $p_i$ aequale sit; expressionem simplicem ipsius $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$, unde illud sponte sequitur, primus ill. *Hamilton* dedit.

*generaliter exhibere non possumus, nisi omnium $a_i$, $b_i$ expressiones per $q_i$, $p_i$, $t$ datae sint. At si vel unius novimus elementi, quod ad systema elementorum canonicum pertinet, expressionem per $q_i$, $p_i$, $t$, directe invenire licet differentialia ipsius $\varphi$ secundum elementum coniugatum sumta et ipsa per $q_i$, $p_i$, $t$ expressa, siquidem bina $a_i$ et $b_i$ elementa coniugata dicimus.* Nam si elementum canonicum datum est $a_i$, habetur e (6.):

$$\frac{\partial\varphi}{\partial b_i} = [\varphi, a_i],$$

unde ponendo $\frac{\partial\varphi}{\partial b_i}$, $\frac{\partial^2\varphi}{\partial b_i^2}$, etc. loco $\varphi$, prodit:

$$\frac{\partial^2\varphi}{\partial b_i^2} = \left[\frac{\partial\varphi}{\partial b_i}, a_i\right], \quad \frac{\partial^3\varphi}{\partial b_i^3} = \left[\frac{\partial^2\varphi}{\partial b_i^2}, a_i\right], \quad \text{etc.}$$

unde successive omnium $\frac{\partial^n\varphi}{\partial b_i^n}$ innotescunt valores per $q_i$, $p_i$, $t$ expressi. Si tantum unum habetur integrale, $\psi = a_i$, poterit constans ipsi $\psi$ aequalis pro elemento canonico accipi; excipias tamen casum, quo $\psi = H$, quod fieri potest, si $H$ ipsam $t$ non involvit, quippe quo casu, quoties $\varphi = \text{Const.}$ est integrale alterum, habetur $[\varphi, a_i] = 0$ neque aliquid novi inde prodit.

Ad illustrandas aequationes (6.), quae magnas partes agere debent in ulterioribus et altioribus disquisitionibus, quas integrationes propositae flagitant, ut omnia quae hic adhuc latent, enucleentur, directe eas de aequationibus §. 69 propositis deducam. Quod fit per considerationes sequentes. Sit enim $a_i$ data functio ipsarum $q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m, t$, ac consideremus ipsam $t$, siquidem $t$ in functione $a_i$ invenitur, ut constantem datam atque omnes $a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m$ praeter unam $b_i$ ut constantes arbitrarias, erunt $q_1, q_2, \ldots q_m, p_1, p_2, \ldots p_m$ functiones ipsius $b_i$, quae satisfaciunt aequationibus differentialibus (13.*), (14.*) §. 69:

$$\frac{dq_1}{db_i} = \frac{\partial a_i}{\partial p_1}, \quad \frac{dq_2}{db_i} = \frac{\partial a_i}{\partial p_2}, \quad \ldots \quad \frac{dq_m}{db_i} = \frac{\partial a_i}{\partial p_m},$$

$$\frac{dp_1}{db_i} = -\frac{\partial a_i}{\partial q_1}, \quad \frac{dp_2}{db_i} = -\frac{\partial a_i}{\partial q_2}, \quad \ldots \quad \frac{dp_m}{db_i} = -\frac{\partial a_i}{\partial q_m},$$

quae plane eandem formam habent atque aequationes differentiales propositae, modo functio $a_i$ functionis $H$ atque variabilis $b_i$ variabilis $t$ locum tenet. Per regulas autem vulgares differentiationis ex aequationibus praecedentibus cuiuslibet functionis ipsarum $q_{i'}$, $p_{i'}$ differentialia prima, secunda, tertia etc. successive eruuntur, continuo differentialium $\frac{dq_{i'}}{db_i}$, $\frac{dp_{i'}}{db_i}$ valores substituendo;

quemadmodum tritum est, ex systemate aequationum differentialium

$$\frac{dq_{i'}}{dt} = \frac{\partial H}{\partial p_{i'}}, \quad \frac{dp_{i'}}{dt} = -\frac{\partial H}{\partial q_{i'}}$$

cuiuslibet functionis differentialia cuiuslibet ordinis per ipsas $q_{i'}$, $p_{i'}$, $t$, expressa inveniri posse. Si ex. gr. functionis $\varphi$ differentiale primum secundum $b_i$ quaeris, eruis:

$$\begin{aligned}\frac{d\varphi}{db_i} &= \frac{\partial\varphi}{\partial q_1}\frac{dq_1}{db_i} + \frac{\partial\varphi}{\partial q_2}\frac{dq_2}{db_i} + \cdots + \frac{\partial\varphi}{\partial q_m}\frac{dq_m}{db_i} \\ &+ \frac{\partial\varphi}{\partial p_1}\frac{dp_1}{db_i} + \frac{\partial\varphi}{\partial p_2}\frac{dp_2}{db_i} + \cdots + \frac{\partial\varphi}{\partial p_m}\frac{dp_m}{db_i} \\ &= \frac{\partial\varphi}{\partial q_1}\frac{\partial a_i}{\partial p_1} + \frac{\partial\varphi}{\partial q_2}\frac{\partial a_i}{\partial p_2} + \cdots + \frac{\partial\varphi}{\partial q_m}\frac{\partial a_i}{\partial p_m} \\ &- \frac{\partial\varphi}{\partial p_1}\frac{\partial a_i}{\partial q_1} - \frac{\partial\varphi}{\partial p_2}\frac{\partial a_i}{\partial q_2} - \cdots - \frac{\partial\varphi}{\partial p_m}\frac{\partial a_i}{\partial q_m} \\ &= [\varphi, a_i],\end{aligned}$$

quae est altera aequationum (6.); eademque methodo demonstratur altera.

Observo adhuc, in formulis §. 69 et antecedentibus, quae ex iis derivatae sunt, ubique $a$, $b$ atque $q$, $p$ inter se permutari posse.

Ipsis $a_i$, $b_i$ per alias quantitates $\alpha$, $\beta$, $\gamma$ etc. expressis, quaeramus adhuc valorem expressionis:

$$\begin{aligned}(\alpha, \beta) &= \frac{\partial q_1}{\partial\beta}\frac{\partial p_1}{\partial\alpha} + \frac{\partial q_2}{\partial\beta}\frac{\partial p_2}{\partial\alpha} + \cdots + \frac{\partial q_m}{\partial\beta}\frac{\partial p_m}{\partial\alpha} \\ &- \frac{\partial q_1}{\partial\alpha}\frac{\partial p_1}{\partial\beta} - \frac{\partial q_2}{\partial\alpha}\frac{\partial p_2}{\partial\beta} - \cdots - \frac{\partial q_m}{\partial\alpha}\frac{\partial p_m}{\partial\beta}.\end{aligned}$$

Fit $(\alpha, \beta) =$

$$\begin{aligned}&\Sigma\left\{\left(\frac{\partial q_{i'}}{\partial a_i}\frac{\partial a_i}{\partial\beta} + \frac{\partial q_{i'}}{\partial b_i}\frac{\partial b_i}{\partial\beta}\right)\left(\frac{\partial p_{i'}}{\partial a_k}\frac{\partial a_k}{\partial\alpha} + \frac{\partial p_{i'}}{\partial b_k}\frac{\partial b_k}{\partial\alpha}\right)\right\} \\ -&\Sigma\left\{\left(\frac{\partial q_{i'}}{\partial a_k}\frac{\partial a_k}{\partial\alpha} + \frac{\partial q_{i'}}{\partial b_k}\frac{\partial b_k}{\partial\alpha}\right)\left(\frac{\partial p_{i'}}{\partial a_i}\frac{\partial a_i}{\partial\beta} + \frac{\partial p_{i'}}{\partial b_i}\frac{\partial b_i}{\partial\beta}\right)\right\},\end{aligned}$$

indicibus $i'$, $i$, $k$ tributis valoribus 1, 2, ... $m$. Evolutis productis ex aequatione praecedente eruimus $(\alpha, \beta) =$

$$\Sigma\left\{(a_k, a_i)\frac{\partial a_i}{\partial\beta}\frac{\partial a_k}{\partial\alpha} + (b_k, a_i)\frac{\partial a_i}{\partial\beta}\frac{\partial b_k}{\partial\alpha} + (a_k, b_i)\frac{\partial b_i}{\partial\beta}\frac{\partial a_k}{\partial\alpha} + (b_k, b_i)\frac{\partial b_i}{\partial\beta}\frac{\partial b_k}{\partial\alpha}\right\},$$

indicibus $i$ et $k$ sub signo summatorio tributis valoribus 1, 2, ... $m$. Sed e formulis (2.) eavnescunt sub signo summatorio termini omnes, pro quibus $k$ et $i$ inter se diversi sunt; unde quum e (4.) sit

$$(a_i, b_i) = 1,$$

atque sponte pateat fieri

$$(a_i, a_i) = 0, \quad (b_i, b_i) = 0,$$

sequitur

$$(\alpha, \beta) = \Sigma \left\{ \frac{\partial a_i}{\partial \alpha} \frac{\partial b_i}{\partial \beta} - \frac{\partial a_i}{\partial \beta} \frac{\partial b_i}{\partial \alpha} \right\},$$

sive

$$(7.) \quad \left\{ \begin{aligned} (\alpha, \beta) &= \frac{\partial q_1}{\partial \beta} \frac{\partial p_1}{\partial \alpha} + \frac{\partial q_2}{\partial \beta} \frac{\partial p_2}{\partial \alpha} + \cdots + \frac{\partial q_m}{\partial \beta} \frac{\partial p_m}{\partial \alpha} \\ &\quad - \frac{\partial q_1}{\partial \alpha} \frac{\partial p_1}{\partial \beta} - \frac{\partial q_2}{\partial \alpha} \frac{\partial p_2}{\partial \beta} - \cdots - \frac{\partial q_m}{\partial \alpha} \frac{\partial p_m}{\partial \beta} \\ &= \frac{\partial a_1}{\partial \alpha} \frac{\partial b_1}{\partial \beta} + \frac{\partial a_2}{\partial \alpha} \frac{\partial b_2}{\partial \beta} + \cdots + \frac{\partial a_m}{\partial \alpha} \frac{\partial b_m}{\partial \beta} \\ &\quad - \frac{\partial a_1}{\partial \beta} \frac{\partial b_1}{\partial \alpha} - \frac{\partial a_2}{\partial \beta} \frac{\partial b_2}{\partial \alpha} - \cdots - \frac{\partial a_m}{\partial \beta} \frac{\partial b_m}{\partial \alpha}. \end{aligned} \right.$$

Statuamus, ipsam $\beta$ ex elementis canonicis esse sive haberi $\beta = a_i$ aut $\beta = b_i$, atque reliquorum elementorum expressiones hoc elementum non continere. Quo casu formula antecedens abit in sequentes simplices:

$$(8.) \quad \left\{ \begin{aligned} (\alpha, a_i) &= -\frac{\partial b_i}{\partial \alpha}, \\ (\alpha, b_i) &= \frac{\partial a_i}{\partial \alpha}. \end{aligned} \right.$$

Quibus adnotatis pauca de formulis generalibus perturbatoriis addam, quae de systemate elementorum quocunque valent.

*Formularum perturbatoriarum systemata quae ill. Lagrange et Poisson posuerunt, demonstrantur et alterum ex altero derivantur.*

73.

Loco ipsorum $a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m$ habeatur systema elementorum quodcunque, $\alpha_1, \alpha_2, \ldots \alpha_{2m}$. Quorum respectu formulas perturbatorias sub duabus maxime formis proponere convenit. In altera, quae ill[i] *Lagrange* est, differentialia partialia functionis perturbatricis $\Omega$ secundum elementa sumta lineariter exprimuntur per differentialia elementorum; in altera, quae ill[i] *Poisson* est, differentialia elementorum perturbatorum lineariter exprimuntur per differentialia partialia functionis perturbatricis $\Omega$ secundum elementa sumta. In altera forma expressionum linearium coefficientes sunt functiones $(\alpha_i, \alpha_k)$, in altera functiones $[\alpha_i, \alpha_k]$. Plerumque adnotari solet, alteram formam ex altera per solam resolutionem aequationum $2m$ linearium obtineri posse. Sed nemo, quantum scio, hanc resolutionem reapse tentavit eaque via directa alteram formam de altera derivavit. Quod quum utile sit et difficultatis speciem quandam habeat, ego sequentibus exponam; antea autem formulas perturbatorias generales de formulis supra traditis deducam, licet eaedem directe ex ipsis aequationibus differentialibus peti possint, sicuti plerumque fit.

Spectentur *primum* elementa canonica $a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m$ ut functiones aliorum elementorum quorumcunque $\alpha_1, \alpha_2, \ldots \alpha_{2m}$; erit e §. 52:

$$\begin{aligned}\frac{\partial\Omega}{\partial\alpha_n} &= \Sigma_i \left\{\frac{\partial\Omega}{\partial a_i}\frac{\partial a_i}{\partial\alpha_n}+\frac{\partial\Omega}{\partial b_i}\frac{\partial b_i}{\partial\alpha_n}\right\} \\ &= \Sigma_i \left\{\frac{db_i}{dt}\frac{\partial a_i}{\partial\alpha_n}-\frac{da_i}{dt}\frac{\partial b_i}{\partial\alpha_n}\right\} \\ &= \Sigma_{i,k} \left\{\frac{\partial b_i}{\partial\alpha_k}\frac{\partial a_i}{\partial\alpha_n}-\frac{\partial a_i}{\partial\alpha_k}\frac{\partial b_i}{\partial\alpha_n}\right\}\frac{d\alpha_k}{dt},\end{aligned}$$

quibus in summis ipsi $i$ valores $1, 2, \ldots m$, ipsi $k$ valores $1, 2, \ldots 2m$ tribuendi sunt. Unde e (7.) §. antec. fit

$$(1.)\quad \begin{cases}\dfrac{\partial\Omega}{\partial\alpha_n} = \Sigma_k(\alpha_n, \alpha_k)\dfrac{d\alpha_k}{dt} \\ \quad = (\alpha_n, \alpha_1)\dfrac{d\alpha_1}{dt}+(\alpha_n, \alpha_2)\dfrac{d\alpha_2}{dt}+\cdots+(\alpha_n, \alpha_{2m})\dfrac{d\alpha_{2m}}{dt}.\end{cases}$$

Spectentur *deinde* $\alpha_1, \alpha_2, \ldots \alpha_{2m}$ ut functiones ipsarum $a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_m$: habetur e §. 52:

$$\begin{aligned}\frac{d\alpha_n}{dt} &= \Sigma_i \left\{\frac{\partial\alpha_n}{\partial a_i}\frac{da_i}{dt}+\frac{\partial\alpha_n}{\partial b_i}\frac{db_i}{dt}\right\} \\ &= \Sigma_i \left\{-\frac{\partial\alpha_n}{\partial a_i}\frac{\partial\Omega}{\partial b_i}+\frac{\partial\alpha_n}{\partial b_i}\frac{\partial\Omega}{\partial a_i}\right\} \\ &= \Sigma_{i,k} \left\{-\frac{\partial\alpha_n}{\partial a_i}\frac{\partial\alpha_k}{\partial b_i}+\frac{\partial\alpha_n}{\partial b_i}\frac{\partial\alpha_k}{\partial a_i}\right\}\frac{\partial\Omega}{\partial\alpha_k},\end{aligned}$$

quibus in summis ipsi $i$ rursus valores $1, 2, \ldots m$, ipsi $k$ valores $1, 2, \ldots 2m$ tribuendi sunt. Unde e (5.) §. antec. scribendo $\alpha_n, \alpha_k$ loco $\varphi, \psi$ prodit:

$$(2.)\quad \begin{cases}\dfrac{d\alpha_n}{dt} = \Sigma_k[\alpha_n, \alpha_k]\dfrac{\partial\Omega}{\partial\alpha_k} \\ \quad = [\alpha_n, \alpha_1]\dfrac{\partial\Omega}{\partial\alpha_1}+[\alpha_n, \alpha_2]\dfrac{\partial\Omega}{\partial\alpha_2}+\cdots+[\alpha_n, \alpha_{2m}]\dfrac{\partial\Omega}{\partial\alpha_{2m}}.\end{cases}$$

Formulae (1.) ab ill. *Lagrange,* formulae (2.) ab ill. *Poisson* traditae sunt. Aliae de aliis derivari possunt ope theorematis sequentis:

### Theorema:

*Sint* $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$ *functiones quaecunque a se invicem independentes quantitatum* $\alpha_1, \alpha_2, \dots \alpha_{2m}$, *ita ut invicem spectari possint* $\alpha_1, \alpha_2, \dots \alpha_{2m}$ *ut functiones a se invicem independentes quantitatum* $q_1, q_2, \dots q_m, p_1, p_2, \dots p_m$; *statuatur in suppositione priori:*

$$(\alpha_i, \alpha_k) = \frac{\partial q_1}{\partial \alpha_k}\frac{\partial p_1}{\partial \alpha_i} + \frac{\partial q_2}{\partial \alpha_k}\frac{\partial p_2}{\partial \alpha_i} + \dots + \frac{\partial q_m}{\partial \alpha_k}\frac{\partial p_m}{\partial \alpha_i}$$
$$- \frac{\partial q_1}{\partial \alpha_i}\frac{\partial p_1}{\partial \alpha_k} - \frac{\partial q_2}{\partial \alpha_i}\frac{\partial p_2}{\partial \alpha_k} - \dots - \frac{\partial q_m}{\partial \alpha_i}\frac{\partial p_m}{\partial \alpha_k},$$

*in suppositione posteriori:*

$$[\alpha_i, \alpha_k] = \frac{\partial \alpha_i}{\partial q_1}\frac{\partial \alpha_k}{\partial p_1} + \frac{\partial \alpha_i}{\partial q_2}\frac{\partial \alpha_k}{\partial p_2} + \dots + \frac{\partial \alpha_i}{\partial q_m}\frac{\partial \alpha_k}{\partial p_m}$$
$$- \frac{\partial \alpha_i}{\partial p_1}\frac{\partial \alpha_k}{\partial q_1} - \frac{\partial \alpha_i}{\partial p_2}\frac{\partial \alpha_k}{\partial q_2} - \dots - \frac{\partial \alpha_i}{\partial p_m}\frac{\partial \alpha_k}{\partial q_m};$$

*quibus statutis significationibus, si proponuntur* $2m$ *aequationes lineares sequentes:*

$$\begin{aligned}
v_1 &= \quad * \quad + (\alpha_1, \alpha_2)u_2 + (\alpha_1, \alpha_3)u_3 + \dots + (\alpha_1, \alpha_{2m})u_{2m}, \\
v_2 &= (\alpha_2, \alpha_1)u_1 + \quad * \quad + (\alpha_2, \alpha_3)u_3 + \dots + (\alpha_2, \alpha_{2m})u_{2m}, \\
v_3 &= (\alpha_3, \alpha_1)u_1 + (\alpha_3, \alpha_2)u_2 + \quad * \quad + \dots + (\alpha_3, \alpha_{2m})u_{2m}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
v_{2m} &= (\alpha_{2m}, \alpha_1)u_1 + (\alpha_{2m}, \alpha_2)u_2 + (\alpha_{2m}, \alpha_3)u_3 + \dots + \quad * \quad ,
\end{aligned}$$

*eruuntur resolutione harum aequationum valores ipsarum* $u_1, u_2, \dots u_{2m}$ *sequentes:*

$$\begin{aligned}
u_1 &= \quad * \quad + [\alpha_1, \alpha_2]v_2 + [\alpha_1, \alpha_3]v_3 + \dots + [\alpha_1, \alpha_{2m}]v_{2m}, \\
u_2 &= [\alpha_2, \alpha_1]v_1 + \quad * \quad + [\alpha_2, \alpha_3]v_3 + \dots + [\alpha_2, \alpha_{2m}]v_{2m}, \\
u_3 &= [\alpha_3, \alpha_1]v_1 + [\alpha_3, \alpha_2]v_2 + \quad * \quad + \dots + [\alpha_3, \alpha_{2m}]v_{2m}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
u_{2m} &= [\alpha_{2m}, \alpha_1]v_1 + [\alpha_{2m}, \alpha_2]v_2 + [\alpha_{2m}, \alpha_3]v_3 + \dots + \quad *
\end{aligned}$$

*et vice versa harum aequationum resolutione illae obtinentur.*

Demonstratio:

Multiplicemus aequationes propositas per

$$[\alpha_i, \alpha_1], \quad [\alpha_i, \alpha_2], \quad \dots \quad [\alpha_i, \alpha_{2m}]$$

et productorum summationem instituamus. Unde prodibit expressio huiusmodi:

$$[\alpha_i, \alpha_1]v_1 + [\alpha_i, \alpha_2]v_2 + \dots + [\alpha_i, \alpha_{2m}]v_{2m} = A_1 u_1 + A_2 u_2 + \dots + A_{2m} u_{2m},$$

in qua:

$$\begin{aligned} A_k &= [\alpha_i, \alpha_1](\alpha_1, \alpha_k) + [\alpha_i, \alpha_2](\alpha_2, \alpha_k) + \dots + [\alpha_i, \alpha_{2m}](\alpha_{2m}, \alpha_k) \\ &= \Sigma_n [\alpha_i, \alpha_n](\alpha_n, \alpha_k) \\ &= \Sigma_n \left\{ \begin{array}{l} \left\{ \begin{array}{l} \frac{\partial \alpha_i}{\partial q_1}\frac{\partial \alpha_n}{\partial p_1} + \frac{\partial \alpha_i}{\partial q_2}\frac{\partial \alpha_n}{\partial p_2} + \dots + \frac{\partial \alpha_i}{\partial q_m}\frac{\partial \alpha_n}{\partial p_m} \\ - \frac{\partial \alpha_i}{\partial p_1}\frac{\partial \alpha_n}{\partial q_1} - \frac{\partial \alpha_i}{\partial p_2}\frac{\partial \alpha_n}{\partial q_2} - \dots - \frac{\partial \alpha_i}{\partial p_m}\frac{\partial \alpha_n}{\partial q_m} \end{array} \right\} \times \\ \left\{ \begin{array}{l} \frac{\partial q_1}{\partial \alpha_k}\frac{\partial p_1}{\partial \alpha_n} + \frac{\partial q_2}{\partial \alpha_k}\frac{\partial p_2}{\partial \alpha_n} + \dots + \frac{\partial q_m}{\partial \alpha_k}\frac{\partial p_m}{\partial \alpha_n} \\ - \frac{\partial p_1}{\partial \alpha_k}\frac{\partial q_1}{\partial \alpha_n} - \frac{\partial p_2}{\partial \alpha_k}\frac{\partial q_2}{\partial \alpha_n} - \dots - \frac{\partial p_m}{\partial \alpha_k}\frac{\partial q_m}{\partial \alpha_n} \end{array} \right\} \end{array} \right\}, \end{aligned}$$

qua in summa ipsi $n$ valores $1, 2, \dots 2m$ tribuendi sunt. Eandem expressionem facta multiplicatione sic repraesentare licet:

$$\begin{aligned} A_k = \; & \Sigma_{i',k'} \left\{ \frac{\partial \alpha_i}{\partial q_{i'}} \frac{\partial q_{k'}}{\partial \alpha_k} \cdot \Sigma_n \frac{\partial p_{k'}}{\partial \alpha_n} \frac{\partial \alpha_n}{\partial p_{i'}} \right\} \\ & + \Sigma_{i',k'} \left\{ \frac{\partial \alpha_i}{\partial p_{i'}} \frac{\partial p_{k'}}{\partial \alpha_k} \cdot \Sigma_n \frac{\partial q_{k'}}{\partial \alpha_n} \frac{\partial \alpha_n}{\partial q_{i'}} \right\} \\ & - \Sigma_{i',k'} \left\{ \frac{\partial \alpha_i}{\partial q_{i'}} \frac{\partial p_{k'}}{\partial \alpha_k} \cdot \Sigma_n \frac{\partial q_{k'}}{\partial \alpha_n} \frac{\partial \alpha_n}{\partial p_{i'}} \right\} \\ & - \Sigma_{i',k'} \left\{ \frac{\partial \alpha_i}{\partial p_{i'}} \frac{\partial q_{k'}}{\partial \alpha_k} \cdot \Sigma_n \frac{\partial p_{k'}}{\partial \alpha_n} \frac{\partial \alpha_n}{\partial q_{i'}} \right\}, \end{aligned}$$

quibus in summis ipsi $n$ valores $1, 2, \dots 2m$, ipsis $i'$, $k'$ valores $1, 2, \dots m$ tribuendi sunt. Iam vero habetur:

$$\Sigma_n \frac{\partial p_{k'}}{\partial \alpha_n} \frac{\partial \alpha_n}{\partial p_{i'}} = \frac{\partial p_{k'}}{\partial p_{i'}},$$

$$\Sigma_n \frac{\partial q_{k'}}{\partial \alpha_n} \frac{\partial \alpha_n}{\partial q_{i'}} = \frac{\partial q_{k'}}{\partial q_{i'}},$$

$$\Sigma_n \frac{\partial q_{k'}}{\partial \alpha_n} \frac{\partial \alpha_n}{\partial p_{i'}} = \frac{\partial q_{k'}}{\partial p_{i'}},$$

$$\Sigma_n \frac{\partial p_{k'}}{\partial \alpha_n} \frac{\partial \alpha_n}{\partial q_{i'}} = \frac{\partial p_{k'}}{\partial q_{i'}},$$

quarum expressionum tertia et quarta semper evanescunt, prima et secunda evanescunt, si $i'$ et $k'$ inter se diversi sunt, in unitatem abeunt, si $i' = k'$. Unde fit:

$$A_k = \Sigma_{i'} \left\{ \frac{\partial \alpha_i}{\partial q_{i'}} \frac{\partial q_{i'}}{\partial \alpha_k} + \frac{\partial \alpha_i}{\partial p_{i'}} \frac{\partial p_{i'}}{\partial \alpha_k} \right\} = \frac{\partial \alpha_i}{\partial \alpha_k}.$$

Quae expressio quum evanescat nisi sit $i = k$, hoc autem casu in unitatem abeat, videmus, in parte posteriore aequationis:

$$[\alpha_i, \alpha_1] v_1 + [\alpha_i, \alpha_2] v_2 + \cdots + [\alpha_i, \alpha_{2m}] v_{2m} = A_1 u_1 + A_2 u_2 + \cdots + A_{2m} u_{2m}$$

coefficientes $A_1, A_2, \ldots A_{2m}$ praeter unum $A_i$ evanescere omnes, fieri autem $A_i = 1$. Unde aequatio antecedens haec evadit:

$$[\alpha_i, \alpha_1] v_1 + [\alpha_i, \alpha_2] v_2 + \cdots + [\alpha_i, \alpha_{2m}] v_{2m} = u_i,$$

in qua, si ipsi $i$ successive valores $1, 2, \ldots 2m$ tribuuntur, eruuntur ipsarum $u_1, u_2, \ldots u_{2m}$ valores in theoremate proposito assignati.

Prorsus simili methodo vice versa e secundo systemate aequationum in theoremate antecedente propositarum systema primum derivari potest.