11.

Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi.

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Caput tertium.

Theoria Multiplicatoris systematis aequationum differentialium ad varia exempla applicata.

De Multiplicatore systematis aequationum differentialium cuiuslibet ordinis.

§. 14.

Aequationum differentialium systema, quo altissima quaeque variabilium dependentium differentialia per differentialia inferiora ipsasque variabiles exprimuntur, constat in systema redire aequationum differentialium primi ordinis, si cuiusque variabilis dependentis differentialia altissimo inferiora ipsis variabilibus adscribantur. Designantibus enim x, y etc. variabilis independentis t functiones, proponantur inter t, x, y etc. aequationes differentiales,

$$1.\quad \frac{d^p x}{dt^p} = A, \quad \frac{d^q y}{dt^q} = B, \text{ etc.}$$

ipsaeque A, B etc. non altioribus afficiantur differentialibus quam $(p-1)^{\text{to}}$ ipsius x, $(q-1)^{\text{to}}$ ipsius y etc. Patet, habendo pro novis variabilibus dependentibus differentialia, quae *Lagrangiano* more per indices denoto,

$$x' = \frac{dx}{dt}, \quad x'' = \frac{d^2x}{dt^2}, \quad \dots\dots \quad x^{(p-1)} = \frac{d^{p-1}x}{dt^{p-1}},$$

$$y' = \frac{dy}{dt}, \quad y'' = \frac{d^2y}{dt^2}, \quad \dots\dots \quad y^{(q-1)} = \frac{d^{q-1}y}{dt^{q-1}}, \text{ etc.},$$

aequationibus differentialibus (1.) has alias substitui posse ***primi*** ordinis:

$$2.\quad \left\{\begin{array}{l} dt : dx : dx' \dots : dx^{(p-2)} : dx^{(p-1)} \\ \quad : dy : dy' \dots : dy^{(q-2)} : dy^{(q-1)} \text{ etc.} \\ = 1 : x' : x'' \dots : x^{(p-1)} : A \\ \quad : y' : y'' \dots : y^{(q-1)} : B \text{ etc.} \end{array}\right.$$

Quibus in aequationibus variabilium numerus summam ordinum altissimorum differentialium in (1.) unitate superat.

Multiplicator aequationum differentialium primi ordinis (2.), cum quibus aequationes differentiales (1.) conveniunt, etiam a me in sequentibus appellabitur aequationum (1.) Multiplicator. Unde ut omnia theoremata de Multiplicatore aequationum differentialium primi ordinis in duobus Capitibus praecedentibus in medium prolata ad Multiplicatores aequationum differentialium cuiuslibet ordinis (1.) applicentur, sufficit ut pro aequationibus ibi propositis,

$$3.\quad dx:dx_1:dx_2\ldots.:dx_n = X:\ X_1:\ X_2\ldots.:\ X_n,$$

sumantur aequationes (2.).

Si aequationes differentiales primi ordinis (2.) et (3.) inter se comparamus, videmus in illis specialitatem quandam formae locum habere, videlicet quantitates primis differentialibus proportionales, quae generaliter variabilium functiones sunt, maximam partem in ipsas abire variabiles, neque vero in eas quarum differentialibus proportionales ponuntur. Quo habitu speciali fit ut aequationum (2.) Multiplicator, quem aequationum (1.) quoque Multiplicatorem voco, definiatur formula quae, tantopere licet aucto in (2.) variabilium numero, non pluribus constat terminis, quam si ipsae primi ordinis fuissent aequationes differentiales propositae (1.). Consideremus enim formulam ad definiendum aequationum (3.) Multiplicatorem propositam §. 7. (4.),

$$(4.)\quad \frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\ldots.+\frac{\partial X_n}{\partial x_n} = -X\frac{d\log M}{dx}.$$

Si pro aequationibus (3.) sumimus aequationes (2.) fit $x=t$, $X=1$; porro variabilibus x_1, x_2 etc. substituendae sunt

$$x,\ x',\ x'',\ \ldots.\ x^{(p-2)},\ x^{(p-1)},$$
$$y,\ y',\ y'',\ \ldots.\ y^{(q-2)},\ y^{(q-1)},\ \text{etc.};$$

functionibus denique X_1, X_2 etc. substituendae sunt quantitates

$$x',\ x'',\ x''',\ \ldots.\ x^{(p-1)},\ A,$$
$$y',\ y'',\ y''',\ \ldots.\ y^{(q-1)},\ B,\ \text{etc.}$$

Iam in (4.), quoties est X_i una e variabilibus x, x_1, x_2 etc., ab ipsa x_i diversa, evanescit terminus $\frac{\partial X_i}{\partial x_i}$, uti generaliter fit si functio X_i ipsam x_i non implicat. Unde sumendo pro (3.) aequationes (2.), abit aggregatum (4.) in hanc expressionem simplicem,

$$5.\quad \frac{\partial A}{\partial x^{(p-1)}}+\frac{\partial B}{\partial y^{(q-1)}}+\text{etc.} = -\frac{d\log M}{dt}.$$

Hac formula Multiplicator M definitur systematis aequationum differentialium cuiuslibet ordinis (1.).

Sequitur e (5.), quoties simul ipsum A a differentiali $(p-1)^{\text{to}}$ ipsius x, ipsum B a differentiali $(q-1)^{\text{to}}$ ipsius y etc. vacuum sit, sive generalius, *quoties aggregatum*

$$\frac{\partial A}{\partial x^{(p-1)}}+\frac{\partial B}{\partial y^{(q-1)}} \text{ etc.}$$

identice evanescat, statui posse $M=1$. Si aggregatum (5.) non identice evanescit, ad indagandum Multiplicatorem circumspiciendum erit differentiale completum, cui idem aggregatum sua sponte vel etiam per aequationes differentiales propositas aequetur.

Principium ultimi Multiplicatoris systemati aequationum differentialium cuiuslibet ordinis applicatum.

§. 15.

Aequationum differentialium propositarum (1.) §. pr. Integralibus praeter unum omnibus inventis, quantitates

$$(A.)\quad \begin{cases} t,\ x,\ x',\ \ldots\ldots\ x^{(p-1)}, \\ \quad\ y,\ y',\ \ldots\ldots\ y^{(q-1)} \text{ etc.} \end{cases}$$

omnes exprimere licet per duas u et v, pro quibus sumere licet binas e quantitatibus (*A.*) vel earum functiones quaslibet. Differentialia $\frac{du}{dt}$ et $\frac{dv}{dt}$, substituendo differentialibus $x^{(p)}$, $y^{(q)}$ etc. si opus est valores A, B etc., et ipsa aequantur quantitatum t, x, x' etc. functionibus. Quae functiones, Integralium inventorum ope per u et v expressae, si denotantur per

$$U=\frac{du}{dt},\qquad V=\frac{dv}{dt},$$

dabitur inter u et v aequatio differentialis primi ordinis, ultima quae integranda restat,

$$1.\qquad V du - U dv = 0.$$

Secundum ea quae §. 11. tradidi, cognito aequationum differentialium propositarum Multiplicatore M erui potest factor N qui eius ultimae aequationis differentialis (1.) laevam partem efficiat differentiale completum, quem ***ultimum Multiplicatorem*** appello. ***Habendo enim, quod per Integralia inventa licet, quantitates (A.) pro functionibus ipsarum u et v Constantiumque Arbitrariarum quas Integralia implicant, earumque functionum formando Determinans Δ, fit ultimus Multiplicator $N=\Delta.M$.***

Principium ultimi Multiplicatoris, quod propositione antecedente continetur, etiam sic concipi potest, *diviso ultimae aequationis differentialis* (1.) *Multiplicatore per Determinans Δ, conditionem **Eulerianam** pro Multiplicatore valentem transformari in aliam conditionem ab Integralibus reductioni adhibitis independentem, cui formulandae sufficiant solae aequationes differentiales propositae.*

Videlicet aequatio conditionalis, cui aequationis (1.) Multiplicator N satisfacere debet, fit

$$\frac{\partial . NU}{\partial u}+\frac{\partial . NV}{\partial v} = 0.$$

Quae ponendo

$$M = \frac{N}{\Delta}$$

et substituendo Constantibus Arbitrariis functiones quantitatum (A.) aequivalentes transformabitur in hanc,

$$\frac{d \log M}{dt}+\frac{\partial A}{\partial x^{(p-1)}}+\frac{\partial B}{\partial y^{(q-1)}} \text{ etc.} = 0,$$

cui formandae sufficiunt aequationes differentiales propositae (1.).

Sint $\Pi_1 = 0$, $\Pi_2 = 0$ etc. aequationes integrales reductioni adhibitae binaeque aequationes quibus u et v ab ipsis t, x, x' etc. pendent, sive etiam aliae quaecunque aequationes cum illis aequivalentes: constat e Determinantium functionalium proprietatibus, *aequari Δ fractioni, cuius denominator sit functionum Π_1, Π_2 etc. Determinans formatum quantitatum* (A.) *respectu, numerator autem earundem functionum Determinans, quantitatum u et v Constantiumque Arbitrariarum respectu formatum.* Si pro u et v ipsae sumuntur t et x, pro aequationibus $\Pi_1 = 0$, $\Pi_2 = 0$ etc. solae sumendae sunt aequationes integrales simulque t et x in binis Determinantibus formandis de numero variabilium tollendae sunt. Porro aequatio (1.) in hanc abit,

$$dx - V dt = 0,$$

ubi V est ipsius $\frac{dx}{dt}$ valor, Integralium inventorum ope per t et x expressus. Si aequationes $\Pi_1 = 0$, $\Pi_2 = 0$ etc. inventae sunt per integrationem successivam, ita ut in quaque aequatione insequente, in qua nova accedit Constans Arbitraria, simul unius variabilis differentiale altissimum ad ordinem proxime inferiorem sit depressum, alterutrum Determinans in unicum terminum abit. Sic proposita unica aequatione differentiali n^{ti} ordinis inter t et x,

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \dots \frac{d^{n-1}x}{dt^{n-1}}\right),$$

integratione successiva inventae sint aequationes,

$$2.\quad \begin{cases} \dfrac{d^{n-1}x}{dt^{n-1}} = f_1\left(t, x, \dfrac{dx}{dt}, \ldots \dfrac{d^{n-2}x}{dt^{n-2}}, \alpha_1\right), \\ \dfrac{d^{n-2}x}{dt^{n-2}} = f_2\left(t, x, \dfrac{dx}{dt}, \ldots \dfrac{d^{n-3}x}{dt^{n-3}}, \alpha_1, \alpha_2\right), \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \dfrac{dx}{dt} = f_{n-1}(t, x, \alpha_1, \alpha_2, \ldots \alpha_{n-1}), \end{cases}$$

in quibus $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$ sunt Constantes Arbitrariae: simpliciter erit

$$\Delta = \frac{\partial f_1}{\partial \alpha_1} \cdot \frac{\partial f_2}{\partial \alpha_2} \ldots \frac{\partial f_{n-1}}{\partial \alpha_{n-1}},$$

cum alterum Determinans in ipsam unitatem abeat. Si functio f ab ipso $\frac{d^{n-1}x}{dt^{n-1}}$ vacuum est, fit aequationis differentialis propositae Multiplicator $= 1$. Quo igitur casu hoc eruitur ultimum Integrale:

$$\int \frac{\partial f_1}{\partial \alpha_1} \cdot \frac{\partial f_2}{\partial \alpha_2} \ldots \frac{\partial f_{n-1}}{\partial \alpha_{n-1}} \{dx - f_{n-1}(t, x, \alpha_1, \alpha_2, \ldots \alpha_{n-1})\, dt\} = \text{Const.},$$

ubi quantitas sub integrationis signo, per t et x expressa, fit differentiale completum. Ut per solas t et x exprimatur valor producti

$$\frac{\partial f_1}{\partial \alpha_1} \cdot \frac{\partial f_2}{\partial \alpha_2} \ldots \frac{\partial f_{n-1}}{\partial \alpha_{n-1}},$$

sufficit ut in eo *successive* substituantur differentialium $\frac{d^{n-2}x}{dt^{n-2}}, \frac{d^{n-3}x}{dt^{n-3}}, \ldots \frac{dx}{dt}$ valores $f_2, f_3, \ldots f_{n-1}$.

Formula symbolica qua Multiplicator systematis aequationum differentialium impliciti definiri potest.

§. 16.

Aequationes differentiales, e quibus petantur altissimorum differentialium valores,

$$1.\quad x^{(p)} = A, \quad y^{(q)} = B, \quad \text{etc.}$$

ponamus forma dari implicita,

$$2.\quad \varphi = 0, \quad \psi = 0, \quad \text{etc.}$$

E quibus aequationibus ut eruantur valores differentialium partialium

$$\frac{\partial A}{\partial x^{(p-1)}}, \quad \frac{\partial B}{\partial y^{(q-1)}}, \quad \text{etc.},$$

quarum summa aequat ipsum $-\frac{d \log M}{dt}$, statuo

$$3.\quad \begin{cases} \frac{\partial \varphi}{\partial x^{(p)}} = a, & \frac{\partial \varphi}{\partial y^{(q)}} = a_1, \quad \text{etc.}, \\ \frac{\partial \psi}{\partial x^{(p)}} = b, & \frac{\partial \psi}{\partial y^{(q)}} = b_1, \quad \text{etc. etc.}, \end{cases}$$

nec non

$$4.\quad \begin{cases} \frac{\partial \varphi}{\partial x^{(p-1)}} = \alpha, & \frac{\partial \varphi}{\partial y^{(q-1)}} = \alpha_1, \quad \text{etc.}, \\ \frac{\partial \psi}{\partial x^{(p-1)}} = \beta, & \frac{\partial \psi}{\partial y^{(q-1)}} = \beta_1, \quad \text{etc. etc.}, \end{cases}$$

formoque aequationes

$$5.\quad \begin{cases} au + a_1 u_1 \text{ etc. } + \alpha v + \alpha_1 v_1 \text{ etc.} = 0, \\ bu + b_1 u_1 \text{ etc. } + \beta v + \beta_1 v_1 \text{ etc.} = 0. \end{cases}$$

Resolutione aequationum (5.) si determinantur u, u_1 etc. ut functiones lineares quantitatum v, v_1 etc., erit quod ex elementis calculi differentialis sequitur,

$$6.\quad \frac{\partial A}{\partial x^{(p-1)}} = \frac{\partial u}{\partial v}, \quad \frac{\partial B}{\partial y^{(q-1)}} = \frac{\partial u_1}{\partial v_1}, \quad \text{etc.}$$

unde prodit

$$7.\quad d \log M = -\left\{\frac{\partial u}{\partial v} + \frac{\partial u_1}{\partial v_1} \text{ etc.}\right\} dt.$$

Iam e formulis, quas de aequationum linearium resolutione et Determinantium proprietatibus tradidi, sequitur, *si in aequationibus linearibus* (5.) *ponatur*

$$8.\quad \begin{cases} \alpha\, dt = \delta a, & \alpha_1\, dt = \delta a_1, \quad \text{etc.}, \\ \beta\, dt = \delta b, & \beta_1\, dt = \delta b_1, \quad \text{etc. etc.}, \end{cases}$$

fieri

$$9.\quad -\left\{\frac{\partial u}{\partial v} + \frac{\partial u_1}{\partial v_2} \text{ etc.}\right\} dt = \delta \log \Sigma \pm a b_1 \ldots .$$

Unde formula, qua Multiplicator M definitur, proponi potest hac forma *symbolica,*

$$10.\quad d \log M = \delta \log \Sigma \pm a b_1 \ldots .$$

Cui formulae ea inest significatio ut variando per regulas notas ipsum $\lg \Sigma \pm a b_1 \ldots$ atque elementorum variationibus singulis substituendo valores (8.), obtineatur expressio ipsi $d \log M$ aequalis.

Si statuitur

$$11.\quad \begin{cases} \alpha\, dt - \lambda\, da = \Delta a, & \alpha_1\, dt - \lambda\, da_1 = \Delta a_1, \quad \text{etc.}, \\ \beta\, dt - \lambda\, db = \Delta b, & \beta_1\, dt - \lambda\, db_1 = \Delta b_1, \quad \text{etc. etc.}, \end{cases}$$

characteristicae δ substituendum est $\lambda d + \Delta$, unde abit (10.) in hanc formulam,

$$12.\quad d \log M = \lambda\, d . \log \Sigma \pm a b_1 \ldots + \Delta . \log \Sigma \pm a b_1 \ldots,$$

sive, designante λ Constantem,

$$13.\quad d \log \frac{M}{\{\Sigma \pm a b_1 \ldots\}^{\lambda}} = \Delta \log \Sigma \pm a b_1 \ldots .$$

Quae formula cum commodo adhibetur, quoties variationum Δa, Δb etc. valores valoribus variationum δa, δb etc. simpliciores sunt.

Sint n aequationes differentiales inter t et variabiles dependentes x_1, x_2, ... x_n propositae,

$$14. \quad \varphi_1 = 0, \quad \varphi_2 = 0, \quad \ldots\ldots \quad \varphi_n = 0,$$

sintque altissima differentialia in iis obvenientia et quorum valores ex iis petere liceat,

$$x_1^{(m_1)}, \quad x_2^{(m_2)}, \quad \ldots\ldots \quad x_n^{(m_n)}.$$

Statuendo secundum antecedentia,

$$15. \quad \begin{cases} \dfrac{\partial \varphi_i}{\partial x_k^{(m_k)}} = a_k^{(i)}, \\ \dfrac{\partial \varphi_i}{\partial x_k^{(m_k-1)}}\, dt = \delta a_k^{(i)} = \lambda\, d a_k^{(i)} + \Delta a_k^{(i)}, \end{cases}$$

fit

$$16. \quad \begin{cases} d \log M = \delta \log \Sigma \pm a_1' a_2'' \ldots\ldots a_n^{(n)}, \\ d \log \dfrac{M}{\{\Sigma \pm a_1' a_2'' \ldots\ldots a_n^{(n)}\}^{\lambda}} = \Delta \log \Sigma \pm a_1' a_2'' \ldots\ldots a_n^{(n)}. \end{cases}$$

Accuratius examinemus casum quo fit

$$17. \quad a_k^{(i)} = a_i^{(k)},$$

unde elementa $a_k^{(i)}$ ad numerum $\frac{n(n+1)}{2}$ reducere licet. Differentialia partialia uncis includendo aut non includendo, prout ista reductio facta est aut non facta est, habetur, si i et k inter se diversi sunt,

$$\left(\frac{\partial R}{\partial a_k^{(i)}}\right) = \frac{\partial R}{\partial a_i^{(k)}} + \frac{\partial R}{\partial a_k^{(i)}}, \quad \left(\frac{\partial R}{\partial a_i^{(i)}}\right) = \frac{\partial R}{\partial a_i^{(i)}}.$$

Designante R Determinans

$$R = \Sigma \pm a_1' a_2'' \ldots\ldots a_n^{(n)},$$

constat per notas Determinantium proprietates, si aequationes (17.) locum habeant, etiam fieri

$$18. \quad \frac{\partial R}{\partial a_k^{(i)}} = \frac{\partial R}{\partial a_i^{(k)}},$$

unde

$$19. \quad \left(\frac{\partial R}{\partial a_k^{(i)}}\right) = 2\,\frac{\partial R}{\partial a_k^{(i)}}.$$

Cum in symbolis adhibitis variationes $\delta a_k^{(i)}$ vel $\Delta a_k^{(i)}$ ab ipsis $a_k^{(i)}$ independentes sint, ex aequationibus (17.) non etiam variationum aequalitas sequitur, unde in formanda Determinantis variatione pro diversis haberi debent $\delta a_k^{(i)}$ et $\delta a_i^{(k)}$ vel

$\Delta a_k^{(i)}$ et $\Delta a_i^{(k)}$, ideoque post institutam ipsius $\boldsymbol{R}$ variationem demum aequationum (17.) usus faciendus est. At observandum est, in Determinantis variatione binorum elementorum $a_k^{(i)}$ et $a_i^{(k)}$ variationum tantum summam obvenire, cum per (18.) et (19.) habeatur,

$$\frac{\partial R}{\partial a_k^{(i)}}\delta a_k^{(i)}+\frac{\partial R}{\partial a_i^{(k)}}\delta a_i^{(k)} = \tfrac{1}{2}\left(\frac{\partial R}{\partial a_k^{(i)}}\right)\{\delta a_k^{(i)}+\delta a_i^{(k)}\}.$$

Quae formula docet, in Determinante $\boldsymbol{R}$ etiam ante eius variationem instituendam poni posse $a_k^{(i)}=a_i^{(k)}$, modo ipsi $\delta a_k^{(i)}=\delta a_i^{(k)}$ tribuatur valor $\frac{1}{2}\{\delta a_k^{(i)}+\delta a_i^{(k)}\}$. Quoties igitur aequationes (17.) locum habent sive fit

$$\frac{\partial \varphi_i}{\partial x_k^{(m_k)}} = \frac{\partial \varphi_k}{\partial x_i^{(m_i)}},$$

valebunt adhuc aequationes (16.), etsi Determinantis elementa ad numerum $\frac{n.n+1}{2}$ inter se inaequalium revocentur, dummodo statuatur

$$20.\qquad \tfrac{1}{2}\left\{\frac{\partial \varphi_i}{\partial x_k^{(m_k-1)}}+\frac{\partial \varphi_k}{\partial x_i^{(m_i-1)}}\right\}.dt = \delta a_k^{(i)} = \lambda\, da_k^{(i)}+\Delta a_k^{(i)}.$$

Quod si igitur aequationes differentiales propositae (14.) *ita comparatae sunt, ut habeatur*

$$\frac{\partial \varphi_i}{\partial x_k^{(m_k)}} = \frac{\partial \varphi_k}{\partial x_i^{(m_i)}},$$

$$\tfrac{1}{2}\left\{\frac{\partial \varphi_i}{\partial x_k^{(m_k-1)}}+\frac{\partial \varphi_k}{\partial x_i^{(m_i-1)}}\right\}dt = \lambda\, d.\frac{\partial \varphi_i}{\partial x_k^{(m_k)}},$$

designante λ Constantem, evanescet variatio Δ dabiturque Multiplicator

$$M = \left\{\Sigma\pm\frac{\partial \varphi_1}{\partial x_1^{(m_1)}}.\frac{\partial \varphi_2}{\partial x_2^{(m_2)}}\cdots\cdot\frac{\partial \varphi_n}{\partial x_n^{(m_n)}}\right\}^{\lambda}.$$

Cuius propositionis applicatio infra dabitur.

Observo ipsum $\boldsymbol{R}$ pro Determinante functionali haberi posse; erit enim $\boldsymbol{R}$ functionum $\varphi_1, \varphi_2, \ldots \varphi_n$ Determinans, si sola altissima differentialia $x_1^{(m_1)}$, $x_2^{(m_2)}$, etc. pro variabilibus sumuntur quarum respectu Determinans formetur. Quarum variabilium valores cum supponamus ex aequationibus (14.) peti posse, non fieri potest ut Determinans $\boldsymbol{R}$ identice evanescat; alioquin enim functiones φ_1, φ_2, etc. earum variabilium respectu non a se invicem independentes forent. V. *Comm. de Det. Funct.* §§. 3 sqq. Si vero per ipsas (14.) evanescit Determinans $\boldsymbol{R}$, id indicio est, duo valorum variabilium systemata inter se aequalia evadere, unde aequationum praeparatione quadam opus est qua radicibus duplicibus liberentur.

Iam praecepta generalia variis applicabo exemplis.

De Multiplicatore systematis aequationum differentialium linearium.

§. 17.

Proponantur aequationes differentiales lineares primi ordinis,

$$1.\quad \begin{cases} \frac{dx_1}{dt} = A_1' x_1 + A_2' x_2 \ldots\ldots + A_n' x_n = X_1, \\ \frac{dx_2}{dt} = A_1'' x_1 + A_2'' x_2 \ldots\ldots + A_n'' x_n = X_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dx_n}{dt} = A_1^{(n)} x_1 + A_2^{(n)} x_2 \ldots\ldots + A_n^{(n)} x_n = X_n, \end{cases}$$

quarum Coëfficientes $A_k^{(i)}$ solius t functiones designant. Systematis aequationum (1.) Multiplicator M definitur formula differentiali,

$$\frac{d\log M}{dt} = -\left\{\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \ldots\ldots + \frac{\partial X_n}{\partial x_n}\right\}$$
$$= -\{A_1' + A_2'' \ldots\ldots + A_n^{(n)}\},$$

unde

$$2.\quad M = e^{-\int \{A_1' + A_2'' \ldots\ldots + A_n^{(n)}\} dt}.$$

Hac formula cognito M, sequitur e §. 15., *si aequationes differentiales lineares* (1.) *per quascunque $n-1$ aequationes integrales, $n-1$ Constantibus Arbitrariis affectas, ad unicam aequationem differentialem primi ordinis inter duas variabiles reducantur, eius quoque ultimae aequationis integrationem per Quadraturas absolvi posse.* Quod hactenus non constabat nisi aequationes quoque integrales reductioni adhibitae lineares erant.

Aequationibus (1.) alterum systema aequationum differentialium linearium coniugatum est,

$$3.\quad \begin{cases} \frac{dy_1}{dt} = -\{A_1' y_1 + A_1'' y_2 \ldots\ldots + A_1^{(n)} y_n\}, \\ \frac{dy_2}{dt} = -\{A_2' y_1 + A_2'' y_2 \ldots\ldots + A_2^{(n)} y_n\}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dy_n}{dt} = -\{A_n' y_1 + A_n'' y_2 \ldots\ldots + A_n^{(n)} y_n\}. \end{cases}$$

Aequationibus (1.) respective per $y_1, y_2, \ldots\ldots y_n$ atque aequationibus (3.) respective per $x_1, x_2, \ldots\ldots x_n$ multiplicatis omniumque aequationum provenientium additione facta, termini ad dextram positi omnino abeunt, expressio ad laevam autem fit differentiale aggregati $x_1 y_1 + x_2 y_2 \ldots\ldots + x_n y_n$; unde integratione facta eruitur,

$$4.\quad x_1 y_1 + x_2 y_2 \ldots\ldots + x_n y_n = \text{Const.}$$

Quot habentur aequationum (3.) solutiones particulares, tot formula (4.) suppeditantur aequationum (1.) Integralia, et quot habentur solutiones particulares aequationum (1.), tot eadem formula suppeditantur aequationum (3.) Integralia. Aequationum (3.) Multiplicator invenitur

$$N = e^{\int \{A'_1 + A''_2 \dots + A^{(n)}_n\} dt},$$

unde *binorum systematum aequationum differentialium linearium inter se coniugatorum Multiplicatores M et N valoribus reciprocis gaudent.*

Functionum $y_1, y_2, \dots y_n$ denotemus n systemata a se independentia per

$$y^{(k)}_1, \; y^{(k)}_2, \; \dots \; y^{(k)}_n,$$

tribuendo successive indici superiori k valores $1, 2, \dots n$. Unde aequationum (1.) proveniunt n Integralia huiusmodi,

$$f_k = x_1 y^{(k)}_1 + x_2 y^{(k)}_2 \dots + x_n y^{(k)}_n = \alpha_k,$$

designantibus $\alpha_1, \alpha_2, \dots \alpha_n$ Constantes Arbitrarias. Secundum Multiplicatoris definitionem, initio huius Commentationis adhibitam, fit

$$5. \quad M = \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n}$$
$$= \Sigma \pm y'_1 y''_2 \dots y^{(n)}_n.$$

Unde obtinetur formula,

$$6. \quad \Sigma \pm y'_1 y''_2 \dots y^{(n)}_n = e^{-\int \{A'_1 + A''_2 \dots + A^{(n)}_n\} dt}.$$

Quae sic directe demonstratur.

Designante enim R Determinans ad laevam, fit

$$dR = \Sigma\Sigma \frac{\partial R}{\partial y^{(k)}_i} dy^{(k)}_i$$
$$= -\Sigma\Sigma \frac{\partial R}{\partial y^{(k)}_i} \{A'_i y^{(k)}_1 + A''_i y^{(k)}_2 \dots + A^{(n)}_i y^{(k)}_n\} dt,$$

extensa duplici summatione ad omnes indicum i et k valores $1, 2, \dots n$. Summando primum indicis k respectu, evanescunt termini in A'_i, A''_i etc. ducti praeter eos qui in $A^{(i)}_i$ ducuntur,

$$-A^{(i)}_i \left\{ \frac{\partial R}{\partial y'_i} y'_i + \frac{\partial R}{\partial y''_i} y''_i \dots + \frac{\partial R}{\partial y^{(n)}_i} y^{(n)}_i \right\} dt$$
$$= -A^{(i)}_i . R\, dt,$$

sicuti notis Determinantium proprietatibus patet. Hinc altera summatio indicis i respectu instituta suggerit,

$$dR = -\{A'_1 + A''_2 \dots A^{(n)}_n\} dt,$$

cuius aequationis integratione formula (6.) obtinetur.

Si aequationes differentiales lineares proponuntur quae altiora quam prima differentialia involvunt, secundum §. 14. (5.) statim earum quoque Multiplicator obtinetur. Brevitatis causa duas tantum consideremus aequationes,

$$7.\quad \begin{cases} \dfrac{d^p x}{dt^p} = Ax + A_1\dfrac{dx}{dt}\dots + A_{p-1}\dfrac{d^{p-1}x}{dt^{p-1}} \\ \qquad + Bx + B_1\dfrac{dy}{dt}\dots + B_{q-1}\dfrac{d^{q-1}y}{dt^{q-1}}, \\ \dfrac{d^q x}{dt^q} = A'x + A'_1\dfrac{dx}{dt}\dots + A'_{p-1}\dfrac{d^{p-1}x}{dt^{p-1}} \\ \qquad + B'y + B'_1\dfrac{dy}{dt}\dots + B'_{q-1}\dfrac{d^{q-1}y}{dt^{q-1}}, \end{cases}$$

in quibus Coëfficientes A, A_1 etc. solius t functiones designant; fit earum aequationum Multiplicator,

$$M = e^{-\int\{A_{p-1}+B'_{q-1}\}dt}.$$

Ponamus, addendo aequationes (7.) respective per λ et μ multiplicatas produci aequationem per se integrabilem: secundum conditiones integrabilitatis fieri debet,

$$8.\quad \begin{cases} \dfrac{d^p\lambda}{dt^p} = -\dfrac{d^{p-1}(A_{p-1}\lambda + A'_{p-1}\mu)}{dt^{p-1}} + \dfrac{d^{p-2}(A_{p-2}\lambda + A'_{p-2}\mu)}{dt^{p-2}}\dots \pm (A\lambda + A'\mu), \\ \dfrac{d^q\mu}{dt^q} = -\dfrac{d^{q-1}(B_{q-1}\lambda + B'_{q-1}\mu)}{dt^{q-1}} + \dfrac{d^{q-2}(B_{q-2}\lambda + B'_{q-2}\mu)}{dt^{q-2}}\dots \pm (B\lambda + B'\mu), \end{cases}$$

quod est aequationum differentialium systema proposito coniugatum. Quod, si p et q inter se inaequales sunt, non ea gaudet forma qua §. 14. supposui aequationes differentiales exhibitas esse, videlicet ut altissima differentialia inveniantur per inferiora ipsasque variabiles expressa. Si $p > q$, ut ea forma obtineatur, aequatio posterior $p-q-1$ vicibus iteratis differentianda est et aequationum ope provenientium eliminanda sunt e priore ipsius μ differentialia superiora $(q-1)^{\text{to}}$. Hac eliminatione priorem aequationem novi non ingrediuntur termini $(p-1)^{\text{to}}$ ipsius λ differentiali affecti, unde in ea immutatus manet unicus terminus differentiale $\dfrac{d^{p-1}\lambda}{dt^{p-1}}$ implicans,

$$-A_{p-1}\frac{d^{p-1}\lambda}{dt^{p-1}}.$$

Porro in aequatione posteriore unicus extat terminus ipso $\dfrac{d^{q-1}\mu}{dt^{q-1}}$ affectus,

$$-B'_{q-1}\frac{d^{q-1}\mu}{dt^{q-1}}.$$

Unde secundum §. 14. aequationum (8.), dicto modo praeparatarum, eruitur Multiplicator

$$N = e^{\int \{A_{p-1}+B'_{q-1}\}dt}.$$

Videmus igitur, bina quoque systemata coniugata (7.) et (8.) Multiplicatoribus reciprocis gaudere. Similiter pro pluribus aequationibus demonstratur, in systemate coniugato per eliminationes, ad formam normalem obtinendam instituendas, hos non mutari terminos qui valorem ipsius $\frac{d\log N}{dt}$ afficiunt, unde facile sequitur, binorum systematum coniugatorum Multiplicatores semper evadere inter se reciprocos.

Observo, formam normalem aequationibus (8.) conciliari posse sine differentiationibus et eliminationibus, cum earum loco hoc pateat substitui posse systema aequationum differentialium linearium primi ordinis inter $p+q+1$ variabiles,

$$9.\quad \begin{cases} \frac{d\lambda}{dt} = -\{A_{p-1}\lambda + A'_{p-1}\mu + \lambda_1\}, \\ \frac{d\lambda_1}{dt} = -\{A_{p-2}\lambda + A'_{p-2}\mu + \lambda_2\}, \\ \dots\dots\dots\dots \\ \frac{d\lambda_{p-1}}{dt} = -\{A\lambda + A'\mu\}, \\ \frac{d\mu}{dt} = -\{B_{q-1}\lambda + B'_{q-1}\mu + \mu_1\}, \\ \frac{d\mu_2}{dt} = -\{B_{q-2}\lambda + B'_{q-2}\mu + \mu_2\}, \\ \dots\dots\dots\dots \\ \frac{d\mu_{q-1}}{dt} = -\{B\lambda + B'\mu\}. \end{cases}$$

Aequationes (9.) eodem gaudent Multiplicatore N supra invento. Quod adnotari meretur. Nam valor supra inventus N Multiplicatori aequationum (8.) conveniebat supponendo eas locum tenere aequationum differentialium primi ordinis, in quibus praeter t, λ, μ pro variabilibus habeantur

$$(A.)\quad \begin{cases} \frac{d\lambda}{dt},\ \frac{d^2\lambda}{dt^2},\ \dots\ \frac{d^{p-1}\lambda}{dt^{p-1}}, \\ \frac{d\mu}{dt},\ \frac{d^2\mu}{dt^2},\ \dots\ \frac{d^{q-1}\mu}{dt^{q-1}}; \end{cases}$$

dum aequationes (9.) sunt inter t, λ, μ aliasque variabiles

$$(B.)\quad \begin{cases} \lambda_1,\ \lambda_2,\ \dots\ \lambda_{p-1}, \\ \mu_1,\ \mu_2,\ \dots\ \mu_{q-1}. \end{cases}$$

Aliis autem variabilibus introductis vidimus in secundo Capite mutari Multiplicatorem, videlicet eum dividi per novarum variabilium Determinans, ipsarum formatum variabilium respectu quarum loco introductae sunt. Unde, cum utrique aequationum systemati *idem* conveniat Multiplicator N, sequitur, si quantitates ($B.$) per t, λ, μ et quantitates ($A.$) exprimantur, Determinans quantitatum ($B.$), ipsarum ($A.$) respectu formatum, aequari Constanti, ac reapse aequale invenitur unitati.

Aequationes differentiales secundi ordinis quarum assignare licet Multiplicatorem. Exempla *Euleriana.*

§. 18.

Paullo immorabor applicationi theoriae novi Multiplicatoris ad aequationes differentiales secundi ordinis inter duas variabiles, qui est casus simplicissimus post aequationes differentiales primi ordinis, ad quas ***Eulerianus*** Multiplicator refertur. Ac primum per theoremata §§. 14, 15 tradita patet,

„*si proponatur aequatio* $\frac{d^2y}{dx^2}+A\frac{dy}{dx}+B=0$, *in qua A solius x, B utriusque x et y functiones quaecunque sunt, atque integratione prima eruatur* $\frac{dy}{dx}=u$, *designante u variabilium x et y et Constantis Arbitrariae α functionem, fore alterum Integrale,*

$$\int e^{\int A\,dx}\cdot\frac{\partial u}{\partial\alpha}(dy-u\,dx) = Const."$$

Quantitatem sub maiore integrationis signo esse differentiale completum, sic verificari potest. Nam ut aequatio differentialis proposita proveniat differentiatione aequationis $\frac{dy}{dx}=u$, locum habere debet aequatio ***identica,***

$$\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}+Au+B = 0.$$

Qua ipsius α respectu differentiata et per $e^{\int A\,dx}$ multiplicata prodit,

$$\frac{\partial.e^{\int A\,dx}\frac{\partial u}{\partial\alpha}}{\partial x}+\frac{\partial.e^{\int A\,dx}u\frac{\partial u}{\partial\alpha}}{\partial y} = 0,$$

quae est conditio requisita, ut quantitas

$$e^{\int A\,dx}\frac{\partial u}{\partial\alpha}(dy-u\,dx)$$

differentiale completum sit.

Generalius *e* §§. 14, 15 sequitur, *si proponatur aequatio,*

$$1.\quad \frac{d^2y}{dx^2}+\frac{1}{2}\frac{\partial\varphi}{\partial y}\left(\frac{dy}{dx}\right)^2+\frac{\partial\varphi}{\partial x}\cdot\frac{dy}{dx}+B = 0,$$

in qua et φ *et* B *variabilium* x *et* y *functiones quaecunque sunt, atque integratione prima inventum sit* $\frac{dy}{dx}=u$, *designante* u *variabilium* x *et* y *et Constantis Arbitrariae* α *functionem, fieri aequationem inter* x *et* y *quaesitam,*

$$2.\quad \int e^{\varphi}\frac{\partial u}{\partial\alpha}(dy-u\,dx) = \text{Const.}$$

Aequationis (1.) tractavit *Eulerus* specimina quibus ei integratio prima successit (Cf. Calc. Integr. Vol. I. Sect. I. Cap. VI. pgg. 162 sqq.). At aequationes differentiales primi ordinis, ad quas ea ratione pervenit, tanta irrationalitate erant implicatae, ut de integratione directa desperans alia artificia circumspexerit. Atque missum facto Integrali invento contigit ei, aequationes differentiales secundi ordinis propositas differentiando alias deducere lineares, Coëfficientibus constantibus affectas, quarum nota integratio propositarum quoque ei suppeditavit integrationem completam. At per antecedentem formulam (2.) illarum aequationum differentialium primi ordinis quamvis complicatarum assignare licet Multiplicatores. Adiungam ipsam variabilium separationem, qua elucescat, revera adiectis illis Multiplicatoribus aequationes sponte integrabiles fore.

Exempla *Euleriana* forma paullo generaliori exhibebo, quod sine calculi complicatione fieri potest.

Exemplum I.

$$y^2\frac{d^2y}{dx^2}+y\left(\frac{dy}{dx}\right)^2+by-cx = 0.$$

(b et c Constantes.)

Secundum *Eulerum* aequationis propositae fit Integrale primum, quod si placet differentiando comprobare licet,

$$y^3\left(\frac{dy}{dx}\right)^3+bxy^2\left(\frac{dy}{dx}\right)^2+(by-3cx)yy\frac{dy}{dx}$$
$$+cy^3+b^2y^2x-2bcyx^2+c^2x^3 = \alpha,$$

designante α Constantem Arbitrariam. Cuius aequationis resolutione eruatur

$$y\frac{dy}{dx} = yu = v,$$

designante v radicem aequationis cubicae

$$3.\quad v^3+bxv^2+y(by-3cx)v$$
$$+cy^3+b^2y^2x-2bcyx^2+c^2x^3 = \alpha.$$

Comparando aequationem differentialem propositam cum (1.) fit

$$\varphi = 2\log y, \quad e^{\varphi} = y^2,$$

unde secundum (2.) invenitur alterum Integrale

$$\int y^2 \frac{\partial u}{\partial \alpha}(dy - u\,dx) = \int \frac{\partial v}{\partial \alpha}(y\,dy - v\,dx) = \text{Const.}$$

Fit autem e (3.)

$$\frac{\partial v}{\partial \alpha} = \frac{1}{3vv + 2bxv + y(by - 3cx)}.$$

Quem aequationis $y\,dy - v\,dx = 0$ Multiplicatorem esse, propter ipsius v irrationalitatem non facile cognoscitur, et minus adhuc separatio variabilium in promtu est. Quam sic assequor.

Aequationem (3.) bene vidit *Eulerus* hac ratione exhiberi posse,

$$4. \quad f.f'.f'' = \alpha,$$

posito

$$5. \quad \begin{cases} f = v + \lambda y + \frac{c}{\lambda}x, \\ f' = v + \lambda' y + \frac{c}{\lambda'}x, \\ f'' = v + \lambda'' y + \frac{c}{\lambda''}x, \end{cases}$$

designantibus λ, λ', λ'' radices diversas aequationis cubicae,

$$6. \quad \lambda^3 + b\lambda - c = 0,$$

unde $\lambda + \lambda' + \lambda'' = 0$, $\lambda\lambda'\lambda'' = c$. Ex aequationibus (4.) et (5.) sequitur

$$\frac{\partial f}{\partial \alpha} = \frac{\partial f'}{\partial \alpha} = \frac{\partial f''}{\partial \alpha} = \frac{\partial v}{\partial \alpha}$$

$$= \frac{1}{f'f'' + f''f + ff'},$$

unde expressio

$$\frac{y\,dy - v\,dx}{f'f'' + f''f + ff'}$$

fieri debet differentiale completum. Invenitur autem e (5.):

$$d(f' - f'') = (\lambda' - \lambda'')(dy - \lambda\,dx),$$
$$d(f'' - f) = (\lambda'' - \lambda)(dy - \lambda' dx),$$
$$d(f - f') = (\lambda - \lambda')(dy - \lambda'' dx),$$
$$\lambda f.d(f' - f'') + \lambda' f'.d(f'' - f) + \lambda'' f''.d(f - f')$$
$$= \Delta.(y\,dy - v\,dx),$$

siquidem ponitur

$$\Delta = \lambda^2(\lambda' - \lambda'') + \lambda'^2(\lambda'' - \lambda) + \lambda''^2(\lambda - \lambda')$$
$$= (\lambda - \lambda')(\lambda - \lambda'')(\lambda' - \lambda'') = 0,$$

atque adnotatur fieri

$$\lambda^3(\lambda'-\lambda'')+\lambda'^3(\lambda''-\lambda)+\lambda''^3(\lambda-\lambda')$$
$$= \Delta(\lambda+\lambda'+\lambda'') = 0.$$

Hinc substitutendo $\lambda'' = -(\lambda+\lambda')$ fit

$$\Delta(y\,dy - v\,dx) = \lambda\{(f+f'')df' - d.ff''\}$$
$$- \lambda'\{(f'+f'')df - d.f'f''\},$$

unde denuo substituendo, quod e (4.) sequitur,

$$d.ff'' = -f''.\frac{df'}{f'}, \quad d.f'f'' = -f'f''.\frac{df}{f},$$

eruitur

$$\frac{y\,dy - v\,dx}{f'f''+f''f+ff'} = \frac{1}{\Delta}\left\{\frac{\lambda\,df'}{f'} - \frac{\lambda'\,df}{f}\right\}.$$

Quod per se integrabile est atque nihilo aequiparatum integratumque suppeditat:

$$\frac{\log f}{\lambda} - \frac{\log f'}{\lambda'} = \text{Const.},$$

quod alterum Integrale est.

Exemplum II.

$$2y^3\frac{d^2y}{dx^2}+y^2\left(\frac{dy}{dx}\right)^2 - ay^2+bx^2-c = 0.$$

(a, b, c Constantes.)

Secundum ***Eulerum*** huius aequationis integratione prima obtinetur $y\,dy - v\,dx = 0$, designante v radicem aequationis biquadraticae,

7. $$(aa-4b)y^2 - 2(abx^2+av^2-4bxv) + \left(\frac{c-bx^2+v^2}{y}\right)^2 = \alpha,$$

atque α Constantem Arbitrariam. Comparando aequationem differentialem propositam cum (1.) fit

$$\varphi = \log y, \quad e^\varphi = y,$$

unde e (2.) eruitur aequatio integralis inter x et y quaesita,

$$\int y\frac{\partial u}{\partial\alpha}\{dy - u\,dx\} = \int\frac{\partial v}{\partial\alpha}.\frac{y\,dy - v\,dx}{y} = \text{Const.}$$

Ponamus $a = \lambda+\lambda'$, $b = \lambda\lambda'$, abit (7.) in hanc formam,

8. $$(\lambda-\lambda')^2y^2 - 2\{\lambda(v-\lambda'x)^2+\lambda'(v-\lambda x)^2\}$$
$$+\left\{\frac{c-\lambda\lambda'x^2+v^2}{y}\right\}^2 = \alpha.$$

Ponatur

9. $$v-\lambda'x = (\lambda-\lambda')p, \quad v-\lambda x = (\lambda'-\lambda)p',$$

unde

$$10.\quad \begin{cases} x = p+p', \qquad v = \lambda p+\lambda' p', \\ \sqrt{\lambda}.p+\sqrt{\lambda'}.p' = \dfrac{v+\sqrt{(\lambda\lambda')}\,x}{\sqrt{\lambda}+\sqrt{\lambda'}}, \qquad \sqrt{\lambda}.p-\sqrt{\lambda'}.p' = \dfrac{v-\sqrt{(\lambda\lambda')}\,x}{\sqrt{\lambda}-\sqrt{\lambda'}}; \end{cases}$$

abit (8.) in hanc aequationem,

$$11.\quad y^2+\left\{\frac{c}{\lambda-\lambda'}+\lambda p^2-\lambda' p'^2\right\}^2\frac{1}{y^2} = 2\left\{\lambda p^2+\lambda' p'^2+\frac{\alpha}{2(\lambda-\lambda')^2}\right\}.$$

Hinc fit

$$12.\quad y = \sqrt{(\varepsilon+\lambda pp)}+\sqrt{(\varepsilon'+\lambda' p'p')},$$

siquidem ponitur

$$13.\quad \varepsilon = \frac{\alpha}{4(\lambda-\lambda')^2}+\frac{c}{2(\lambda-\lambda')}, \qquad \varepsilon' = \frac{\alpha}{4(\lambda-\lambda')^2}+\frac{c}{2(\lambda'-\lambda)}.$$

E formulis (9.) et (13.) sequitur

$$\frac{\partial p}{\partial \alpha} = -\frac{\partial p'}{\partial \alpha} = \frac{1}{\lambda-\lambda'}\cdot\frac{\partial v}{\partial \alpha},$$

$$\frac{\partial \varepsilon}{\partial \alpha} = \frac{\partial \varepsilon'}{\partial \alpha} = \frac{1}{4(\lambda-\lambda')^2};$$

unde e (12.) obtinetur,

$$14.\quad \frac{1}{y}\cdot\frac{\partial v}{\partial \alpha} = \frac{1}{8(\lambda-\lambda')\{\lambda' p'\sqrt{(\varepsilon+\lambda pp)}-\lambda p\sqrt{(\varepsilon'+\lambda' p'p')}\}},$$

qui fieri debet Multiplicator aequationis $y\,dy-v\,dx=0$. Ac reapse invenitur e (10.) et (12.),

$$\begin{aligned} y\,dy-v\,dx &= \left\{\frac{\lambda p\,dp}{\sqrt{(\varepsilon+\lambda pp)}}+\frac{\lambda' p'\,dp'}{\sqrt{(\varepsilon'+\lambda' p'p')}}\right\}\{\sqrt{(\varepsilon+\lambda pp)}+\sqrt{(\varepsilon'+\lambda' p'p')}\} \\ &\quad -\{\ dp \ + \ dp'\ \}\{\ \lambda p \ + \ \lambda' p'\ \} \\ &= \{\lambda p\sqrt{(\varepsilon'+\lambda' p'p)}-\lambda' p'\sqrt{(\varepsilon+\lambda pp)}\}\left\{\frac{dp}{\sqrt{(\varepsilon+\lambda pp)}}-\frac{dp'}{\sqrt{(\varepsilon'+\lambda' p'p')}}\right\}. \end{aligned}$$

Unde per factorem (14.) atque substitutionem (9.) aequationem differentialem, $y\,dy-v\,dx=0$, in aliam mutamus, in qua variabiles separatae sunt,

$$\frac{dp}{\sqrt{(\varepsilon+\lambda pp)}}-\frac{dp'}{\sqrt{(\varepsilon'+\lambda' p'p')}} = 0.$$

Cuius integratione prodit:

$$\frac{\{\sqrt{\lambda}.p+\sqrt{(\varepsilon+\lambda pp)}\}^{\sqrt{\lambda'}}}{\{\sqrt{\lambda'}.p'+\sqrt{(\varepsilon'+\lambda' p'p')}\}^{\sqrt{\lambda}}} = \text{Const.}$$

Ponendo autem

$$(\sqrt{\lambda}+\sqrt{\lambda'})y+v+\sqrt{(\lambda\lambda')}x = A,$$
$$(\sqrt{\lambda}-\sqrt{\lambda'})y+v-\sqrt{(\lambda\lambda')}x = B,$$
$$(\sqrt{\lambda}-\sqrt{\lambda'})y-v+\sqrt{(\lambda\lambda')}x = C,$$

fit e (10.) et (11.) post calculos faciles,

$$\sqrt{\lambda}.p+\sqrt{(\varepsilon+\lambda pp)} = \frac{AB+c}{2(\lambda-\lambda')y},$$
$$\sqrt{\lambda'}.p+\sqrt{(\varepsilon'+\lambda'p'p')} = \frac{AC-c}{2(\lambda-\lambda')y}.$$

Unde aequatio integralis inventa sic exhiberi potest,

$$\frac{(AB+c)^{\sqrt{\lambda'}}}{(AC-c)^{\sqrt{\lambda}}} = \beta.y^{\sqrt{\lambda'}-\sqrt{\lambda}},$$

ubi β est nova Constans Arbitraria atque quantitas v, quae ipsas A, B, C afficit, est radix aequationis biquadraticae (7.), porro λ et λ' sunt radices diversae aequationis quadraticae $\lambda^2-a\lambda+b=0$.

Integrationem his duobus exemplis praestitam etiam assequi licuisset ponendo cum *Eulero* $dx=y\,dt$, et aequationem differentialem secundi ordinis exemplo primo propositam *semel*, exemplo secundo propositam *bis* differentiando, ita ut t pro variabili independente habeatur. Quo facto respective pervenitur ad aequationes differentiales lineares tertii et quarti ordinis, quae Coëfficientibus gaudent constantibus notisque methodis integrantur.

De Multiplicatore systematis aequationum differentialium vulgarium quod mediante solutione completa unius aequationis differentialis partialis primi ordinis integratur.

§. 19.

Systema aequationum differentialium vulgarium proponatur hoc,

$$1.\quad\begin{cases} \dfrac{dq_1}{dt}=\dfrac{\partial\varphi}{\partial p_1}, & \dfrac{dp_1}{dt}=-\left\{\dfrac{\partial\varphi}{\partial q_1}+p_1\dfrac{\partial\varphi}{\partial V}\right\},\\ \dfrac{dq_2}{dt}=\dfrac{\partial\varphi}{\partial p_2}, & \dfrac{dp_2}{dt}=-\left\{\dfrac{\partial\varphi}{\partial q_2}+p_2\dfrac{\partial\varphi}{\partial V}\right\},\\ \dots\dots & \dots\dots\\ \dfrac{dq_n}{dt}=\dfrac{\partial\varphi}{\partial p_n}, & \dfrac{dp_n}{dt}=-\left\{\dfrac{\partial\varphi}{\partial q_n}+p_n\dfrac{\partial\varphi}{\partial V}\right\},\\ \dfrac{dV}{dt}=p_1\dfrac{\partial\varphi}{\partial p_1}+p_2\dfrac{\partial\varphi}{\partial p_2}\dots\dots+p_n\dfrac{\partial\varphi}{\partial p_n}, \end{cases}$$

ubi φ est functio quaecunque quantitatum $q_1, q_2, \dots\dots q_n, V, p_1, p_2, \dots\dots p_n$. Designante M aequationum (1.) Multiplicatorem, secundum formulas nostras gerales fit

$$\frac{d\log M}{dt} = -\Sigma\frac{\partial^2\varphi}{\partial p_i\,\partial q_i}+\Sigma\left\{\frac{\partial^2\varphi}{\partial q_i\,\partial p_i}+p_i\frac{\partial^2\varphi}{\partial V\,\partial p_i}\right\}+n\frac{\partial\varphi}{\partial V}$$
$$-\frac{\partial\left\{p_1\frac{\partial\varphi}{\partial p_1}+p_2\frac{\partial\varphi}{\partial p_2}\ldots\ldots+p_n\frac{\partial\varphi}{\partial p_n}\right\}}{\partial V},$$

tribuendo indici i valores $1, 2, \ldots\ldots n$. Unde reiectis terminis se destruentibus obtinetur,

$$2.\qquad \frac{d\log M}{dt} = n\frac{\partial\varphi}{\partial V}.$$

Quae evanescit expressio si φ ipsa V vacat. ***Quoties igitur functio φ ab ipsa V vacua est, aequationum (1.) Multiplicatorem unitati aequare licet.***

Aequationum (1.) habetur Integrale unum,

$$3.\qquad \varphi = h,$$

designante h Constantem. In ea aequatione ponatur

$$4.\qquad p_1=\frac{\partial V}{\partial q_1},\quad p_2=\frac{\partial V}{\partial q_2},\quad\ldots\ldots\quad p_n=\frac{\partial V}{\partial q_n},$$

obtinetur aequatio differentialis partialis primi ordinis, in quà V est functio quaesita atque $q_1, q_2, \ldots\ldots q_n$ sunt variabiles independentes. Faciamus inventam esse eius aequationis differentialis partialis solutionem *quamcunque* V, dico aequationes (4.) totidem esse aequationes integrales, quibus aequationes differentiales vulgares (1.) gaudere possint. Nam differentiando ex. gr. earum primam $\frac{\partial V}{\partial q_1}-p_1=0$ et substituendo aequationes differentiales (1.) prodit,

$$5.\qquad \Sigma\frac{\partial^2 V}{\partial q_1\,\partial q_i}\cdot\frac{\partial\varphi}{\partial p_i}+\frac{\partial\varphi}{\partial q_1}+p_1\frac{\partial\varphi}{\partial V} = 0.$$

Cui aequationi satisfit substituendo ipsorum p_1, p_2, etc. valores (4.). Nimirum e suppositione facta aequatio (3.) identica evadit substituendo (4.) solutionisque V valorem, eam autem aequationem identicam ipsius q_1 respectu differentiando prodit aequatio in quam abit (5.) per aequationes (4.). ***Itaque aequationes (4.) una cum ipsa aequatione, qua V per $q_1, q_2, \ldots\ldots q_n$ definiri ponitur, constituunt systema $n+1$ aequationum integralium idque tale e quo differentiando ipsasque aequationes differentiales propositas substituendo deducere non licet aequationes integrales novas.*** Scilicet aequationes provenientes (5.) per illas $n+1$ aequationes identicas fieri vidimus.

Constans h ubi servat significationem generalem ingredi debet solutionem quamcunque V unde, data V, differentiale quoque partiale $\frac{\partial V}{\partial h}$ assignare licebit, quod per z designabo. Erit per (1.), (3.), (4.),

$$6.\qquad \frac{dz}{dt} = \Sigma\frac{\partial^2 V}{\partial h\,\partial q_i}\cdot\frac{\partial\varphi}{\partial p_i} = \frac{\partial\varphi}{\partial h}-\frac{\partial\varphi}{\partial V}z = 1-\frac{\partial\varphi}{\partial V}z.$$

Si solutio V aliquam involvit Constantem Arbitrariam α atque ponitur $\frac{\partial V}{\partial \alpha} = y$, similiter erit

$$7. \quad \frac{dy}{dt} = \Sigma \frac{\partial^2 V}{\partial \alpha\, \partial q_i} \cdot \frac{\partial \varphi}{\partial p_i} = \frac{\partial \varphi}{\partial \alpha} - \frac{\partial \varphi}{\partial V} y = -\frac{\partial \varphi}{\partial V} y.$$

Scilicet functio φ, substituendo datam solutionem V atque ponendo $p_i = \frac{\partial V}{\partial q_i}$, identice aequatur Constanti h ideoque post eam substitutionem differentiata ipsius h respectu unitati aequatur, differentiata ipsius α respectu evanescit. E (2.) et (7.) sequitur,

$$d \log M = -n\, d \log y,$$

ideoque fit

$$8. \quad y^n M = \left(\frac{\partial V}{\partial \alpha}\right)^n M = \beta,$$

designante β Constantem. Haec formula docet, Multiplicatori M competere valorem qui per aequationes integrales (3.) et (4.) aequetur quantitati $\left\{\frac{\partial V}{\partial \alpha}\right\}^{-n}$. Observo adhuc, e binis formulis (6.) et (7.) sequi

$$y\, dz - z\, dy = y\, dt,$$

unde, designante U functionem quantitatum y et z homogeneam rationalem $(-1)^{ti}$ ordinis, assignari poterit integrale $\int U\, dt$. Si solutio V plures Constantes Arbitrarias involvit, totidem habebuntur aequationes (8.), binarumque divisione obtinebuntur aequationes integrales, inventis (3.) et (4.) accedentes. Si functio φ ab ipsa V vacua est ideoque $M = 1$, aequationes (8.) per se sunt aequationes integrales.

Si habetur solutio *completa* $V = F$, n Constantes Arbitrarias $\alpha_1, \alpha_2, \ldots \alpha_n$ involvens, poniturque $\frac{\partial F}{\partial \alpha_i} = u_i$, fit systema aequationum integralium completarum,

$$9. \quad \begin{cases} F - V = 0, \quad \frac{\partial F}{\partial q_1} - p_1 = 0, \quad \frac{\partial F}{\partial q_2} - p_2 = 0, \quad \ldots \quad \frac{\partial F}{\partial q_n} - p_n = 0, \\ \qquad \frac{u_1}{u_n} - \beta_1 = 0, \quad \frac{u_2}{u_n} - \beta_2 = 0, \quad \ldots \quad \frac{u_{n-1}}{u_n} - \beta_{n-1} = 0, \end{cases}$$

designantibus $\beta_1, \beta_2, \ldots \beta_{n-1}$ alias Constantes Arbitrarias. Si ex his aequationibus petuntur valores quantitatum h, α_i, β_i, atque functionum iis aequivalentium formantur Determinantia *partialia*, in quibus una quantitatum q_i, p_i, V pro Constante, reliquae pro variabilibus habentur, ea aequare debent quantitates ad dextram aequationum differentialium (1.) positas, in *Multiplicatorem* ductas. Supersedere resolutioni aequationum (9.) et immediate functionum $F - V$, $\frac{\partial F}{\partial q_1} - p_1$ etc. sumere possumus Determinantia partialia, dummodo ea dividimus per earundem functionum Determinans, quantitatum h, α_i, β_i respectu formatum. Qua de re Cap. I. egi. Determinantia functionalia hic obvenientia in alia simpliciora redeunt, propterea quod quantitates V, p_1, p_2, ... p_n tantum in $n+1$ prioribus aequationum (9.), quantitates β_1, β_2, ... β_{n-1} tantum in $n-1$ posterioribus, singulae in singulis reprehenduntur. Sic Determinans, quantitatum h, α_i, β_i respectu formatum, quod per ∇ designabo, aequatur Determinanti functionum ab ipsis β_i vacuarum,

$$F, \frac{\partial F}{\partial q_1}, \frac{\partial F}{\partial q_2}, \dots \frac{\partial F}{\partial q_n},$$

solarum h et α_1, α_2, ... α_n respectu formato. Determinans partiale, in quo q_n pro Constante habetur et quod per (q_n) designabo, aequatur Determinanti functionum

$$\frac{u_1}{u_n}, \frac{u_2}{u_n}, \dots \frac{u_{n-1}}{u_n},$$

formato solarum respectu q_1, q_2, ... q_{n-1}. Per theorema autem in Comment. *de Determinantibus functionalibus* comprobato, quod Determinantia spectat functionum communi denominatore praeditarum, fit

$$(q_n) = u_n^{-n} Q_n = \left(\frac{\partial F}{\partial \alpha_n}\right)^{-n} Q_n,$$

posito

$$Q_n = \Sigma \pm \frac{\partial u_1}{\partial q_1} \cdot \frac{\partial u_2}{\partial q_2} \dots \frac{\partial u_{n-1}}{\partial q_{n-1}} u_n,$$

ubi formantur Determinantis Q_n termini permutando omnimodis functiones u_1, u_2, ... u_n. Substituendo autem valores $u_i = \frac{\partial F}{\partial \alpha_i}$ et differentiationum ordinem invertendo sequitur, ***Determinans Q_n fieri Determinans functionum***

$$F, \frac{\partial F}{\partial q_1}, \frac{\partial F}{\partial q_2}, \dots \frac{\partial F}{\partial q_{n-1}},$$

quantitatum α_1, α_2, ... α_n respectu formatum. Iam aequationem identicam,

$$\varphi\left(q_1, q_2, \dots q_n, F, \frac{\partial F}{\partial q_1}, \frac{\partial F}{\partial q_2}, \dots \frac{\partial F}{\partial q_n}\right) = h,$$

differentiando respectu quantitatum h, α_1, α_2, ... α_n, quibus ipsae F, $\frac{\partial F}{\partial q_1}$ etc. afficiuntur, scribendoque V et p_i ipsarum F et $\frac{\partial F}{\partial q_i}$ loco, obtinentur inter incognitas $\frac{\partial \varphi}{\partial V}$ et $\frac{\partial \varphi}{\partial p_i}$ aequationes $n+1$ lineares, quarum resolutione invenitur

$$\frac{\partial \varphi}{\partial p_n} = \frac{Q_n}{\nabla},$$

unde

$$\frac{(q_n)}{\nabla} = \frac{\partial \varphi}{\partial p_n} \left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n}.$$

Eadem ratione generaliter, ubi vocamus (q_i) functionum (9.) Determinans partiale in quo q_i pro Constante habetur, invenitur

$$10.\quad \frac{(q_i)}{\nabla} = \left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \cdot \frac{\partial \varphi}{\partial p_i}.$$

Vocando W functionum

$$\frac{\partial F}{\partial q_1},\ \frac{\partial F}{\partial q_2},\ \dots\ \frac{\partial F}{\partial q_n}$$

Determinans, quantitatum $\alpha_1, \alpha_2, \dots \alpha_n$ respectu formatum, earundem $n+1$ aequationum linearium resolutione eruitur,

$$\frac{\partial \varphi}{\partial V} = \frac{W}{\nabla}.$$

Functionum (9.) Determinans partiale (p_n), in quo p_n pro Constante habetur, aequatur Determinanti functionum

$$\frac{\partial F}{\partial q_n},\ \frac{u_1}{u_n},\ \frac{u_2}{u_n},\ \dots\ \frac{u_{n-1}}{u_n},$$

quantitatum $q_1, q_2, \dots q_n$ respectu formato. Invertendo autem ordinem differentiationum in differentialibus ipsius $\frac{\partial F}{\partial q_n}$ atque similes adhibendo formulas earum quibus supra (q_n) ad Q_n revocavi, redit $u_n^n(p_n)$ in differentiam Determinantis P_n functionum

$$F,\ \frac{\partial F}{\partial q_1},\ \frac{\partial F}{\partial q_2},\ \dots\ \frac{\partial F}{\partial q_n},$$

quantitatum $q_n, \alpha_1, \alpha_2, \dots \alpha_n$ respectu formati, atque Determinantis functionalis modo adhibiti W per $\frac{\partial F}{\partial q_n}$ multiplicati, sive fit

$$\left(\frac{\partial F}{\partial \alpha_n}\right)^n (p_n) = P_n - \frac{\partial F}{\partial q_n}.W = P_n - p_n W.$$

Adiiciendo autem $n+1$ aequationibus linearibus commemoratis aliam provenientem ex aequatione $\varphi = h$, quantitatis q_n respectu differentiata, eruitur per eliminationem quantitatum $\frac{\partial \varphi}{\partial V}, \frac{\partial \varphi}{\partial p_1}, \frac{\partial \varphi}{\partial p_2}, \dots \frac{\partial \varphi}{\partial p_n}$,

$$\nabla \frac{\partial \varphi}{\partial q_n} + P_n = 0.$$

Unde fit

$$\frac{(p_n)}{\nabla} = \left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \left\{\frac{P_n}{\nabla} - p_n \frac{W}{\nabla}\right\} = -\left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \cdot \left\{\frac{\partial \varphi}{\partial q_n} + p_n \frac{\partial \varphi}{\partial V}\right\};$$

eademque ratione obtinetur generaliter, ubi (p_i) est functionum (9.) Determinans partiale in quo habetur p_i pro Constante,

$$11.\quad \frac{(p_i)}{\nabla} = -\left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \cdot \left\{\frac{\partial \varphi}{\partial q_n} + p_n \frac{\partial \varphi}{\partial V}\right\}.$$

Quae paullo difficiliora erant indagatu. Postremo functionum (9.) Determinans partiale (V), in quo habetur V pro Constante, aequale erit functionum

$$F,\ \frac{u_1}{u_n},\ \frac{u_2}{u_n},\ \ldots\ \frac{u_{n-1}}{u_n}$$

Determinanti, quantitatum $q_1,\ q_2,\ \ldots\ q_n$ respectu formato. Quod adhibendo notationem supra traditam fieri patet

$$(V) = \frac{\partial F}{\partial q_1}(q_1) + \frac{\partial F}{\partial q_2}(q_2) \ldots + \frac{\partial F}{\partial q_n}(q_n),$$

unde secundum (10.) invenitur:

$$12.\quad \frac{(V)}{\nabla} = \left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \left\{p_1 \frac{\partial \varphi}{\partial p_1} + p_2 \frac{\partial \varphi}{\partial p_2} \ldots + p_n \frac{\partial \varphi}{\partial p_n}\right\}.$$

Formulae (10.), (11.), (12.) docent, functionum ad laevam aequationum (9.) positarum Determinantia partialia aequari quantitatibus ad dextram aequationum differentialium (1.) positis, per factorem communem $\left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n}$ multiplicatis. Ea Determinantia partialia autem sunt ut differentialia dq_i, dp_i, dV. Unde antecedentibus continetur demonstratio directa, aequationes differentiales propositas e formulis (9.) differentiatis per aequationum linearium resolutionem fluere easque Multiplicatore gaudere $\left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n}$, qualis e formula (8.) obtinebatur. Quam demonstrationem hic breviter indicasse placuit, cum ad illustrandam Determinantium theoriam faciat.

Casu quo φ ab ipsa V vacua est cum cognitus sit Multiplicator, videamus, quid sit quod ea cognitione lucremur in exemplo simplicissimo quo $n = 2$. Tributo Constanti h valore particulari, substituamus aequationi $\varphi = h$ aliam qua ipsius p_2 valor per q_1, q_2, p_1 exhibetur, ita ut aequationes differentiales proponantur sequentes,

$$13.\quad dq_1 : dq_2 : dp_1 = \frac{\partial p_2}{\partial p_1} : -1 : -\frac{\partial p_2}{\partial q_1}.$$

Quarum Multiplicatorem patet ***unitati*** aequari, cum summa differentialium quantitatum ad dextram, respective secundum q_1, q_2, p_1 sumtorum, evanescat. Unde si post primam integrationem exprimitur p_1 per q_1, q_2 et Constantem Arbitrariam α, secundum principium ultimi Multiplicatoris fit alterum Integrale,

$$14.\quad \int \frac{\partial p_1}{\partial \alpha}\left\{dq_1 + \frac{\partial p_2}{\partial p_1} dq_2\right\} = \text{Const.}$$

Sub integrationis signo haberi differentiale completum, e ***Lagrangiana*** aequationum differentialium partialium theoria sic probatur. Nam cum expressis p_1 et p_2 per q_1 et q_2 fieri debeat $p_1 dq_1 + p_2 dq_2$ differentiale completum atque p_2 per

q_1, q_2, p_1 expressum detur, pro p_1 talis sumi debet quantitatum q_1 et q_2 functio quae satisfaciat conditioni,

$$\frac{\partial p_1}{\partial q_2} - \frac{\partial p_2}{\partial p_2} \cdot \frac{\partial p_1}{\partial q_1} - \frac{\partial p_2}{\partial q_1} = 0.$$

Qualem functionem, e theoria aequationum differentialium partialium primi ordinis *linearium* constat, e quocunque Integrali aequationum differentialium vulgarium (13.) erui. Quod ubi Constantem Arbitrariam α implicat, eandem implicabunt valores ipsarum p_1 et p_2 per q_1 et q_2 exhibiti, qui expressionem $p_1 dq_1 + p_2 dq_2$ integrabilem reddebant. Quae secundum Constantem α differentiata rursus prodire debet expressio integrabilis, sive expressio

$$\frac{\partial p_1}{\partial \alpha} dq_1 + \frac{\partial p_2}{\partial p_1} \cdot \frac{\partial p_1}{\partial \alpha} dq_2 = \frac{\partial p_1}{\partial \alpha} \left\{ dq_1 + \frac{\partial p_2}{\partial p_1} dq_2 \right\}$$

evadere debet differentiale completum. Q. D. E. Simul videmus, Integrale (14.) obtineri aequiparando novae Constanti Arbitrariae differentiale partiale solutionis $V = \int \{p_1 dq_1 + p_2 dq_2\}$, ipsius α respectu sumtum, id quod cum supra expositis convenit.

De Multiplicatore aequationum differentialium vulgarium systematis quod mediante solutione completa problematis *Pfaffiani* integratur. Conditiones ut aequatio differentialis vulgaris linearis primi ordinis inter p variabiles per pauciores quam $\frac{1}{2}p$ aequationes integrari possit.

§. 20.

Problema ***Pfaffianum*** voco integrationem singularis aequationis differentialis linearis primi ordinis inter numerum variabilium parem per semissem aequationum finitarum numerum. Sit aequatio differentialis singularis proposita,

$$1. \quad 0 = X_1 dx_1 + X_2 dx_2 \dots + X_{2m} dx_{2m},$$

designantibus X_1, X_2 etc. variabilium x_1, x_2, ... x_{2m} functiones quascunque. Qua integrata per numerum m aequationum, totidem Constantibus Arbitrariis affectarum, demonstravi ***Diar. Crell. Vol. XVII. pgg. 148 sqq.***, praestari integrationem completam systematis aequationum differentialium sequentis,

$$2. \quad \begin{cases} X_1 dt = \quad * \quad + a_{1,2} dx_2 + a_{1,3} dx_3 \dots + a_{1,2m} dx_{2m}, \\ X_2 dt = -a_{1,2} dx_1 \quad * \quad + a_{2,3} dx_3 \dots + a_{2,2m} dx_{2m}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ X_{2m} dt = -a_{1,2m} dx_1 - a_{2,2m} dx_2 \dots\dots\dots \quad * \quad, \end{cases}$$

ubi

$$3. \quad a_{i,k} = -a_{k,i} = \frac{\partial X_i}{\partial x_k} - \frac{\partial X_k}{\partial x_i}, \qquad a_{i,i} = 0.$$

Dedi in ***Diario Crell. Vol. II. pgg. 354 sqq.*** resolutionem algebraicam generalem aequationum linearium ad instar aequationum (2.) formatarum. Cuius ope exhibitis aequationibus differentialibus forma proportionum nobis usitata,

$$4. \quad dx_1 : dx_2 \ldots : dx_{2m} = A_1 : A_2 \ldots : A_{2m},$$

investigemus formulam qua aequationum (4.) Multiplicator definiatur sive valorem expressionis

$$5. \quad \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} \ldots + \frac{\partial A_{2m}}{\partial x_{2m}} = -A_i \frac{d \log M}{dx_i}.$$

Auspicabor ab aequationum linearium (2.) resolutione quae sic proponi potest.

Deriventur de producto

$$a_{1,2}\, a_{3,4} \ldots a_{2m-1,2m}$$

alii similes termini, mutando indices $2, 3, \ldots 2m-1, 2m$ respective in $3, 4, \ldots 2m, 2$, eandemque indicum commutationem repetendo, donec ad terminum primitivum reditur, id quod suggerit $2m-1$ terminos diversos. Ea ratione, indicum certo ordine proposito, si quisque eorum in proxime sequentem, ultimus in primum mutatur idque repetitur dum ad ordinem indicum primitivum reditur, dicam ***indices cyclum percurrere.*** Postquam e producto proposito $2m-1$ termini deducti sunt per cyclum, quem indices $2, 3, \ldots 2m$ fecimus percurrere, rursus in eorum terminorum unoquoque ponamus indices $2m-3$ postremos cyclum percurrere, unde nanciscimur terminorum numerum $(2m-1)(2m-3)$. In eorum terminorum unoquoque rursus ponamus indices $2m-5$ postremos cyclum percurrere, erit terminorum diversorum provenientium numerus totalis $(2m-1)(2m-3)(2m-5)$. Ita pergendo donec postremo soli tres indices postremi cyclum percurrant, producta $3.5 \ldots (2m-1)$ ex uno proposito deducta erunt, quorum omnium aggregatum R vocemus. Sit ex. gr. $m = 3$, erit R aggregatum ***quindecim*** terminorum,

$$\begin{aligned} & a_{1,2}\, a_{3,4}\, a_{5,6} + a_{1,2}\, a_{3,5}\, a_{6,4} + a_{1,2}\, a_{3,6}\, a_{4,5} \\ + & a_{1,3}\, a_{4,5}\, a_{6,2} + a_{1,3}\, a_{4,6}\, a_{2,5} + a_{1,3}\, a_{4,2}\, a_{5,6} \\ + & a_{1,4}\, a_{5,6}\, a_{2,3} + a_{1,4}\, a_{5,2}\, a_{3,6} + a_{1,4}\, a_{5,3}\, a_{6,2} \\ + & a_{1,5}\, a_{6,2}\, a_{3,4} + a_{1,5}\, a_{6,3}\, a_{4,2} + a_{1,5}\, a_{6,4}\, a_{2,3} \\ + & a_{1,6}\, a_{2,3}\, a_{4,5} + a_{1,6}\, a_{2,4}\, a_{5,3} + a_{1,6}\, a_{2,5}\, a_{3,4}, \end{aligned}$$

quorum quinque in prima verticali ex eorum uno derivantur, identidem mutando indices 2, 3, 4, 5, 6 in 3, 4, 5, 6, 2; terni iuxta positi indicibus tribus posterioribus cyclum percurrentibus ex uno eorum fluunt. Aggregatum R fit denominator communis expressionum algebraicarum quibus valores incognitarum exhibentur. Numeratorum autem Coëfficientes, qui ducuntur in terminos ad laevam aequationum linearium constitutos, sunt ipsius R differentialia, quantitatum $a_{i,k}$

respectu sumta, ita ut aequationum (2.) resolutione proveniant valores,

$$5^*.\quad \begin{cases} R\frac{dx_1}{dt} = \quad * \quad -\frac{\partial R}{\partial a_{1,2}}X_2 - \dots - \frac{\partial R}{\partial a_{1,2m}}X_{2m}, \\ R\frac{dx_2}{dt} = \frac{\partial R}{\partial a_{1,2}}X_1 \quad * \quad - \dots - \frac{\partial R}{\partial a_{2,2m}}X_{2m}, \\ \dots\dots\dots\dots\dots \\ R\frac{dx_{2m}}{dt} = \frac{\partial R}{\partial a_{1,2m}}X_1 + \frac{\partial R}{\partial a_{2,2m}}X_2 + \dots\dots \quad * \quad . \end{cases}$$

Aggregatum R gaudet proprietatibus plane analogis earum quae de Determinantibus circumferuntur. Quarum gravissima ea est ut *binis indicum* $1, 2, \dots 2m$ *inter se permutatis simul omnes ipsius R termini valores oppositos induant ideoque ipsum R in valorem oppositum abeat.* Porro fit

$$6.\quad R = a_{1,i}\frac{\partial R}{\partial a_{1,i}} + a_{2,i}\frac{\partial R}{\partial a_{2,i}} \dots + a_{2m,i}\frac{\partial R}{\partial a_{2m,i}},$$

et quoties i et k inter se diversi sunt,

$$7.\quad 0 = a_{1,i}\frac{\partial R}{\partial a_{1,k}} + a_{2,i}\frac{\partial R}{\partial a_{2,k}} \dots + a_{2m,i}\frac{\partial R}{\partial a_{2m,k}},$$

ubi terminus in $a_{k,i}$ ductus ommittendus est. Designantibus i, i', i'' etc. indices inter se diversos, si sumuntur differentialia partialia

$$\frac{\partial R}{\partial a_{i,i'}},\quad \frac{\partial^2 R}{\partial a_{i,i'}\,\partial a_{i'',i'''}},\quad \text{etc.}$$

ea erunt aggregata ad instar aggregati R formata, respective reiectis Coëfficientium binis, quatuor etc. seriebus cum horizontalibus tum verticalibus, eritque

$$8.\quad \frac{\partial^2 R}{\partial a_{i,i'}\,\partial a_{i'',i'''}} = \frac{\partial^2 R}{\partial a_{i,i''}\,\partial a_{i''',i'}} = \frac{\partial^2 R}{\partial a_{i,i'''}\,\partial a_{i',i''}}.$$

His rebus praemissis, quarum demonstrationem aliis relinquo vel ad alium locum relego, Multiplicator quaesitus sic invenitur. Sequitur e (5*.), siquidem signo summatorio subscribuntur indices quorum respectu summatio instituenda est,

$$9.\quad R\frac{dx_i}{dt} = A_i = \sum_\alpha \frac{\partial R}{\partial a_{\alpha,i}}X_\alpha,$$

unde

$$10.\quad \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} \dots + \frac{\partial A_{2m}}{\partial x_{2m}} = -\sum_{\alpha,i}\frac{\partial.\frac{\partial R}{\partial a_{\alpha,i}}}{\partial x_i} - \sum_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}}.\frac{\partial X_\alpha}{\partial x_i},$$

ubi indicibus α et i tribuuntur valores $1, 2, \dots 2m$, solis ommissis valoribus $i = \alpha$. Examinemus formulae (10.) summam priorem. Aggregati $\frac{\partial R}{\partial a_{\alpha,i}}$ cum terminus nullus afficiatur elemento cuius alter index est α aut i, fit

$$\frac{\partial.\frac{\partial R}{\partial a_{\alpha,i}}}{\partial x_i} = \sum_{k,l} \frac{\partial^2 R}{\partial a_{\alpha,i}\,\partial a_{k,l}}\cdot\frac{\partial a_{k,l}}{\partial x_i},$$

summatione duplici ad omnes $\frac{(2m-2)(2m-3)}{1\,.\,2}$ combinationes extensa, quibus indices k et l valores obtinent et inter se et ab ipsis α et i diversos. E formula antecedente sequitur,

$$\sum_i \frac{\partial.\frac{\partial R}{\partial a_{\alpha,i}}}{\partial x_i} = \sum_{i,k,l} \frac{\partial^2 R}{\partial a_{\alpha,i}\,\partial a_{k,l}}\cdot\frac{\partial a_{k,l}}{\partial x_i},$$

ubi indicum i, k, l valores in quoque termino sub signo summatorio et inter se et ab indice α diversi sunt, ipsi i valores $1, 2, \ldots. 2m$ conveniunt, binorum k et l valores non inter se permutari debent. Unde triplex summa conflatur e $\frac{(2m-1)(2m-2)(2m-3)}{1\,.\,2\,.\,3}$ terminis huiusmodi,

$$\frac{\partial^2 R}{\partial a_{\alpha,i}\,\partial a_{k,l}}\left\{\frac{\partial a_{k,l}}{\partial x_i}+\frac{\partial a_{l,i}}{\partial x_k}+\frac{\partial a_{i,k}}{\partial x_l}\right\},$$

qui obtinentur sumendo pro indicibus i, k, l ternos diversos ex indicibus $1, 2, \ldots. \alpha-1, \alpha+1, \ldots. 2m$. At substituendo quantitatum $a_{i,k}$ valores (3.), ternorum terminorum uncis inclusorum summa,

$$\frac{\partial a_{k,l}}{\partial x_i}+\frac{\partial a_{l,i}}{\partial x_k}+\frac{\partial a_{i,k}}{\partial x_l},$$

identice evanescit, ideoque pro quoque ipsius α valore fit,

$$11.\qquad \sum_i \frac{\partial.\frac{\partial R}{\partial a_{\alpha,i}}}{\partial x_i} = 0,$$

sive formulae (10.) prior summa evanescit. Alterius summae valor facile invenitur permutando indices α et i formulamque (6.) in auxilium vocando, quae summata pro omnibus indicis i valoribus suppeditat,

$$\sum_{\alpha,i} a_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}} = 2m.R.$$

Hinc enim fit,

$$\sum_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}}\cdot\frac{\partial X_\alpha}{\partial x_i} = \tfrac{1}{2}\sum_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}}\left\{\frac{\partial X_\alpha}{\partial x_i}-\frac{\partial X_i}{\partial x_\alpha}\right\} = \tfrac{1}{2}\sum_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}}a_{\alpha,i} = mR.$$

Unde iam formula (10.) in hanc abit,

$$12.\qquad \frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2}\ldots.+\frac{\partial A_{2m}}{\partial x_{2m}} = mR.$$

Cuius formulae pars laeva cum secundum (5.) et (9.) ipsi $-R\frac{d\log M}{dt}$ aequetur, aequationum differentialium (4.) Multiplicatorem statuere licet

$$13.\quad M = e^{-mt}.$$

Docet ea formula, aequationibus differentialibus (4.) complete integratis ad eruendam relationem inter t et variabiles x_i nulla amplius opus esse Quadratura, sed valorem integralis

$$\int \frac{R\,dx_i}{A_i} = t + \text{Const.}$$

exhiberi posse per logarithmum Determinantis functionum quae Constantibus Arbitrariis aequantur.

Ponamus quod semper licet $X_{2m} = -1$ sintque Coëfficientes reliqui omnes $X_1, X_2, \ldots, X_{2m-1}$ a variabili x_{2m} vacui, redit problema ***Pfaffianum*** in hoc, *ut expressio differentialis* $2m-1$ *variabilium*

$$X_1\,dx_1 + X_2\,dx_2 \ldots . + X_{2m-1}\,dx_{2m-1}$$

per $m-1$ *aequationes finitas reddatur differentiale completum* dx_{2m}. Scilicet ea re effecta, obtinetur m^{ta} aequatio per solas Quadraturas,

$$x_{2m} + \text{Const.} = \int \{X_1\,dx_1 + X_2\,dx_2 \ldots . + X_{2m-1}\,dx_{2m-1}\}.$$

Eo casu evanescunt omnes quantitates $a_{i,2m}$ ideoque ipsum quoque dt, unde aequationes differentiales (2.) in has abeunt,

$$14.\quad \begin{cases} 0 = \quad * \quad + a_{1,2}\,dx_2 + a_{1,3}\,dx_3 \ldots . + a_{1,2m-1}\,dx_{2m-1}, \\ 0 = a_{2,1}\,dx_1 \quad * \quad + a_{2,3}\,dx_3 \ldots . + a_{2,2m-1}\,dx_{2m-1}, \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ 0 = a_{2m-1,1}\,dx_1 + a_{2m-1,2}\,dx_2 + a_{2m-1,3}\,dx_3 \ldots \ldots \quad * \quad . \end{cases}$$

Quarum una e reliquis fluit, sicuti sequitur summando aequationes respective per $\frac{\partial R}{\partial a_{1,2m}}, \frac{\partial R}{\partial a_{2,2m}}, \ldots . \frac{\partial R}{\partial a_{2m-1,2m}}$ multiplicatas. Evanescentibus $a_{i,2m}$ evanescunt et ipsum R et omnia ipsius R differentialia $\frac{\partial R}{\partial a_{i,k}}$, in quibus neuter indicum i et k ipsi $2m$ aequatur. Unde e (9.) fit,

$$A_1 = \frac{\partial R}{\partial a_{1,2m}}, \quad A_2 = \frac{\partial R}{\partial a_{2,2m}}, \quad \ldots . \quad A_{2m-1} = \frac{\partial R}{\partial a_{2m-1,2m}},$$

$$A_{2m} = X_1 A_1 + X_2 A_2 \ldots . + X_{2m-1} A_{2m-1}.$$

Cum A_{2m} a variabili x_{2m} vacua sit, formula (12.) abit in hanc,

$$15.\quad \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_{2m}} \ldots . + \frac{\partial A_{2m-1}}{\partial x_{2m-1}} = 0.$$

Quae docet, *aequationum differentialium quae e* (14.) *proveniunt*,

$$16.\quad dx_1 : dx_2 \ldots . : dx_{2m-1} = A_1 : A_2 \ldots . : A_{2m-1}$$

Multiplicatorem aequari unitati.

Principium ultimi Multiplicatoris applicemus exemplo simplicissimo quo $m = 2$ sive quo aequationes differentiales proponuntur,

$$17.\quad dx_1 : dx_2 : dx_3 = \frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2} : \frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3} : \frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1}.$$

Inventa per primam integrationem variabilis x_3 expressione per x_1, x_2 et Constantem Arbitrariam α, secundum principium illud fit altera aequatio integralis,

$$18.\quad \int \frac{\partial x_3}{\partial \alpha} \left\{ \left(\frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3} \right) dx_1 + \left(\frac{\partial X_3}{\partial x_2} - \frac{\partial X_2}{\partial x_3} \right) dx_2 \right\} = \text{Const.}$$

Quantitatem sub integrationis signo differentiale completum esse, sic verificari potest. Substituta variabilis x_3 expressione per integrationem primam inventa in formula $X_1 dx_1 + X_2 dx_2 + X_3 dx_3$, obtinetur

$$\left(X_1 + X_3 \frac{\partial x_3}{\partial x_1} \right) dx_1 + \left(X_2 + X_3 \frac{\partial x_3}{\partial x_2} \right) dx_2.$$

Eadem expressione substituta in aequationibus differentialibus, prodit aequatio,

$$\frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1} = \frac{\partial x_3}{\partial x_1} \left\{ \frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2} \right\} + \frac{\partial x_3}{\partial x_2} \left\{ \frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3} \right\},$$

quae est conditio ut formula differentialis antecedens sit differentiale aliquod completum dx_4. Si ipsius x_3 expressio implicat Constantem Arbitrariam α, fit

$$\begin{aligned} d.\frac{\partial x_4}{\partial \alpha} &= \frac{\partial \left\{ X_1 + X_3 \frac{\partial x_3}{\partial x_1} \right\}}{\partial \alpha} dx_1 + \frac{\partial \left\{ X_2 + X_3 \frac{\partial x_3}{\partial x_2} \right\}}{\partial \alpha} dx_2 \\ &= \frac{\partial x_3}{\partial \alpha} \left\{ \left(\frac{\partial X_1}{\partial x_3} + \frac{\partial X_3}{\partial x_3} \cdot \frac{\partial x_3}{\partial x_1} \right) dx_1 + \left(\frac{\partial X_2}{\partial x_3} + \frac{\partial X_3}{\partial x_3} \cdot \frac{\partial x_3}{\partial x_2} \right) dx_2 \right\} \\ &\qquad + X_3 \left\{ \frac{\partial^2 x_3}{\partial x_1 \partial \alpha} dx_1 + \frac{\partial^2 x_3}{\partial x_2 \partial \alpha} dx_2 \right\} \\ &= \frac{\partial x_3}{\partial \alpha} \left\{ \left(\frac{\partial X_1}{\partial x_3} - \frac{\partial X_3}{\partial x_1} \right) dx_1 + \left(\frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2} \right) dx_2 \right\} \\ &\qquad + \frac{\partial x_3}{\partial \alpha} dX_3 + X_3 d \frac{\partial x_3}{\partial \alpha}. \end{aligned}$$

Unde sequitur, quod propositum erat, quantitatem sub integrationis signo aequari differentiali completo, videlicet differentiali

$$d.X_3 \frac{\partial x_3}{\partial \alpha} - d.\frac{\partial x_4}{\partial \alpha}.$$

Quod si igitur functio x_4 inventa est, aequationem integralem (18.) sic quoque repraesentare licet,

$$19.\quad X_3 \frac{\partial x_3}{\partial \alpha} - \frac{\partial x_4}{\partial \alpha} = \text{Const.}$$

Quae de formulis quoque generalibus deduci potuit, quas loco citato tradidi de aequationum differentialium (2.) systemate per solutionem completam aequationis (1.) integrando. Qua de integratione hac occasione novas addam propositiones novasque demonstrationes sequentes.

§. 21.

Ac primum comprobabo propositionem, *si aequatio differentialis singularis*

$$20.\quad X_1\,dx_1+X_2\,dx_2\ldots\ldots+X_p\,dx_p = 0$$

integretur per m aequationes quascunque, earum ope fieri, ut de quibusque m e numero p aequationum differentialium sequentium,

$$21.\quad \begin{cases} X_1\,dt = \quad * \quad +a_{1,2}\,dx_2+a_{1,3}\,dx_3\ldots\ldots+a_{1,p}\,dx_p, \\ X_2\,dt = a_{2,1}\,dx_1 \quad * \quad +a_{2,3}\,dx_3\ldots\ldots+a_{2,p}\,dx_p, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_p\,dt = a_{p,1}\,dx_1+a_{p,2}\,dx_2+a_{p,3}\,dx_3\ldots\ldots\ldots\ldots * \end{cases}$$

reliquae p—m sponte fluant, ipsis $a_{k,k'}$ designantibus quantitates $\frac{\partial X_k}{\partial x_{k'}}-\frac{\partial X_{k'}}{\partial x_k}$.
Cuius propositionis demonstrationem sic adorno.

Designo

per h, h' etc. indices $1, 2, \ldots\ldots m$,
per i, i' etc. indices $m+1, m+2, \ldots\ldots p$,
per k, k' etc. indices $1, 2, 3, \ldots\ldots p$.

Aequando $x_1, x_2, \ldots\ldots x_m$ *quibuscunque* reliquarum variabilium $x_{m+1}, x_{m+2}, \ldots\ldots x_p$ functionibus, abeunt aequationes (21.) in sequentes:

$$22.\quad 0 = u_k = X_k\,dt-\sum_i b_{k,i}\,dx_i,$$

siquidem statuitur,

$$23.\quad b_{k,i} = a_{k,1}\frac{\partial x_1}{\partial x_i}+a_{k,2}\frac{\partial x_2}{\partial x_i}\ldots\ldots+a_{k,m}\frac{\partial x_m}{\partial x_i}+a_{k,i}$$
$$= a_{k,i}+\sum_h a_{k,h}\frac{\partial x_h}{\partial x_i}.$$

Ponamus porro

$$24.\quad v_i = X_1\frac{\partial x_1}{\partial x_i}+X_2\frac{\partial x_2}{\partial x_i}\ldots\ldots+X_m\frac{\partial x_m}{\partial x_i}+X_i,$$

erit substituendo (22.):

$$25.\quad \frac{\partial x_1}{\partial x_{i'}}u_1+\frac{\partial x_2}{\partial x_{i'}}u_2\ldots\ldots+\frac{\partial x_m}{\partial x_{i'}}u_m+u_{i'} = v_{i'}\,dt-\sum_i c_{i',i}\,dx_i,$$

posito

$$26.\quad c_{i',i} = \frac{\partial x_1}{\partial x_{i'}}b_{1,i}+\frac{\partial x_2}{\partial x_{i'}}b_{2,i}\ldots\ldots+\frac{\partial x_m}{\partial x_{i'}}b_{m,i}+b_{i',i}$$
$$= b_{i',i}+\sum_h\frac{\partial x_h}{\partial x_{i'}}b_{h,i}.$$

Substituendo ipsorum $b_{k,i}$ valores (23.), induit $c_{i',i}$ valorem sequentem,

$$27.\quad c_{i',i} = a_{i',i} + \sum_h a_{i',h}\frac{\partial x_h}{\partial x_i} + \sum_h a_{h,i}\frac{\partial x_h}{\partial x_{i'}} + \sum_{h,h'} a_{h',h}\frac{\partial x_h}{\partial x_i}\cdot\frac{\partial x_{h'}}{\partial x_{i'}},$$

sive reponendo quantitatum $a_{k,k'}$ valores,

$$28.\quad c_{i',i} = \frac{\partial X_{i'}}{\partial x_i} - \frac{\partial X_i}{\partial x_{i'}} + \sum_h\left\{\frac{\partial X_{i'}}{\partial x_h} - \frac{\partial X_h}{\partial x_{i'}}\right\}\frac{\partial x_h}{\partial x_i} + \sum_h\left\{\frac{\partial X_h}{\partial x_i} - \frac{\partial X_i}{\partial x_h}\right\}\frac{\partial x_h}{\partial x_{i'}}$$
$$+ \sum_{h,h'}\left\{\frac{\partial X_{h'}}{\partial x_h} - \frac{\partial X_h}{\partial x_{h'}}\right\}\frac{\partial x_h}{\partial x_i}\cdot\frac{\partial x_{h'}}{\partial x_{i'}}.$$

Includamus uncis differentialia partialia, in quibus solae x_i sive $x_{m+1}, x_{m+2}, \dots x_p$ pro independentibus habentur atque quantitates x_h sive $x_1, x_2, \dots x_m$ pro earum functionibus: erit

$$29.\quad \left(\frac{\partial X_k}{\partial x_i}\right) = \frac{\partial X_k}{\partial x_i} + \sum_h \frac{\partial X_k}{\partial x_h}\cdot\frac{\partial x_h}{\partial x_i},$$

unde

$$30.\quad c_{i',i} = \left(\frac{\partial X_{i'}}{\partial x_i}\right) - \left(\frac{\partial X_i}{\partial x_{i'}}\right) + \sum_h\left\{\left(\frac{\partial X_h}{\partial x_i}\right)\frac{\partial x_h}{\partial x_{i'}} - \left(\frac{\partial X_h}{\partial x_{i'}}\right)\frac{\partial x_h}{\partial x_i}\right\}.$$

Id quod sequitur, indicibus h et h' in summa duplici $\sum_{h,h'}\frac{\partial X_{h'}}{\partial x_h}\cdot\frac{\partial x_h}{\partial x_i}\cdot\frac{\partial x_{h'}}{\partial x_{i'}}$ inter se permutatis nec non in (29.) scripto h' ipsius h loco. Inventam autem ipsius $c_{i',i}$ expressionem (30.) ope formulae (24.) sic exhibere licet,

$$31.\quad c_{i',i} = \left(\frac{\partial v_{i'}}{\partial x_i}\right) - \left(\frac{\partial v_i}{\partial x_{i'}}\right),$$

reiectis qui se mutuo destruunt terminis,

$$X_h\frac{\partial^2 x_h}{\partial x_{i'}\partial x_i} - X_h\frac{\partial^2 x_h}{\partial x_i\partial x_{i'}}.$$

Quo ipsius $c_{i',i}$ valore substituto in (25.), eruimus formulam, quae valet *quaecunque sint quantitates x_h reliquarum x_i functiones,*

$$32.\quad \frac{\partial x_1}{\partial x_{i'}}u_1 + \frac{\partial x_2}{\partial x_{i'}}u_2 \dots + \frac{\partial x_m}{\partial x_{i'}}u_m + u_{i'} = v_{i'}dt + \sum_i\left\{\left(\frac{\partial v_i}{\partial x_{i'}}\right) - \left(\frac{\partial v_{i'}}{\partial x_i}\right)\right\}dx_i.$$

Quantitatibus x_h per variabiles x_i expressis cum fiat e (24.)

$$33.\quad X_1dx_1 + X_2dx_2 \dots + X_pdx_p = v_{m+1}dx_{m+1} + v_{m+2}dx_{m+2} \dots + v_pdp,$$

si per m aequationes, quibus quantitates x_h per variabiles $x_{m+1}, x_{m+2}, \dots x_p$ determinantur, aequatio differentialis (20.) integratur, singuli termini ad dextram formulae (33.) per se evanescere debent, sive fieri debet

$$34.\quad v_{m+1} = v_{m+2} \dots = v_p = 0.$$

Unde etiam aequationis (32.) pars laeva evanescere debet sive, scribendo i ipsius i' loco, pro quolibet ipsius i valore fieri debet,

$$34^*.\quad \frac{\partial x_1}{\partial x_i}u_1 + \frac{\partial x_2}{\partial x_i}u_2 \dots + \frac{\partial x_m}{\partial x_i}u_m + u_i = 0.$$

Quae formula docet, si per m aequationes integretur aequatio differentialis (20.), earum aequationum ope fieri, ut ex aequationibus

$$u_1 = 0, \quad u_2 = 0, \quad \dots\dots \quad u_m = 0$$

reliquae

$$u_{m+1} = 0, \quad u_{m+2} = 0, \quad \dots\dots \quad u_p = 0$$

sponte fluant. Q. D. E.

Si $p > 2m$, inter coëfficientes X_1, X_2 etc. certae quaedam locum habere debent relationes, cum determinando m functiones $x_1, x_2, \dots\dots x_m$ satisfieri debeat pluribus conditionibus, videlicet $p-m$ aequationibus,

$$0 = v_i = X_1 \frac{\partial x_1}{\partial x_i} + X_2 \frac{\partial x_2}{\partial x_i} \dots\dots + X_m \frac{\partial x_m}{\partial x_i} + X_i.$$

Quae relationes obtineri possunt e formula (32.). Nam secundum eam formulam aequationibus differentialibus (21.) sive aequationibus

$$u_1 = 0, \quad u_2 = 0, \quad \dots\dots \quad u_p = 0$$

satisfit per numerum $2m$ aequationum, videlicet per m aequationes, quibus $x_1, x_2, \dots\dots x_m$ per reliquas variabiles determinantur, atque m aequationes differentiales $u_1 = 0$, $u_2 = 0$, ... $u_m = 0$. Unde inter quantitates X_1, X_2 etc. tales locum habere debent relationes, ut de p aequationum (21.) numero $2m$ reliquae $p-2m$ sponte fluant sive, ope $2m$ aequationum differentialium $u_1=0, u_2=0, \dots\dots u_{2m}=0$ eliminatis $2m$ differentialibus $dx_1, dx_2, \dots\dots dx_{2m}$, reliquae $p-2m$ aequationes differentiales, $u_{2m+1}=0$, $u_{2m+2}=0$, ... $u_p=0$, identicae evadant. Secundum observationem olim a me factam in Diar. *Crell.* Vol. II. pag. 357, hae $p-2m$ aequationes post eam eliminationem formam induunt eandem atque propositae (21.), videlicet formam huiusmodi,

$$\begin{array}{lllll} F_1\,dt = & * & + f_{1,2}\,dx_{2m+2} & + f_{1,3}\,dx_{2m+3} \dots\dots & + f_{1,p-2m}\,dx_p, \\ F_2\,dt = & f_{2,1}\,dx_{2m+1} & * & + f_{2,3}\,dx_{2m+3} \dots\dots & + f_{2,p-2m}\,dx_p, \\ \dots & \dots & \dots & \dots & \dots \\ F_{p-2m}\,dt = & f_{p-2m,1}\,dx_{2m+1} & + f_{p-2m,2}\,dx_{2m+2} & \dots\dots\dots\dots & *\ , \end{array}$$

ubi $f_{i,k} = -f_{k,i}$. Quae aequationes ut identicae evadant, evanescere debent et $p-2m$ quantitates F_i et $\frac{(p-2m)(p-2m-1)}{2}$ quantitates $f_{i,k}$. Unde *locum habere debent* $\frac{(p-2m)(p-2m+1)}{1\,.\,2}$ *conditiones ut aequatio differentialis linearis primi ordinis inter p variabiles* (20.) *per* $m<\frac{1}{2}p$ *aequationes integrari possit, eaedemque sunt conditiones quibus efficitur, ut p aequationes lineares* (21.) *ex earum numero* $2m$ *fluant.* Si $p=2m+1$, prodit una conditio iam a Cl. ***Pfaff*** olim exhibita, quae si $m=1$ notam conditionem integrabilitatis suppeditat. Si $p=2m+2$, locum habere debent tres conditiones, quas pro $m=1$ accuratius examinemus.

Sit igitur propositum indagare conditiones, ut aequatio differentialis linearis inter *quatuor* variabiles,

$$35.\quad X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 = 0,$$

unica aequatione integrari possit. Qua aequatione si exprimitur una variabilium x_4 per x_1, x_2, x_3, proposita (35.) identica fieri debet, id quod aequationes poscit sequentes,

$$36.\quad \frac{\partial x_4}{\partial x_1} = -\frac{X_1}{X_4},\quad \frac{\partial x_4}{\partial x_2} = -\frac{X_2}{X_4},\quad \frac{\partial x_4}{\partial x_3} = -\frac{X_3}{X_4}.$$

Secunda et tertia earum aequationum suppeditat,

$$X_4^2 \frac{\partial^2 x}{\partial x_2 \partial x_3} = X_2 \left\{\frac{\partial X_4}{\partial x_3} - \frac{X_3}{X_4}\cdot\frac{\partial X_4}{\partial x_4}\right\} - X_4 \left\{\frac{\partial X_2}{\partial x_3} - \frac{X_3}{X_4}\cdot\frac{\partial X_2}{\partial x_4}\right\}$$
$$= X_3 \left\{\frac{\partial X_4}{\partial x_2} - \frac{X_2}{X_4}\cdot\frac{\partial X_4}{\partial x_4}\right\} - X_4 \left\{\frac{\partial X_3}{\partial x_2} - \frac{X_2}{X_4}\cdot\frac{\partial X_3}{\partial x_4}\right\}.$$

Unde ponendo $a_{i,k} = \frac{\partial X_i}{\partial x_k} - \frac{\partial X_k}{\partial x_i}$ similesque aequationes de tertia et prima, de prima et secunda aequationum (36.) deducendo obtinentur tres primae aequationum sequentium, quibus duas alias addidi ex iis provenientes,

$$37.\quad \begin{cases} 0 = \quad * \quad + a_{3,4} X_2 + a_{4,2} X_3 + a_{2,3} X_4, \\ 0 = a_{4,3} X_1 \quad * \quad + a_{1,4} X_3 + a_{3,1} X_4, \\ 0 = a_{2,4} X_1 + a_{4,1} X_2 \quad * \quad + a_{1,2} X_4, \\ 0 = a_{3,2} X_1 + a_{1,3} X_2 + a_{2,1} X_3 \quad * \quad, \\ 0 = a_{2,3} a_{1,4} + a_{3,1} a_{2,4} + a_{1,2} a_{3,4}. \end{cases}$$

Ad easdem autem relationes secundum propositionem generalem supra conditam pervenire debemus, si quaerimus conditiones ut quatuor aequationum linearium,

$$\begin{aligned} X_1\, dt &= \quad * \quad + a_{1,2}\, dx_2 + a_{1,3}\, dx_3 + a_{1,4}\, dx_4, \\ X_2\, dt &= a_{2,1}\, dx_1 \quad * \quad + a_{2,3}\, dx_3 + a_{2,4}\, dx_4, \\ X_3\, dt &= a_{3,1}\, dx_1 + a_{3,2}\, dx_2 \quad * \quad + a_{3,4}\, dx_4, \\ X_4\, dt &= a_{4,1}\, dx_1 + a_{4,2}\, dx_2 + a_{4,3}\, dx_3 \quad * \end{aligned}$$

binae e duabus reliquis fluant. Quod re vera fieri, facile comprobatur. Aequationum (37.) quatuor primae sunt notae conditiones integrabilitatis aequationis differentialis linearis primi ordinis inter tres variabiles, ex eadem aequatione (35.) provenientis si successive x_1, x_2, x_3, x_4 constantes ponuntur. Quatuor illarum aequationum ternae cum quartam secum ducant, sequitur, *si tres aequationes,*

$$\begin{aligned} X_2\, dx_2 + X_3\, dx_3 + X_4\, dx_4 &= 0, \\ X_1\, dx_1 + X_3\, dx_3 + X_4\, dx_4 &= 0, \\ X_1\, dx_1 + X_2\, dx_2 + X_4\, dx_4 &= 0, \end{aligned}$$

habitis respective x_1, x_2, x_3 *pro Constantibus, conditioni integrabilitatis* *satisfaciant, hanc quoque aequationem,*

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 = 0,$$

si in ea x_4 pro Constante habeatur, conditioni integrabilitatis satisfacturam esse, nec non aequationem, $X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 = 0$, in qua omnes quatuor quantitates x_1, x_2, x_3, x_4 variabiles sunt, unica aequatione integrari posse. Ut ipsa absolvatur integratio, opus erit integratione completa trium aequationum differentialium primi ordinis inter duas variabiles, id quod simili ratione demonstratur atque in tractatibus Calculi Integralis probatur, ad integrandam aequationem differentialem linearem primi ordinis inter tres variabiles, conditioni integrabilitatis satisfacientem, requiri integrationem completam duarum aequationum differentialium primi ordinis inter duas variabiles. Quae res in tractatibus ita proponi solet, ut alteram ne condere quidem liceat aequationem differentialem, nisi iam antea altera complete integrata habeatur. At observo, si aequatio differentialis inter tres variabiles x_1, x_2, x_3, conditioni integrabilitatis satisfaciens, est $X_1 dx_1 + X_2 dx_2 + X_3 dx_3 = 0$, pro duabus aequationibus inter duas variabiles integrandis sumi posse has, quae *separatim* tractari possint.

$$X_1 dx_1 + X_2 dx_2 = 0, \qquad X_2^0 dx_2 + X_3^0 dx_3 = 0,$$

quae e proposita proveniunt, prima habendo x_3 pro Constante, secunda ponendo $x_1 = 0$. Scilicet post integrationem secundae in locum ipsius x_2 substituenda est ea quantitatum x_1, x_2, x_3 functio, quae per integrationem primae aequiparatur valori variabilis x_2 qui ipsi $x_1 = 0$ respondet. Similiter, si proponitur integrare aequationem inter quatuor variabiles,

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 = 0,$$

conditionibus (37.) locum habentibus, pro tribus aequationibus inter duas variabiles, quae integrandae sunt, sumi possunt sequentes separatim tractandae,

$$X_1 dx_1 + X_2 dx_2 = 0, \quad X_2^0 dx_2 + X_3^0 dx_3 = 0, \quad X_3^{00} dx_3 + X_4^{00} dx_4 = 0,$$

in quibus designant X_2^0 et X_3^0 valores in quos X_2 et X_3 abeunt pro $x_1 = 0$, porro X_3^{00} et X_4^{00} valores in quos X_3 et X_4 pro $x_1 = x_2 = 0$ abeunt; deinde in prima aequatione x_3 et x_4, in secunda x_4 pro Constantibus habendae sunt. Integrata tertia aequatione, ipsi x_3 ea substituenda est quantitatum x_2, x_3, x_4 functio, quae per integrationem secundae aequat variabilis x_3 valorem ipsi $x_2 = 0$ respondentem; ac deinde ipsi x_2 ea quantitatum x_1, x_2, x_3, x_4 functio substituenda est, quae per aequationis primae integrationem aequat variabilis x_2 valorem ipsi $x_1 = 0$ respondentem.

Propositis p aequationibus differentialibus vulgaribus inter $p+1$ variabiles quibuscunque, aequationes m inter ipsas variabiles sunt integrales propositarum, si efficiunt, ut harum numerus m e reliquis $p-m$ fluat; porro tale constituunt aequationum integralum systema, e quo per differentiationem aequationumque differentialium substitutionem aliae novae non obtineantur, si earum adiumento non plures quam m aequationes differentiales e reliquis fluunt. Antecedentibus vidimus, per m aequationes, quibus integretur aequatio differentialis vulgaris linearis inter p variabiles (20.), fieri ut e p aequationum differentialium vulgarium (21.) numero m reliquae $p-m$ sponte fluant. Unde si $p-m=m$ sive $p=2m$, qui est casus problematis *Pfaffiani*, sequitur, *quascunque m* aequationes, quibus integretur aequatio differentialis linearis primi ordinis inter $2m$ variabiles,

$$0 = X_1\,dx_1 + X_2\,dx_2 \ldots\ldots + X_{2m}\,dx_{2m},$$

haberi posse pro integralibus systematis $2m$ aequationum differentialium vulgarium,

$$X_k\,dt = a_{k,1}\,dx_1 + a_{k,2}\,dx_2 \ldots\ldots + a_{k,2m}\,dx_{2m},$$

ex iisque per differentiationem novas deduci non posse aequationes integrales. Si $m < \frac{1}{2}p$ atque aequatio (20.) integrari potest m aequationibus, vidimus p aequationum (21.) tantum $2m$ a se independentes esse, reliquas $p-2m$ ex iis sponte fluere; unde ex arbitrio iis addere licet $p-2m$ aequationes differentiales, ut habeatur systema p aequationum differentialium inter $p+1$ variabiles. Eo casu aequationes m, quibus aequatio (20.) integrari supponitur, rursus haberi possunt pro aequationibus eius systematis integralibus, quaecunque sint $p-2m$ aequationes differentiales ipsis (21.) ex arbitrio adiectae, cum illae m aequationes efficiant, quod e (32.) sequebatur, ut m aequationes differentiales $u_{m+1}=0$, $u_{m+2}=0$, ... $u_{2m}=0$ ex aliis systematis aequationibus differentialibus, $u_1=0$, $u_2=0$, ... $u_m=0$ obtineantur.

Designantibus A_1, A_2 etc. quascunque variabilium x_1, x_2, ... x_p functiones, quoties aequationum differentialium,

$$dx_1 : dx_2 \ldots\ldots : dx_p = A_1 : A_2 \ldots\ldots A_p,$$

dantur aequationes integrales m, quarum differentiatione aliae novae non prodeunt, earumque ope exprimuntur x_1, x_2, ... x_m ut functiones variabilium x_{m+1}, x_{m+2}, ... x_p, eas functiones satisfacere constat systemati aequationum differentialium partialium linearium primi ordinis sequenti,

$$38.\quad \begin{cases} A_1 = A_{m+1}\dfrac{\partial x_1}{\partial x_{m+1}} + A_{m+2}\dfrac{\partial x_1}{\partial x_{m+2}} \ldots\ldots + A_p\dfrac{\partial x_1}{\partial x_p}, \\ A_2 = A_{m+1}\dfrac{\partial x_2}{\partial x_{m+1}} + A_{m+2}\dfrac{\partial x_2}{\partial x_{m+2}} \ldots\ldots + A_p\dfrac{\partial x_2}{\partial x_p}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_m = A_{m+1}\dfrac{\partial x_m}{\partial x_{m+1}} + A_{m+2}\dfrac{\partial x_m}{\partial x_{m+2}} \ldots\ldots + A_p\dfrac{\partial x_m}{\partial x_p}. \end{cases}$$

Qua de re pluribus egi in alia Commentatione Diar. *Crell.* Vol. XXIII. inserta. Systema (38.) ita est comparatum ut in quaque aequatione eiusdem functionis reperiantur differentialia partialia secundum diversas variabiles independentes sumta, atque differentialia partialia diversarum functionum secundum eandem variabilem independentem in diversis aequationibus sumta eodem afficiantur Coëfficiente. Eiusmodi systematis hoc, a cuius solutione problema *Pfaffianum* pendet,

$$39. \quad v_{m+1} = 0, \quad v_{m+2} = 0, \quad \dots \quad v_{2m} = 0,$$

quodammodo inversum est, sicuti e functionis v_i expressione (24.) patet; quippe in quaque huius systematis aequatione diversarum functionum differentialia reprehenduntur secundum eandem variabilem sumta, atque eiusdem functionis differentialia, secundum diversas variabiles independentes in diversis aequationibus sumta, eodem afficiuntur Coëfficiente. Secundum antecedentia e systemate (39.) sequitur aliud eius inversum formae systematis (38.). Nam ubi aequationes (2.) ad formam aequationum (9.) revocamus, sequitur ex antecedentibus, m aequationes quae systemati (39.) satisfaciant sive quibus (1.) integretur, ipsarum (9.) fieri aequationes integrales, quarum differentiatione aliae novae non prodeant, ideoque easdem systemati aequationum (38.) satisfacere. Unde haec obtinetur

Propositio.

„*E systemate aequationum differentialium partialium linearium primi ordinis huiusmodi,*

$$39^*. \quad \begin{cases} X_{m+1} = X_1 \dfrac{\partial x_1}{\partial x_{m+1}} + X_2 \dfrac{\partial x_2}{\partial x_{m+1}} \dots + X_m \dfrac{\partial x_m}{\partial x_{m+1}}, \\ X_{m+2} = X_1 \dfrac{\partial x_1}{\partial x_{m+2}} + X_2 \dfrac{\partial x_2}{\partial x_{m+2}} \dots + X_m \dfrac{\partial x_m}{\partial x_{m+2}}, \\ \dots\dots\dots\dots\dots\dots \\ X_{2m} = X_1 \dfrac{\partial x_1}{\partial x_{2m}} + X_2 \dfrac{\partial x_2}{\partial x_{2m}} \dots + X_m \dfrac{\partial x_m}{\partial x_{2m}}, \end{cases}$$

hoc sequitur alterum formae quodammodo inversae,

$$\begin{aligned} A_1 &= A_{m+1} \frac{\partial x_1}{\partial x_{m+1}} + A_{m+2} \frac{\partial x_1}{\partial x_{m+2}} \dots + A_{2m} \frac{\partial x_1}{\partial x_{2m}}, \\ A_2 &= A_{m+1} \frac{\partial x_2}{\partial x_{m+1}} + A_{m+2} \frac{\partial x_2}{\partial x_{m+2}} \dots + A_{2m} \frac{\partial x_2}{\partial x_{2m}}, \\ &\dots\dots\dots\dots\dots\dots \\ A_m &= A_{m+1} \frac{\partial x_m}{\partial x_{m+1}} + A_{m+2} \frac{\partial x_m}{\partial x_{m+2}} \dots + A_{2m} \frac{\partial x_m}{\partial x_{2m}}, \end{aligned}$$

ubi, posito $a_{k,k'} = \dfrac{\partial X_k}{\partial x_{k'}} - \dfrac{\partial X_{k'}}{\partial x_k}$ *ac designante* R *aggregatum, e* 1.3.5...

...(2m — 1) terminis huiusmodi

$$a_{1,2}\, a_{3\,4} \,\ldots\ldots\, a_{2m-1,2m}$$

ratione supra descripta conflatum, fit

$$A_k = \frac{\partial R}{\partial a_{1,k}} X_1 + \frac{\partial R}{\partial a_{2,k}} X_2 \ldots\ldots + \frac{\partial R}{\partial a_{2m,k}} X_{2m},$$

omisso termino in X_k ducto."

Huius memorabilis propositionis si demonstrationem cupis ab aequationum differentialium vulgarium consideratione independentem, rem sic adornare licet.

Sit rursus

$$v_i = X_1 \frac{\partial x_1}{\partial x_i} + X_2 \frac{\partial x_2}{\partial x_i} \ldots\ldots + X_m \frac{\partial x_m}{\partial x_i} + X_i,$$

ac designantibus

$$y,\; y_1,\; \ldots\ldots y_{2m}$$

quantitates *indefinitas,* ponatur

$$U_k = X_k \cdot y - a_{k,1} y_1 - a_{k,2} y_2 \ldots\ldots - a_{k,2m} y_{2m},$$

$$Y_h = y_h - \frac{\partial x_h}{\partial x_{m+1}} y_{m+1} - \frac{\partial x_h}{\partial x_{m+2}} y_{m+2} \ldots\ldots - \frac{\partial x_h}{\partial x_{2m}} y_{2m},$$

$$u_k = U_k + a_{k,1} Y_1 + a_{k,2} Y_2 \ldots\ldots + a_{k,m} Y_m.$$

Eodem modo atque (32.) probavimus, demonstratur, quaecunque sint x_1, x_2, ... x_m reliquarum variabilium x_{m+1}, x_{m+2}, ... x_{2m} functiones, fieri

$$\frac{\partial x_1}{\partial x_i} u_1 + \frac{\partial x_2}{\partial x_i} u_2 \ldots\ldots + \frac{\partial x_m}{\partial x_i} u_m + u_i = v_i y + \sum_{i'} \left\{ \left(\frac{\partial v_{i'}}{\partial x_i}\right) - \left(\frac{\partial v_i}{\partial x_{i'}}\right) \right\} y_{i'}.$$

Partes ad dextram signi aequalitatis evanescunt, ubi pro x_1, x_2, ... x_m sumuntur functiones satisfacientes m aequationibus $v_i = 0$, quae sunt ipsae functiones in theoremate tradito propositae, quas a se independentes esse subintelligo. Hinc si quantitatum u_k expressiones substituuntur atque statuitur

$$L_{i,h} = \frac{\partial x_1}{\partial x_i} a_{1,h} + \frac{\partial x_2}{\partial x_i} a_{2,h} \ldots\ldots + \frac{\partial x_m}{\partial x_i} a_{m,h} + a_{i,h},$$

sequitur per m aequationes $v_i = 0$ obtineri m sequentes,

$$40.\quad 0 = \frac{\partial x_1}{\partial x_i} U_1 + \frac{\partial x_2}{\partial x_i} U_2 \ldots\ldots + \frac{\partial x_m}{\partial x_i} U_m + L_{i,1} Y_1 + L_{i,2} Y_2 \ldots\ldots + L_{i,m} Y_m.$$

Supponamus, quantitatum indefinitarum y, y_1 etc. functiones lineares U_1, U_2, ... U_{2m} a se independentes esse, sive quantitatem, supra per R designatam,

$$\Sigma\, a_{1,2}\, a_{3,4} \ldots\ldots a_{2m-1,2m}$$

neque per se neque substituendo functionum x_k valores evanescere. Quae secundum supra tradita est conditio ut aequatio

$$X_1\,dx_1 + X_2\,dx_2 \ldots\ldots + X_{2m}\,dx_{2m} = 0$$

non paucioribus quam m aequationibus integrari possit. Eo casu etiam m functiones ipsarum Y_1, Y_2, ... Y_m lineares, quas per H_i designabo,

$$L_{i,1}Y_1 + L_{i,2}Y_2 \ldots\ldots + L_{i,m}Y_m = H_i,$$

a se independentes erunt, sive non dabuntur factores ab ipsis y_k independentes λ_1, λ_2 etc., qui efficiant

$$\lambda_1 H_{m+1} + \lambda_2 H_{m+2} \ldots\ldots + \lambda_m H_{2m} = 0.$$

Nam si eiusmodi dantur factores, secundum (40.) aut $x_1, x_2, \ldots\ldots x_i$ non a se independentes sunt aut datur aequatio inter functiones lineares $U_1, U_2, \ldots\ldots U_m$, quod utrumque contra suppositionem est. Functiones autem a se independentes H_{m+1}, H_{m+2}, ... H_{2m} omnes simul evanescere non possunt nisi simul evanescunt omnes Y_1, Y_2, ... Y_m. Iam igitur cum pro ipsarum y, y_1 etc. valoribus

$$y = R,\quad y_1 = A_1,\quad y_2 = A_2,\quad \ldots\ldots\quad y_{2m} = A_{2m}$$

omnes simul evanescant U_1, U_2, ... U_{2m}, siquidem quantitatum A_k, R valores sunt ipsi in Propositione tradita assignati, ideoque omnes secundum (40.) evanescant H_i, pro valoribus illis omnes quoque Y_1, Y_2, ... Y_m evanescere debent, sive pro ipsius h valoribus 1, 2, ... m fieri debet,

$$0 = A_h - \frac{\partial x_h}{\partial x_{m+1}}A_{m+1} - \frac{\partial x_h}{\partial x_{m+2}}A_{m+2} \ldots\ldots - \frac{\partial x_h}{\partial x_{2m}}A_{2m},$$

quae est propositio demonstranda.

Propositionis antecedentis pro casu simplicissimo $m=2$ hoc addam exemplum:

„Ubi semper ponitur $a_{\alpha,\beta} = \frac{\partial X_\alpha}{\partial x_\beta} - \frac{\partial X_\beta}{\partial x_\alpha}$, ex aequationibus

$$-X_3 = X_1\frac{\partial x_1}{\partial x_3} + X_2\frac{\partial x_2}{\partial x_3},$$

$$-X_4 = X_1\frac{\partial x_1}{\partial x_4} + X_2\frac{\partial x_2}{\partial x_4}$$

fluunt sequentes,

$$a_{3,4}X_2 + a_{4,2}X_3 + a_{2,3}X_4$$
$$= (a_{2,4}X_1 + a_{4,1}X_2 + a_{1,2}X_4)\frac{\partial x_1}{\partial x_3} + (a_{3,2}X_1 + a_{1,3}X_2 + a_{2,1}X_3)\frac{\partial x_1}{\partial x_4},$$

$$a_{4,3}X_1 + a_{1,4}X_3 + a_{3,1}X_4$$
$$= (a_{2,4}X_1 + a_{4,1}X_2 + a_{1,2}X_4)\frac{\partial x_2}{\partial x_3} + (a_{3,2}X_1 + a_{1,3}X_2 + a_{2,1}X_3)\frac{\partial x_2}{\partial x_4}."$$

Si $p > 2m$ atque variabilium independentium x_{m+1}, x_{m+2}, ... x_p functiones x_1, x_2, ... x_m ita determinari possunt, ut $p-m$ aequationibus $v_i = 0$ satisfaciant, habentur ***complura systemata*** aequationum differentialium partialium, ad instar aequationum (38.) formata. Videlicet e numero m aequationum

$$v_1 = 0,\ v_2 = 0,\ \dots\ v_p = 0$$

per Propositionem antecedentem deducere licet alterum m aequationum differentialium partialium systema (38.), eaque ratione aliud aliudque systema (38.) obtinebitur, prout aliae $p-2m$ e $p-m$ variabilibus independentibus Constantium loco habentur.

Ponamus iam esse $x_1, x_2, \dots x_m$ variabilium $x_{m+1}, x_{m+2}, \dots x_p$ functiones ***involventes Constantem Arbitrariam*** α, sitque

$$41.\quad w = X_1\frac{\partial x_1}{\partial\alpha} + X_2\frac{\partial x_2}{\partial\alpha} \dots + X_m\frac{\partial x_m}{\partial\alpha},$$

porro

$$v_i = X_1\frac{\partial x_1}{\partial x_i} + X_2\frac{\partial x_2}{\partial x_i} \dots + X_m\frac{\partial x_m}{\partial x_i} + X_i,$$

$$\begin{aligned} u_k &= X_k\,dt - \{a_{k,1}\,dx_1 + a_{k,2}\,dx_2 \dots + a_{k,p}\,dx_p\} \\ &= X_k\,dt - dX_k + \frac{\partial X_1}{\partial x_k}dx_1 + \frac{\partial X_2}{\partial x_k}dx_2 \dots + \frac{\partial X_p}{\partial x_k}dx_p \\ &= X_k\,dt - dX_k + \sum_i \frac{\partial X_i}{\partial x_k}dx_i + \sum_{hi}\frac{\partial X_h}{\partial x_k}\cdot\frac{\partial x_h}{\partial x_i}dx_i. \end{aligned}$$

Quae ubi substituuntur in formula,

$$\begin{aligned} & dw - \left\{\frac{\partial x_1}{\partial\alpha}dX_1 + \frac{\partial x_2}{\partial\alpha}dX_2 \dots + \frac{\partial x_m}{\partial\alpha}dX_m\right\} \\ &= X_1\,d\frac{\partial x_1}{\partial\alpha} + X_2\,d\frac{\partial x_2}{\partial\alpha} \dots + X_m\,d\frac{\partial x_m}{\partial\alpha} \\ &= \sum_{hi} X_h\frac{\partial^2 x_h}{\partial\alpha\,\partial x_i}dx_i, \end{aligned}$$

obtinetur

$$\begin{aligned} 42.\quad & dw - w\,dt + \frac{\partial x_1}{\partial\alpha}u_1 + \frac{\partial x_2}{\partial\alpha}u_2 \dots + \frac{\partial x_m}{\partial\alpha}u_m \\ &= \sum_i\left\{\left(\frac{\partial X_i}{\partial\alpha}\right) + \sum_h\left[\left(\frac{\partial X_h}{\partial\alpha}\right)\frac{\partial x_h}{\partial x_i} + X_h\frac{\partial^2 x_h}{\partial\alpha\,\partial x_i}\right]\right\}dx_i \\ &= \sum_i\left(\frac{\partial v_i}{\partial\alpha}\right)dx_i, \end{aligned}$$

siquidem uncis differentialia partialia includendo innuitur, ante differentiationes substitutos esse functionum $x_1, x_2, \dots x_m$ valores. Si m aequationibus, quibus $x_1, x_2, \dots x_m$ determinantur, integratur aequatio,

$$0 = X_1\,dx_1 + X_2\,dx_2 \dots + X_p\,dp,$$

locum habere debent $p-m$ aequationes $v_i=0$, unde aequationis (42.) dextra pars evanescit sive fit

$$43.\quad dw-w\,dt+\frac{\partial x_1}{\partial\alpha}u_1+\frac{\partial x_2}{\partial\alpha}u_2\,\ldots\,+\frac{\partial x_m}{\partial\alpha}u_m = 0.$$

Si $p \geqq 2m$, vidimus supra, m aequationibus illis fieri ut de m aequationibus differentialibus $u_h=0$ fluant $p-m$ reliquae $u_i=0$, ita ut m aequationes illae sint aequationes integrales systematis aequationum differentialium $u_k=0$, quarum $p-2m$ e reliquis fluunt. Formula (43.) docet, si insuper inter variabiles $t,\ x_{m+1},\ x_{m+2},\ \ldots\ x_p$ statuatur aequatio $w=\beta e^t$ sive

$$44.\quad X_1\frac{\partial x_1}{\partial\alpha}+X_2\frac{\partial x_2}{\partial\alpha}\,\ldots\,+X_m\frac{\partial x_m}{\partial\alpha} = \beta e^t,$$

designante β Constantem Arbitrariam, ipsas m aequationes differentiales $u_h=0$ in earum $m-1$ redire, ideoque (44.) esse novam eiusdem systematis $u_k=0$ aequationem integralem. Si m aequationes, quibus aequatio

$$X_1\,dx_1+X_2\,dx_2\,\ldots\,+X_p\,dx_p = 0$$

integratur, plures involvunt Constantes Arbitrarias, per (44.) totidem obtinentur systematis $u_k=0$ aequationes integrales, quas diversae ingrediuntur Constantes Arbitrariae β, et e quarum binis per solam divisionem eliminatur t. Quae manent aequationes integrales, quaecunque $p-2m$ aequationes differentiales adiiciantur systemati $u_k=0$, quippe quod tantum $2m$ aequationum differentialium vices gerit. Ubi Constantes Arbitrariae sunt numero m, habetur problematis ***Pfaffiani*** solutio completa, simulque m aequationes (44.) iunctae m aequationibus, quibus aequatio (20.) integratur, suppeditant systematis aequationum differentialium (21.) integrationem completam.

Si $p=2m$, aequationes Constantem Arbitrariam α involventes, quibus aequatio

$$X_1\,dx_1+X_2\,dx_2\,\ldots\,+X_{2m}\,dx_{2m} = 0$$

integratur et quibus determinabantur functiones $x_1,\ x_2,\ \ldots\ x_m$, sunt aequationes integrales systematis aequationum differentialium (2.), sive resolutione earum provenientium (4.):

$$dx_1:dx_2\,\ldots\,:dx_{2m} = A_1:A_2\,\ldots\,:A_{2m}.$$

Quarum Multiplicatorem, docent formulae (13.) et (44.), per illas m aequationes integrales induere valorem,

$$M = \left\{X_1\frac{\partial x_1}{\partial\alpha}+X_2\frac{\partial x_2}{\partial\alpha}\,\ldots\,+X_m\frac{\partial x_m}{\partial\alpha}\right\}^{-m}.$$

Si $X_{2m} = -1$ atque omnes $X_1, X_2, \ldots . X_{2m-1}$ variabili x_{2m} vacant, vidimus supra Multiplicatorem Constanti aequari. Ac reapse eo casu evanescente dt e (44.) eruitur,

$$X_1 \frac{\partial x_1}{\partial \alpha} + X_2 \frac{\partial x_2}{\partial \alpha} \ldots . + X_m \frac{\partial x_m}{\partial \alpha} = \beta,$$

quae ipsarum (4.) aequatio integralis est. Quae pro $m = 2$ cum formula (19.) convenit, quam supra alia via erui.

Methodum ad solvendum problema *Pfaffianum* ab ipso autore adhibitam, data occasione observo, per plures et altiores procedere integrationes quam methodus vera et genuina poscat. Quam novam methodum pro exemplo simplice explicabo. Ad aequationem differentialem

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 = 0$$

per duas aequationes integrandam poscit *Pfaffiana* methodus integrationem completam systematis trium aequationum differentialium primi ordinis inter quatuor variabiles ac deinde unius aequationis differentialis primi ordinis inter duas variabiles. Illius igitur systematis Integrali uno invento, secundum illam methodum restat integratio completa duarum aequationum differentialium primi ordinis inter tres variabiles sive unius aequationis differentialis secundi ordinis inter duas variabiles ac deinde aequationis differentialis primi ordinis inter duas variabiles. At observo, si Integrali illo invento exprimatur x_4 per x_1, x_2, x_3, aequationem differentialem propositam abire in aliam linearem primi ordinis inter tres variabiles, conditioni integrabilitatis satisfacientem; cuius integrationem vidimus absolvi posse per integrationes separatas duarum aequationum differentialium primi ordinis inter duas variabiles. Unde in locum aequationis differentialis secundi ordinis tantum integrandae sunt duae aequationes differentiales separatae primi ordinis, quae est reductio maxime insignis; integrationi autem aequationis differentialis primi ordinis postremo praestandae omnino supersedetur. Tractatio huius rei gravissimae completa ac generalis alii Commentationi reservanda est.

Novum Principium Generale Mechanicum quod e Principio Ultimi Multiplicatoris fluit.

§. 22.

Sint x_i, y_i, z_i Coordinatae orthogonales puncti massa m_i praediti; sint vires massam m_i secundum directiones Coordinatarum sollicitantes X_i, Y_i, Z_i. Ubi systema n punctorum materialium $m_1, m_2, \ldots . m_n$ prorsus liberum est, inter tempus t atque Coordinatas punctorum habentur $3n$ aequationes differentiales secundi ordinis,

$$1.\quad \begin{cases} \dfrac{d^2 x_i}{dt^2} = \dfrac{1}{m_i} X_i, \\ \dfrac{d^2 y_i}{dt^2} = \dfrac{1}{m_i} Y_i, \\ \dfrac{d^2 z_i}{dt^2} = \dfrac{1}{m_i} Z_i. \end{cases}$$

Vires X_i, Y_i, Z_i suppositione maxime generali erunt functiones $3n$ Coordinatarum x_i, y_i, z_i, temporis t atque differentialium primorum Coordinatarum,

$$x'_i = \frac{dx_i}{dt}, \quad y'_i = \frac{dy_i}{dt}, \quad z'_i = \frac{dz_i}{dt},$$

quae sunt punctorum velocitates in Coordinatarum directiones proiectae. Secundum (5.) §. 14. systematis aequationum differentialium dynamicarum (1.) Multiplicator definitur formula,

$$2.\quad \frac{d\log M}{dt} + \Sigma \frac{1}{m_i}\left(\frac{\partial X_i}{\partial x'_i} + \frac{\partial Y_i}{\partial y'_i} + \frac{\partial Z_i}{\partial z'_i}\right) = 0,$$

indice i valente ad omnia puncta materialia systematis.

Quoties vires sollicitantes a solis massarum positionibus in spatio pendent sive praeterea etiam a tempore t, quantitates X_i, Y_i, Z_i ipsa x'_i, y'_i, z'_i omnino non involvunt, ideoque evanescente expressione

$$\Sigma \frac{1}{m_i}\left(\frac{\partial X_i}{\partial x'_i} + \frac{\partial Y_i}{\partial y'_i} + \frac{\partial Z_i}{\partial z'_i}\right),$$

statuere licet

$$M = 1.$$

Hinc secundum principium ultimi Multiplicatoris sequitur, si systema punctorum materialium liberum sit atque vires mobilia propellentes ab eorum velocitatibus non pendeant, ultimam integrationem, vel si vires etiam a tempore non explicite pendeant, *duas ultimas integrationes* revocari posse ad Quadraturas. Videlicet posteriore casu constat tempus t prorsus separari posse et post alias omnes integrationes transactas per Quadraturam inveniri.

Idem iam demonstrabo pro casu generali quo systema n punctorum materialium non est liberum, sed certis obnoxium est conditionibus, quae exprimantur per aequationes inter Coordinatas x_i, y_i, z_i locum habentes,

$$3.\quad \Pi = 0, \quad \Pi_1 = 0, \quad \text{etc.}$$

Aequationes differentiales dynamicas pro motu sic impedito praecepit ill. *Lagrange* haberi sequentes,

$$4.\quad \begin{cases} \dfrac{d^2 x_i}{dt^2} = \dfrac{1}{m_i}\left\{X_i + \lambda\dfrac{\partial\Pi}{\partial x_i} + \lambda_1\dfrac{\partial\Pi_1}{\partial x_i}\text{ etc.}\right\}, \\ \dfrac{d^2 y_i}{dt^2} = \dfrac{1}{m_i}\left\{Y_i + \lambda\dfrac{\partial\Pi}{\partial y_i} + \lambda_1\dfrac{\partial\Pi_1}{\partial y_i}\text{ etc.}\right\}, \\ \dfrac{d^2 z_i}{dt^2} = \dfrac{1}{m_i}\left\{Z_i + \lambda\dfrac{\partial\Pi}{\partial z_i} + \lambda_1\dfrac{\partial\Pi_1}{\partial z_i}\text{ etc.}\right\}, \end{cases}$$

factoribus λ, λ_1 etc. determinatis per aequationes lineares, quae obtinentur substituendo aequationes differentiales (4.) in aequationibus conditionalibus bis differentiatis,

$$\frac{d^2\Pi}{dt^2} = 0,\quad \frac{d^2\Pi_1}{dt^2} = 0,\quad \text{etc.}$$

Ad eas aequationes lineares formandas pono

$$5.\quad \begin{cases} U = \Sigma\left\{x_i'\dfrac{d.\frac{\partial\Pi}{\partial x_i}}{dt} + y_i'\dfrac{d.\frac{\partial\Pi}{\partial y_i}}{dt} + z_i'\dfrac{d.\frac{\partial\Pi}{\partial z_i}}{dt}\right\}, \\ U_1 = \Sigma\left\{x_i'\dfrac{d.\frac{\partial\Pi_1}{\partial x_i}}{dt} + y_i'\dfrac{d.\frac{\partial\Pi_1}{\partial y_i}}{dt} + z_i'\dfrac{d.\frac{\partial\Pi_1}{\partial z_i}}{dt}\right\}, \\ \qquad\text{etc.}\qquad\text{etc.,} \end{cases}$$

fit

$$0 = \frac{d^2\Pi}{dt^2} = \Sigma\left\{\frac{\partial\Pi}{\partial x_i}.\frac{d^2x_i}{dt^2} + \frac{\partial\Pi}{\partial y_i}.\frac{d^2y_i}{dt^2} + \frac{\partial\Pi}{\partial z_i}.\frac{d^2z_i}{dt^2}\right\} + U,$$

$$0 = \frac{d^2\Pi_1}{dt^2} = \Sigma\left\{\frac{\partial\Pi_1}{\partial x_i}.\frac{d^2x_i}{dt^2} + \frac{\partial\Pi_1}{\partial y_i}.\frac{d^2y_i}{dt^2} + \frac{\partial\Pi_1}{\partial z_i}.\frac{d^2z_i}{dt^2}\right\} + U_1,$$

$$\text{etc.}\qquad\qquad\text{etc.}$$

Ubi in his aequationibus substituuntur formulae (4.) atque ponitur,

$$6.\quad \begin{cases} V = U + \Sigma\dfrac{1}{m_i}\left\{\dfrac{\partial\Pi}{\partial x_i}X_i + \dfrac{\partial\Pi}{\partial y_i}Y_i + \dfrac{\partial\Pi}{\partial z_i}Z_i\right\}, \\ V_1 = U_1 + \Sigma\dfrac{1}{m_i}\left\{\dfrac{\partial\Pi_1}{\partial x_i}X_i + \dfrac{\partial\Pi_1}{\partial y_i}Y_i + \dfrac{\partial\Pi_1}{\partial z_i}Z_i\right\}, \\ \qquad\text{etc.}\qquad\text{etc.,} \end{cases}$$

porro

$$7.\quad (\alpha,\beta) = (\beta,\alpha) = \Sigma\frac{1}{m_i}\left\{\frac{\partial\Pi_\alpha}{\partial x_i}.\frac{\partial\Pi_\beta}{\partial x_i} + \frac{\partial\Pi_\alpha}{\partial y_i}.\frac{\partial\Pi_\beta}{\partial y_i} + \frac{\partial\Pi_\alpha}{\partial z_i}.\frac{\partial\Pi_\beta}{\partial z_i}\right\},$$

aequationes, quibus λ, λ_1 determinantur, evadunt sequentes,

$$8.\quad \begin{cases} 0 = V + (0,0)\lambda + (0,1)\lambda_1 \text{ etc.}, \\ 0 = V_1 + (1,0)\lambda + (1,1)\lambda_1 \text{ etc.}, \\ \qquad \text{etc.} \qquad \text{etc.} \end{cases}$$

His de factorum λ, λ_1 etc. valoribus praemissis, aequationum *Lagrangianarum* (4.) investigabo Multiplicatorem.

Ac primum observo, secundum ea quae de viribus sollicitantibus statuta sunt, in dextris partibus aequationum (4.) solos factores λ, λ_1 etc. implicare differentialia prima x'_i, y'_i, z'_i. Unde e (5.) §. 14. Multiplicator M definietur formula,

$$-\frac{d\log M}{dt} = \Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi}{\partial x_i}\cdot\frac{\partial \lambda}{\partial x'_i} + \frac{\partial \Pi}{\partial y_i}\cdot\frac{\partial \lambda}{\partial y'_i} + \frac{\partial \Pi}{\partial z_i}\cdot\frac{\partial \lambda}{\partial z'_i}\right\}$$
$$+\Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_1}{\partial x_i}\cdot\frac{\partial \lambda_1}{\partial x'_i} + \frac{\partial \Pi_1}{\partial y_i}\cdot\frac{\partial \lambda_1}{\partial y'_i} + \frac{\partial \Pi_1}{\partial z_i}\cdot\frac{\partial \lambda_1}{\partial z'_i}\right\}$$
$$\text{etc.} \qquad \text{etc.},$$

quam posito

$$9.\quad \Delta_{\alpha,\beta} = \Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{\partial \lambda_\beta}{\partial x'_i} + \frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{\partial \lambda_\beta}{\partial y'_i} + \frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{\partial \lambda_\beta}{\partial z'_i}\right\},$$

sic exhibere licet

$$10.\quad d\log M = -\{\Delta_{0,0} + \Delta_{1,1} + \text{etc.}\}\, dt.$$

Ad quantitates $\Delta_{0,0}$, $\Delta_{1,1}$ etc. determinandas, aequationes (5.),

$$0 = V_\beta + (\beta, 0)\lambda + (\beta, 1)\lambda_1 \text{ etc.},$$

quarum Coëfficientes $(\beta, 0)$, $(\beta, 1)$ etc. solarum x_i, y_i, z_i functiones sunt, secundum omnes quantitates x'_i, y'_i, z'_i differentientur, aequationesque differentiationibus provenientes respective per quantitates

$$\frac{1}{m_i}\cdot\frac{\partial \Pi_\alpha}{\partial x_i},\quad \frac{1}{m_i}\cdot\frac{\partial \Pi_\alpha}{\partial y_i},\quad \frac{1}{m_i}\cdot\frac{\partial \Pi_\alpha}{\partial z_i}$$

multiplicatae consummentur: prodit

$$11.\quad 0 = u_{\alpha,\beta} + (\beta, 0)\Delta_{\alpha,0} + (\beta, 1)\Delta_{\alpha,1} \text{ etc.},$$

siquidem statuitur

$$u_{\alpha,\beta} = \Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{\partial V_\beta}{\partial x'_i} + \frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{\partial V_\beta}{\partial y'_i} + \frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{\partial V_\beta}{\partial z'_i}\right\}.$$

Cum secundum (6.) habeatur

$$\frac{\partial V_\beta}{\partial x'_i} = \frac{\partial U_\beta}{\partial x'_i},\quad \frac{\partial V_\beta}{\partial y'_i} = \frac{\partial U_\beta}{\partial y'_i},\quad \frac{\partial V_\beta}{\partial z'_i} = \frac{\partial U_\beta}{\partial z'_i},$$

quantitates $u_{\alpha,\beta}$ sic repraesentare licet,

$$u_{\alpha,\beta} = \Sigma \frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{\partial U_\beta}{\partial x'_i} + \frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{\partial U_\beta}{\partial y'_i} + \frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{\partial U_\beta}{\partial y'_i}\right\}.$$

At e (5.) obtinetur, evolutione differentialium $d.\frac{\partial \Pi_\beta}{\partial x_i}$ etc. facta,

$$12.\quad \begin{cases} \dfrac{\partial U_\beta}{\partial x'_i} = 2\dfrac{d.\frac{\partial \Pi_\beta}{\partial x_i}}{dt}, \\[2ex] \dfrac{\partial U_\beta}{\partial y'_i} = 2\dfrac{d.\frac{\partial \Pi_\beta}{\partial y_i}}{dt}, \\[2ex] \dfrac{\partial U_\beta}{\partial z'_i} = 2\dfrac{d.\frac{\partial \Pi_\beta}{\partial z_i}}{dt}, \end{cases}$$

quibus valoribus substitutis fit

$$13.\quad u_{\alpha,\beta} = 2\Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{d.\frac{\partial \Pi_\beta}{\partial x_i}}{dt} + \frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{d.\frac{\partial \Pi_\beta}{\partial y_i}}{dt} + \frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{d.\frac{\partial \Pi_\beta}{\partial z_i}}{dt}\right\}.$$

Cuius aequationis beneficio obtinentur quantitatum (α,β) per formulam (7.) definitarum differentialia,

$$14.\quad \frac{d.(\alpha,\beta)}{dt} = \frac{d.(\beta,\alpha)}{dt} = \tfrac{1}{2}\{u_{\alpha,\beta}+u_{\beta,\alpha}\}.$$

In aequatione (11.) indici β valores 0, 1, 2 etc. tribuendo obtinentur aequationes lineares quibus quantitas $\varDelta_{\alpha,\alpha}$ determinatur. At quantitatum omnium sic inventarum $\varDelta_{\alpha,\alpha}$ aggregatum docui per formulam symbolicam concinnam exhiberi posse, quaecunque sint quantitates $u_{\alpha,\beta}$. Vocetur enim $\boldsymbol{R}$ earum aequationum linearium Determinans sive sit

$$\Sigma \pm (00)(11)(22)\ldots. = \boldsymbol{R},$$

atque statuatur

$$\tfrac{1}{2}\{u_{\alpha,\beta}+u_{\beta,\alpha}\}\,dt = \delta(\alpha,\beta) = \delta(\beta,\alpha):$$

sequitur per ratiocinia similia atque §. 16. adhibui,

$$-\{\varDelta_{0,0}+\varDelta_{1,1}+\text{etc.}\}\,dt = \delta\log\boldsymbol{R}.$$

Unde cum secundum (14.) sit

$$\delta(\alpha,\beta) = d(\alpha,\beta) \quad\text{ideoque}\quad \delta\log\boldsymbol{R} = d\log\boldsymbol{R},$$

eruitur e (10.),

$$-\{\varDelta_{0,0}+\varDelta_{1,1}+\text{etc.}\}\,dt = d\log\boldsymbol{M} = d\log\boldsymbol{R},$$

id quod suppeditat

$$15.\quad \boldsymbol{M} = \boldsymbol{R} = \Sigma\pm(00)(11)(22)\ldots.,$$

qui est Multiplicatoris quaesiti valor.

Operae pretium est adnotare, aequationem inventam $M = R$ non tantum ad casum valere quo functiones X_i, Y_i, Z_i, viribus sollicitantibus aequales, tempus t explicite continent, sed ad hunc quoque casum *quo tempus t ipsas explicite afficit aequationes conditionales* $\Pi = 0$, $\Pi_1 = 0$ etc. Eo casu aequationes dynamicae *Lagrangianae* (4.) eandem servant formam, sed factoribus λ, λ_1 etc. alii competunt valores; quippe quantitatibus U, U_1 etc. ideoque etiam quantitatibus V, V_1 etc. quae aequationum linearium (8.), quibus factores λ, λ_1 etc. determinantur, terminos constantes constituunt, respective addendi sunt termini,

$$2\frac{d.\frac{\partial \Pi}{\partial t}}{dt}, \quad 2\frac{d.\frac{\partial \Pi_1}{\partial t}}{dt}, \quad \text{etc.}$$

At patet, inde non mutari aequationes (12.); unde aequationes quoque (13.) et (14.) immutatae manebunt ideoque formula pro aggregato $\varDelta_{0,0} + \varDelta_{1,1}$ etc. inventa ideoque etiam ipsius Multiplicatoris valor R.

Si vires sollicitantes X_i, Y_i, Z_i solarum functiones sunt Coordinatarum x_i, y_i, z_i, atque inter has solas dantur aequationes conditionales $\Pi = 0$, $\Pi_1 = 0$ etc., valor $M = R$ inventus secundum principium ultimi Multiplicatoris hoc suppeditat theorema:

Novum Principium Generale Mechanicum.

„*Proponatur motus systematis n punctorum materialium, quae in datis superficiebus vel curvis aut dato quocunque modo inter se connexa manere debent, ita ut inter Coordinatas eorum locum habeant k aequationes conditionales; porro vires sollicitantes et magnitudine et directione solis punctorum positionibus datae sint: semper duas ultimas integrationes absolvere licet Quadraturis. Sint enim*

punctorum massae $m_1, m_2, \ldots m_n$;

massae m_i *Coordinatae orthogonales* x_i, y_i, z_i, *earumque differentialia prima* $x_i' = \frac{dx_i}{dt}$, $y_i' = \frac{dy_i}{dt}$, $z_i' = \frac{dz_i}{dt}$;

sint aequationes conditionales $\Pi = 0$, $\Pi_1 = 0$, $\ldots$ $\Pi_{k-1} = 0$ *et differentiatione prima ex iis provenientes* $\Pi' = 0$, $\Pi_1' = 0$, $\ldots$ $\Pi_{k-1}' = 0$, *ubi*

$$\Pi_\alpha' = \Sigma \left\{ \frac{\partial \Pi_\alpha}{\partial x_i} x_i' + \frac{\partial \Pi_\alpha}{\partial y_i} y_i' + \frac{\partial \Pi_\alpha}{\partial z_i} z_i' \right\};$$

inter $6n$ *quantitates* x_i, y_i, z_i, x_i', y_i', z_i' *praeter* $2k$ *aequationes* $\Pi_\alpha = 0$,

$\Pi'_\alpha = 0$, *inventa sint* $6n - k - 2 = \mu$ *Integralia* $F_1 = \alpha_1$, $F_2 = \alpha_2$, ... $F_\mu = \alpha_\mu$, *designantibus* α_1, α_2, ... α_μ *Constantes Arbitrarias; restabit integratio unius aequationis differentialis primi ordinis inter duas quantitates u et v,*

$$v'du - u'dv = 0,$$

ubi u et v esse possunt ipsarum x_i, y_i, z_i, x'_i, y'_i, z'_i *functiones quaecunque atque u' et v' designant valores differentialium* $\frac{du}{dt}$ *et* $\frac{dv}{dt}$, *adiumento aequationum datarum et integratione inventarum nec non ipsarum aequationum differentialium dynamicarum per ipsas u et v expressos. His praemissis, ponatur*

$$(\alpha, \beta) = \Sigma \frac{1}{m_i} \left\{ \frac{\partial \Pi_\alpha}{\partial x_i} . \frac{\partial \Pi_\beta}{\partial x_i} + \frac{\partial \Pi_\alpha}{\partial y_i} . \frac{\partial \Pi_\beta}{\partial y_i} + \frac{\partial \Pi_\alpha}{\partial z_i} . \frac{\partial \Pi_\beta}{\partial z_i} \right\},$$

atque kk quantitatum (α, β) *formetur Determinans R; porro si vocatur* Δ *Determinans functionale* $6n$ *functionum*

$$\Pi,\ \Pi_1,\ \ldots.\ \Pi_{k-1},\ \Pi',\ \Pi'_1,\ \ldots.\ \Pi'_{k-1},$$
$$F_1,\ F_2,\ \ldots.\ F_{6n-k-2},\ u,\ v,$$

$6n$ *quantitatum* x_i, y_i, z_i, x'_i, y'_i, z'_i *respectu formatum, exprimantur R et* Δ *et ipsa per solas u et v; erit aequationis* $v'du - u'dv = 0$ *Multiplicator* $\frac{R}{\Delta}$, *unde nova habetur aequatio integralis,*

$$\int \frac{R}{\Delta} (v'du - u'dv) = \text{Const.},$$

ubi expressio sub integrationis signo est differentiale completum; denique si nova illa aequatione integrali exprimitur v per u, unde evadit etiam u' solius u functio, invenitur simplice Quadratura,

$$t + \text{Const.} = \int \frac{du}{w}.\text{"}$$

Sub forma antecedente principium novum mechanicum ante hos tres annos cum illustri Academia ***Petropolitana*** communicavi. Alias eiusdem formas infra tradam. Ultimam integrationem, qua t per Coordinatas exprimatur, Quadraturis absolvi, res erat nota et sponte patens. At inventum novum, penultimam quoque integrationem Quadraturis perfici posse, constituere mihi videbatur principium mechanicum.

Si tempus t vires sollicitantes sive etiam aequationes conditionales afficit, non amplius ipsum t a reliquis variabilibus separare licet, unde eo casu principium nostrum tantum omnium ultimam integrationem per Quadraturas absolvere docet. Supponendo, inventa esse $6n - 2k - 1$ Integralia,

$$F_1 = \alpha_1,\ F_2 = \alpha_2,\ \ldots.\ F_{6n-2k-1} = \alpha_{6n-2k-1};$$

atque u et v esse ipsius t et $6n$ quantitatum x_i, y_i, z_i, x'_i, y'_i, z'_i functiones, Determinans Δ formandum est $6n$ functionum,

$$F_1,\ F_2,\ \ldots.\ F_{2n-2k-1},\ \Pi,\ \Pi_1,\ \ldots.\ \Pi_{k-1},\ \Pi',\ \Pi'_1,\ \ldots.\ \Pi'_{k-1},\ u,\ v,$$

$6n+1$ quantitatum t, x_i, y_i, z_i, x'_i, y'_i, z'_i respectu; eadem manente ipsius R significatione, rursus exprimenda erunt R, Δ, $u' = \frac{du}{dt}$, $v' = \frac{dv}{dt}$ per u et v, eritque aequatio integralis ultima,

$$\int \frac{R}{\Delta}(v'du - u'dv) = \text{Const.},$$

ubi expressio sub integrationis signo est differentiale completum.

Habemus hic exemplum, quo ad reductionem aequationum differentialium propositarum adhibentur Integralia *particularia;* nam ex aequationibus differentialibus (4.) sequuntur Integralia completa, $\Pi'_\alpha = C_\alpha$, $\Pi_\alpha = C_\alpha t + C'_\alpha$, designantibus C_α, C'_α Constantes Arbitrarias. Neque tamen sunt $\Pi'_\alpha = 0$, $\Pi_\alpha = 0$ aequationes integrales particulares *quaecunque*, sed tales pro quibus secundum §. 12. fit ut Multiplicator quo aequationes differentiales earum beneficio reductae gaudent e Multiplicatore propositarum (4.) deduci possit. Scilicet aequatio quidem integralis particularis est $\Pi'_\alpha = 0$, at functio Π'_α ita comparata est ut Constanti Arbitrariae aequiparata suppeditet Integrale completum; porro si reductioni adhibetur aequatio integralis particularis $\Pi'_\alpha = 0$ ex eaque nova deducitur aequatio integralis $\Pi_\alpha = 0$, rursus innotescit functio Π_α, quae Constanti Arbitrariae aequiparata non quidem aequationum differentialium propositarum (4.), sed reductarum tamen Integrale completum suppeditat. Quod secundum §. 12. poscitur et sufficit.

Designentur $3n$ quantitates $x_i\sqrt{m_i}$, $y_i\sqrt{m_i}$, $z_i\sqrt{m_i}$ per

$$\xi_1,\ \xi_2,\ \ldots.\ \xi_{3n},$$

fit e (7.),

$$(\alpha,\beta) = \frac{\partial \Pi_\alpha}{\partial \xi_1}\cdot\frac{\partial \Pi_\beta}{\partial \xi_1} + \frac{\partial \Pi_\alpha}{\partial \xi_2}\cdot\frac{\partial \Pi_\beta}{\partial \xi_2} \ldots. + \frac{\partial \Pi_\alpha}{\partial \xi_{3n}}\cdot\frac{\partial \Pi_\beta}{\partial \xi_{3n}}.$$

Unde secundum propositionem notam, in Commentatione *de formatione atque proprietatibus Determinantium* §. 13. probatam, quantitatum (α,β) Determinans exhibere licet ut aggregatum quadratorum Determinantium functionum Π, Π_1, ... Π_{k-1}, formatorum respectu quarumque k e numero quantitatum ξ_1, ξ_2, ... ξ_{3n} sumtarum, sive ponere licet

$$16.\qquad R = M = S.\left\{\Sigma \pm \frac{\partial \Pi}{\partial \xi_{m'}}\cdot\frac{\partial \Pi_1}{\partial \xi_{m''}}\ldots.\frac{\partial \Pi_{k-1}}{\partial \xi_{m^{(k)}}}\right\}^2,$$

siquidem m', m'', ... $m^{(k)}$ designant quoscunque k diversos ex indicibus 1, 2, ... $3n$. Ex. gr. pro uno puncto, massa $=1$ praedito, cuius Coordinatae orthogonales sunt x, y, z, et quod moveri debet in superficie cuius aequatio $\Pi = 0$, fit

$$M = R = \left(\frac{\partial \Pi}{\partial x}\right)^2 + \left(\frac{\partial \Pi}{\partial y}\right)^2 + \left(\frac{\partial \Pi}{\partial z}\right)^2;$$

si punctum moveri debet in curva, cuius aequationes sunt $\Pi = 0$, $\Pi_1 = 0$, fit

$$\begin{aligned} M = R = & \left\{\frac{\partial \Pi}{\partial y}\cdot\frac{\partial \Pi_1}{\partial z} - \frac{\partial \Pi}{\partial z}\cdot\frac{\partial \Pi_1}{\partial y}\right\}^2 \\ & + \left\{\frac{\partial \Pi}{\partial z}\cdot\frac{\partial \Pi_1}{\partial x} - \frac{\partial \Pi}{\partial x}\cdot\frac{\partial \Pi_1}{\partial z}\right\}^2 \\ & + \left\{\frac{\partial \Pi}{\partial x}\cdot\frac{\partial \Pi_1}{\partial y} - \frac{\partial \Pi}{\partial y}\cdot\frac{\partial \Pi_1}{\partial x}\right\}^2. \end{aligned}$$

Erat R Determinans aequationum linearium, quibus factores *Lagrangiani* λ, λ_1 etc. determinantur, qui igitur factores indeterminati aut infiniti evadere nequeunt nisi evanescat R. At docet formula (16.), non evanescere posse R nisi singula evanescant Determinantia functionalia

$$\Sigma \pm \frac{\partial \Pi}{\partial \xi_{m'}}\cdot\frac{\partial \Pi_1}{\partial \xi_{m''}}\cdots\frac{\partial \Pi_{k-1}}{\partial \xi_{m^{(k)}}}.$$

Id quod ubi *identice* fit, ipsarum Π, Π_1, ... Π_{k-1} una reliquarum functio est, quo casu aequationes conditionales aut sibi contradicunt aut una quae e reliquis sequitur est superflua. Singula Determinantia illa si non quidem identice evanescunt sed ipsarum aequationum $\Pi = 0$, $\Pi_1 = 0$, ... $\Pi_{k-1} = 0$ adiumento, id indicio est, earum aequationum unam reliquarum ope formam *Quadrati* induere. Eo casu per certas eliminationes et radicis extractionem transformari debent aequationes $\Pi = 0$ etc.; quam praeparationem semper factam esse supponi debet, ut aequationum dynamicarum *Lagrangianarum* usus esse possit.

Si ex antecedentibus semper supponere licet Determinans R non indefinite evanescere, fieri tamen potest ut R evanescat pro punctorum materialium positionibus particularibus determinatis. Quemadmodum si inter tres puncti Coordinatas una vel duae habentur aequationes conditionales repraesentantes superficiem aut curvam apice praeditam, evanescit R si punctum in eo apice collocatur. Ubi agitur de aequilibrio systematis punctorum materialium in eiusmodi positionibus particularibus collocatorum, pro quibus Determinans R evanescit, praecepta statica generalia aut deficiunt aut accuratioribus explicationibus indigent. Nec non si in certo temporis momento systema in motu suo ad tales positiones particulares pervenit, velocitatum intensitates et directiones mutationem finitam in temporis intervallo infinite parvo subeunt. Si, ut in rerum natura fieri solet, conditiones quibus systema subiicitur non exprimuntur per aequationes, sed per inaequalitates $\Pi > 0$, $\Pi_1 > 0$ etc., inde ab eo temporis momento ipsae plerumque aequationes differentiales (4.) cum aliis commutari debent.

De Multiplicatore aequationum differentialium dynamicarum forma *Lagrangiana* secunda exhibitarum.

§. 23.

Ill. *Lagrange* aequationes differentiales dynamicas generales alia quoque forma memorabili exhibuit, Coordinatarum $3n$ loco, k aequationibus conditionalibus satisfacientium, introducendo $3n-k$ quantitates a se independentes

$$q_1, \quad q_2, \quad \dots \quad q_{3n-k}.$$

Quarum ipsae Coordinatae x_i, y_i, z_i tales esse debent functiones, quae substitutae in aequationibus conditionalibus $\Pi = 0$, $\Pi_1 = 0$ etc. sponte iis satisfaciant. Unde etiam aequationem $\Pi_\alpha = 0$ cuiuslibet variabilis q_m respectu differentiando habetur

$$1. \quad \sum_i \left\{ \frac{\partial \Pi_\alpha}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_m} + \frac{\partial \Pi_\alpha}{\partial y_i} \cdot \frac{\partial y_i}{\partial q_m} + \frac{\partial \Pi_\alpha}{\partial z_i} \cdot \frac{\partial z_i}{\partial q_m} \right\} = 0.$$

Statuatur

$$2. \quad \sum_i \left\{ X_i \frac{\partial x_i}{\partial q_m} + Y_i \frac{\partial y_i}{\partial q_m} + Z_i \frac{\partial z_i}{\partial q_m} \right\} = Q_m;$$

consummando $3n$ aequationes (4.) §. pr. respective per $m_i \frac{\partial x_i}{\partial q_m}$, $m_i \frac{\partial y_i}{\partial q_m}$, $m_i \frac{\partial z_i}{\partial q_m}$ multiplicatas, evanescunt secundum (1.) aggregata in factores λ, λ_1 etc. ducta, unde prodit

$$3. \quad \sum_i m_i \left\{ \frac{d^2 x_i}{dt^2} \cdot \frac{\partial x_i}{\partial q_m} + \frac{d^2 y_i}{dt^2} \cdot \frac{\partial y_i}{\partial q_m} + \frac{d^2 z_i}{dt^2} \cdot \frac{\partial z_i}{\partial q_m} \right\} = Q_m.$$

Ponendo $q'_m = \frac{dq_m}{dt}$ et considerando quantitates x'_i ut quantitatum q_m, q'_m functiones, quae dantur formula,

$$x'_i = \frac{\partial x_i}{\partial q_1} q'_1 + \frac{\partial x_i}{\partial q_2} q'_2 \dots + \frac{\partial x_i}{\partial q_{3n-k}} q'_{3n-k},$$

sequitur

$$\frac{\partial x'_i}{\partial q'_m} = \frac{\partial x_i}{\partial q_m}.$$

Porro

$$\frac{\partial x'_i}{\partial q_m} = \frac{\partial^2 x_i}{\partial q_m \partial q_1} q'_1 + \frac{\partial^2 x_i}{\partial q_m \partial q_2} q'_2 \dots + \frac{\partial^2 x_i}{\partial q_m \partial q_{3n-k}} q'_{3n-k} = \frac{d \cdot \frac{\partial x_i}{\partial q_m}}{dt}.$$

Eodem modo pro omnibus tribus Coordinatis fit

$$4.\quad \begin{cases} \dfrac{\partial x'_i}{\partial q'_m} = \dfrac{\partial x_i}{\partial q_m}, & \dfrac{\partial y'_i}{\partial q'_m} = \dfrac{\partial y_i}{\partial q_m}, & \dfrac{\partial z'_i}{\partial q'_m} = \dfrac{\partial z_i}{\partial q_m}, \\ \dfrac{\partial x'_i}{\partial q_m} = \dfrac{d.\dfrac{\partial x_i}{\partial q_m}}{dt}, & \dfrac{\partial y'_i}{\partial q_m} = \dfrac{d.\dfrac{\partial y_i}{\partial q_m}}{dt}, & \dfrac{\partial z'_i}{\partial q_m} = \dfrac{d.\dfrac{\partial z_i}{\partial q_m}}{dt}. \end{cases}$$

Unde aequatio (3.) sic exhiberi potest,

$$Q_m = \sum_i m_i \left\{ \frac{dx'_i}{dt}.\frac{\partial x'_i}{\partial q'_m} + \frac{dy'_i}{dt}.\frac{\partial y'_i}{\partial q'_m} + \frac{dz'_i}{dt}.\frac{\partial z'_i}{\partial q'_m} \right\}$$

$$= \frac{d.\sum_i m_i \left\{ x'_i \dfrac{\partial x'_i}{\partial q'_m} + y'_i \dfrac{\partial y'_i}{\partial q'_m} + z'_i \dfrac{\partial z'_i}{\partial q'_m} \right\}}{dt} - \sum_i m_i \left\{ x'_i \frac{\partial x'_i}{\partial q_m} + y'_i \frac{\partial y'_i}{\partial q_m} + z'_i \frac{\partial z'_i}{\partial q_m} \right\},$$

sive ponendo

$$T = \tfrac{1}{2} \Sigma m_i \{ x'_i x'_i + y'_i y'_i + z'_i z'_i \},$$

fit

$$Q_m = \frac{d.\dfrac{\partial T}{\partial q'_m}}{dt} - \frac{\partial T}{\partial q_m}.$$

Qua in formula ubi T et quantitates Q_m per $6n-2k$ quantitates q_1, q_2, ... q_{3n-k}, q'_1, q'_2, ... q'_{3n-k} exprimuntur atque indici m tribuuntur valores $1, 2, \ldots .\ 3n-k$, obtinentur $3n-k$ aequationes differentiales secundi ordinis inter tempus t atque $3n-k$ variabiles a se independentes q_m,

$$5.\quad \begin{cases} \dfrac{d.\dfrac{\partial T}{\partial q'_1}}{dt} - \dfrac{\partial T}{\partial q_1} - Q_1 = 0, \\ \dfrac{d.\dfrac{\partial T}{\partial q'_2}}{dt} - \dfrac{\partial T}{\partial q_2} - Q_2 = 0, \\ \ldots\ldots\ldots\ldots\ldots \\ \dfrac{d.\dfrac{\partial T}{\partial q'_{3n-k}}}{dt} - \dfrac{\partial T}{\partial q_{3n-k}} - Q_{3n-k} = 0, \end{cases}$$

quae altera est forma *Lagrangiana* aequationum differentialium dynamicarum. Aequationum (5.) iam investigabo Multiplicatorem.

Sint aequationes dynamicae,

$$\varphi_1 = 0, \quad \varphi_2 = 0, \quad \ldots . \quad \varphi_{3n-k} = 0,$$

ubi φ_1, φ_2 etc. designent laevas partes aequationum (5.). Statuamus

$$6.\quad T = \tfrac{1}{2}\Sigma a_{i,i'} q'_i q'_{i'},$$

utroque i et i' ad omnes indices 1, 2, ... $3n-k$ valente et designantibus quantitatibus $a_{i,i'} = a_{i',i}$ solarum q_1, q_2, ... q_{3n-k} functiones. Hinc fit e (5.),

$$\varphi_m = \frac{d\sum\limits_i a_{i,m} q'_i}{dt} - \tfrac{1}{2}\sum_{i,i'} \frac{\partial a_{i,i'}}{\partial q_m} q'_i q'_{i'} - Q_m,$$

unde ponendo $q''_i = \frac{d^2 q_i}{dt^2}$ eruitur,

$$7.\quad \frac{\partial \varphi_m}{\partial q''_h} = a_{h,m} \quad \text{ideoque} \quad \frac{\partial \varphi_m}{\partial q''_h} = \frac{\partial \varphi_h}{\partial q''_m}.$$

Porro si vires sollicitantes X_i, Y_i, Z_i a quantitatibus x'_i, y'_i, z'_i non pendent ideoque etiam quantitates Q_m ipsa q'_1, q'_2 etc. non implicant, fit

$$\frac{\partial \varphi_m}{\partial q'_h} = \frac{d a_{h,m}}{dt} + \sum_i \frac{\partial a_{i,m}}{\partial q_h} q'_i - \sum_i \frac{\partial a_{i,h}}{\partial q_m} q'_i,$$

unde reiectis terminis se mutuo destruentibus fit

$$\tfrac{1}{2}\left\{\frac{\partial \varphi_m}{\partial q'_h} + \frac{\partial \varphi_h}{\partial q'_m}\right\} = \frac{d a_{h,m}}{dt},$$

sive

$$8.\quad \tfrac{1}{2}\left\{\frac{\partial \varphi_m}{\partial q'_h} + \frac{\partial \varphi_h}{\partial q'_m}\right\} = \frac{d.\frac{\partial \varphi_m}{\partial q''_h}}{dt} = \frac{d.\frac{\partial \varphi_h}{\partial q''_m}}{dt}.$$

At e propositione generali, quam sub finem §[i] 16. tradidi, ponendo $\lambda = 1$ sequitur, ubi formulae (8.) locum habeant, aequationum differentialium (5.) fieri Multiplicatorem

$$9.\quad M_1 = \Sigma \pm \frac{\partial \varphi_1}{\partial q''_1} \cdot \frac{\partial \varphi_2}{\partial q''_2} \cdots \frac{\partial \varphi_{3n-k}}{\partial q''_{3n-k}} = \Sigma \pm a_{1,1}\, a_{2,2} \ldots a_{3n-k,\,3n-k}.$$

Si rursus $3n$ quantitatum $x_i\sqrt{m_i}$, $y_i\sqrt{m_i}$, $z_i\sqrt{m_i}$ loco ponimus ξ_1, ξ_2, ... ξ_{3n}, fit

$$10.\quad T = \tfrac{1}{2}\{\xi'_1\xi'_1 + \xi'_2\xi'_2 \ldots \xi'_{3n}\xi'_{3n}\},$$

qua expressione in formula (6.) substituta obtinetur

$$11.\quad a_{i,i'} = \frac{\partial \xi_1}{\partial q_i}\cdot\frac{\partial \xi_1}{\partial q_{i'}} + \frac{\partial \xi_2}{\partial q_i}\cdot\frac{\partial \xi_2}{\partial q_{i'}} \cdots + \frac{\partial \xi_{3n}}{\partial q_i}\cdot\frac{\partial \xi_{3n}}{\partial q_{i'}}.$$

Harum quantitatum Determinans, secundum eandem propositionem quam §. pr. allegavi (*De Determ. form. et propr.* §. 13.), aequatur aggregato quadratorum Determinantium functionalium quarumque $3n-k$ e numero functionum ξ_1, ξ_2, ... ξ_{3n}, quantitatum q_1, q_2, ... q_{3n-k} respectu formatorum, sive fit

$$12. \quad M_1 = \Sigma \pm a_{1,1} a_{2,2} \dots a_{3n-k,\,3n-k}$$
$$= S \left\{ \Sigma \pm \frac{\partial \xi_{m'}}{\partial q_1} \cdot \frac{\partial \xi_{m''}}{\partial q_2} \cdots \frac{\partial \xi_{m^{(3n-k)}}}{\partial q_{3n-k}} \right\}^2,$$

designantibus m', m'' etc. quoscunque $3n-k$ ex indicibus $1, 2, \dots 3n$.

In deducendis aequationibus differentialibus (5.) supposui, aequationes conditionales tempus t non explicite continere. Quod ubi fit, statuendum erit, functiones, quibus $3n$ quantitates x_i, y_i, z_i aequantur, praeter $3n-k$ quantitates q_m etiam ipsum t continere. At hinc non mutabuntur formulae (1.), (3.), (4.), ideoque ipsae aequationes (5.) immutatae manebunt. Unde altera quoque forma *Lagrangiana* aequationum differentialium dynamicarum ad hunc valet casum quo aequationes conditionales tempus explicite continent. Neque eo casu mutationem subeunt formulae (7.) et (8.), unde etiam valor Multiplicatoris inventus immutatus manet. Quod breviter adnotare sufficiat.

De Multiplicatore aequationum differentialium dynamicarum forma tertia exhibitarum. Multiplicatores trium formarum aequationum differentialium dynamicarum inter se comparantur. Principium ultimi multiplicatoris ad tertiam formam relatum.

§. 24.

Quantitatum $q_1', q_2', \dots q_{3n-k}'$ respectu functio T homogenea erat secundi gradus, unde fit

$$2T = q_1'\frac{\partial T}{\partial q_1'} + q_2'\frac{\partial T}{\partial q_2'} \dots + q_{3n-k}'\frac{\partial T}{\partial q_{3n-k}'},$$

sive

$$T = q_1'\frac{\partial T}{\partial q_1'} + q_2'\frac{\partial T}{\partial q_2'} \dots + q_{3n-k}'\frac{\partial T}{\partial q_{3n-k}'} - T.$$

Si variamus quantitates omnes, quarum T functio est, ponimusque

$$1. \quad \frac{\partial T}{\partial q_i'} = p_i,$$

sequitur e valore ipsius T praecedente,

$$2. \quad \delta T = q_1'\,\delta p_1 + q_2'\,\delta p_2 \dots + q_{3n-k}'\,\delta p_{3n-k}$$
$$- \left\{\frac{\partial T}{\partial q_1}\delta q_1 + \frac{\partial T}{\partial q_2}\delta q_2 \dots + \frac{\partial T}{\partial q_{3n-k}}\delta q_{3n-k}\right\},$$

ubi in dextra parte bini termini se mutuo destruentes, $\frac{\partial T}{\partial q_i'}\delta q_i' - \frac{\partial T}{\partial q_i'}\delta q_i'$, omissi sunt. Formula (2.) docet, si per $3n-k$ aequationes, e (6.) §. pr. fluentes,

$$3. \quad p_i = a_{i,1} q_1' + a_{i,2} q_2' \dots + a_{i,3n-k}\, q_{3n-k}',$$

quantitates q'_i per quantitates p_i et q_i exprimantur earumque valores in functione T substituantur, fore ipsius T differentialia partialia quantitatum q_i et p_i respectu sumta, quae uncis includendo distinguamus ab ipsius T differentialibus partialibus quantitatum q_i et q'_i respectu sumtis,

$$4.\quad \left(\frac{\partial T}{\partial q_i}\right) = -\frac{\partial T}{\partial q_i},\quad \left(\frac{\partial T}{\partial p_i}\right) = q'_i.$$

Harum formularum ope aequationes differentiales (5.) §. pr. exhibere licet ut systema $6n-2k$ aequationum differentialium primi ordinis inter t et quantitates $q_1, q_2, \ldots q_{3n-k}, p_1, p_2, \ldots p_{3n-k}$,

$$5.\quad \frac{dq_i}{dt} = \left(\frac{\partial T}{\partial p_i}\right),\quad \frac{dp_i}{dt} = -\left(\frac{\partial T}{\partial q_i}\right)+Q_i.$$

Hae formulae *tertiam* formam aequationum differentialium dynamicarum constituunt. Quas, pro casu quo $3n$ quantitates X_i, Y_i, Z_i sunt differentialia partialia eiusdem functionis U respective secundum x_i, y_i, z_i sumta, primus condidit celeb. ***Hamilton***, Astronomus Regius Hibernensis. Eo casu fit e (2.) §. pr. $Q_i = \frac{\partial U}{\partial q_i}$, unde statuendo $T-U=H$, si vires non a velocitatibus pendent ideoque U ab ipsis p_i vacua est, aequationes differentiales dynamicae evadunt,

$$6.\quad \frac{dq_i}{dt} = \left(\frac{\partial H}{\partial p_i}\right),\quad \frac{dp_i}{dt} = -\left(\frac{\partial H}{\partial q_i}\right).$$

Iam olim quidem ill. ***Poisson*** in celeberrimo opere de Constantium Arbitrariarum variatione id egerat, ut quantitatum q'_i loco in aequationibus differentialibus dynamicis ***Lagrangianis*** secundis introduceret quantitates p_i; quae aequationes si ea substitutione abeunt in

$$7.\quad \frac{dq_i}{dt} = A_i,\quad \frac{dp_i}{dt} = B_i,$$

bene idem cognoverat fore

$$\left(\frac{\partial A_i}{\partial q_k}\right) = -\left(\frac{\partial B_k}{\partial p_i}\right),\quad \left(\frac{\partial A_i}{\partial p_k}\right) = \left(\frac{\partial A_k}{\partial p_i}\right),\quad \left(\frac{\partial B_i}{\partial q_k}\right) = \left(\frac{\partial B_k}{\partial q_i}\right),$$

unde sequebatur, omnes $6n-2k$ quantitates A_i et $-B_i$ esse differentialia partialia eiusdem functionis, ipsarum p_i et q_i respectu sumta. At meritum, eam functionem $H=T-U$ ipsam assignavisse eaque re aequationibus differentialibus dynamicis formam perfectissimam conciliavisse, celeb. ***Hamilton*** debetur.

Casu quo mobilium Coordinatae functionibus aequantur quae praeter quantitates q_i ipsum tempus t implicant, forma simplex aequationum (5.) perit, qua de re hoc quidem loco transformationem *Hamiltonianam* ad eum casum non applicabo.

Facile invenitur aequationum (5.) Multiplicator M_2. Etenim si aequationes (5.) per formulas (7.) designamus, fit

$$\frac{d\log M_2}{dt}+\Sigma\left\{\left(\frac{\partial A_i}{\partial q_i}\right)+\left(\frac{\partial B_i}{\partial p_i}\right)\right\} = 0.$$

At ponendo

$$A_i = \left(\frac{\partial T}{\partial p_i}\right),\quad B_i = -\left(\frac{\partial T}{\partial q_i}\right)+Q_i,$$

sequitur, si vires sollicitantes a velocitatibus non pendent ideoque functiones Q_i quantitates p_1, p_2 etc. non implicant,

$$\left(\frac{\partial A_i}{\partial q_i}\right)+\left(\frac{\partial B_i}{\partial p_i}\right) = 0,$$

ideoque

$$8.\quad M_2 = 1.$$

Si functiones Q_i quoque implicant quantitates p_i, definitur M_2 per formulam,

$$9.\quad \frac{d\log M_2}{dt}+\frac{\partial Q_1}{\partial p_1}+\frac{\partial Q_2}{\partial p_2}\dots+\frac{\partial Q_{3n-k}}{\partial p_{3n-k}} = 0.$$

Iam tres Multiplicatores M, M_1, M_2, pro tribus aequationum differentialium dynamicarum formis inventos, inter se comparemus.

Forma secunda aequationum differentialium dynamicarum proveniebat e prima reducta per $2k$ aequationes integrales,

$$10.\quad \begin{cases} \Pi = 0, & \Pi_1 = 0, & \dots & \Pi_{k-1} = 0, \\ \Pi' = 0, & \Pi'_1 = 0, & \dots & \Pi'_{k-1} = 0. \end{cases}$$

Quae aequationes integrales, licet non completae, ita tamen sunt comparatae ut aequationum differentialium reductarum Multiplicator e Multiplicatore propositarum per eandem formulam obtineatur ac si reductio per aequationes integrales completa facta esset (cf. §§. 10. et 12.). Cum per aequationes (10.) revocentur $6n$ variabiles x_i, y_i, z_i, x'_i, y'_i, z'_i ad $6n-2k$ variabiles q_i et q'_i, secundum ea quae l. c. tradidi duorum Multiplicatorum Quotiens $\frac{M}{M_1}$ aequatur Determinanti $6n$ functionum

$$\Pi,\ \Pi_1,\ \dots\ \Pi_{k-1},\ q_1,\ q_2,\ \dots\ q_{3n-k},$$
$$\Pi',\ \Pi'_1,\ \dots\ \Pi'_{k-1},\ q'_1,\ q'_2,\ \dots\ q'_{3n-k},$$

formato respectu $6n$ quantitatum x_i, y_i, z_i, x'_i, y'_i, z'_i. Expressiones novarum variabilium q_1, q_2 etc. per x_i, y_i, z_i per aequationes (10.) diversas subire possunt mutationes, quibus tamen illius Determinantis valor non mutatur (cf. §. 3. (12.)). Ponamus rursus, ut supra, $3n$ quantitates ξ_i loco quantitatum $\sqrt{m_i}\,x_i$, $\sqrt{m_i}\,y_i$, $\sqrt{m_i}\,z_i$, atque $3n$ quantitates ξ'_i loco quantitatum $\sqrt{m_i}\,x'_i$, $\sqrt{m_i}\,y'_i$, $\sqrt{m_i}\,z'_i$, valor ipsius $\frac{M}{M_1}$ etiam aequari poterit Determinanti earundem $6n$ functionum, formato quantitatum ξ_i et ξ'_i respectu, quippe quod ab illo Determinante functionali tantum discrepat factore constante (cubo producti massarum). Cum $3n$ quantitates ξ'_i non reprehendantur in $3n$ functionibus Π_m et q_m, Determinans Quotienti $\frac{M}{M_1}$ aequale induit formam producti,

$$\Sigma \pm \frac{\partial \Pi}{\partial \xi_1}\cdot\frac{\partial \Pi_1}{\partial \xi_2}\cdots\frac{\partial \Pi_{k-1}}{\partial \xi_k}\cdot\frac{\partial q_1}{\partial \xi_{k+1}}\cdot\frac{\partial q_2}{\partial \xi_{k+2}}\cdots\frac{\partial q_{3n-k}}{\partial \xi_{3n}}$$
$$\times\, \Sigma \pm \frac{\partial \Pi'}{\partial \xi'_1}\cdot\frac{\partial \Pi'_1}{\partial \xi'_2}\cdots\frac{\partial \Pi'_{k-1}}{\partial \xi'_k}\cdot\frac{\partial q'_1}{\partial \xi'_{k+1}}\cdot\frac{\partial q'_2}{\partial \xi'_{k+2}}\cdots\frac{\partial q'_{3n-k}}{\partial \xi'_{3n}}.$$

Cum vero insuper sit

$$\frac{\partial \Pi'_m}{\partial \xi'_i} = \frac{\partial \Pi_m}{\partial \xi_i},\qquad \frac{\partial q'_m}{\partial \xi'_i} = \frac{\partial q_m}{\partial \xi_i},$$

utrumque in se ductum Determinans aequale evadit, unde eruitur

$$11.\qquad \frac{M}{M_1} = \left\{\Sigma \pm \frac{\partial \Pi}{\partial \xi_1}\cdot\frac{\partial \Pi_1}{\partial \xi_2}\cdots\frac{\partial \Pi_{k-1}}{\partial \xi_k}\cdot\frac{\partial q_1}{\partial \xi_{k+1}}\cdot\frac{\partial q_2}{\partial \xi_{k+2}}\cdots\frac{\partial q_{3n-k}}{\partial \xi_{3n}}\right\}^2.$$

Sint

$$m',\ m'',\ \ldots\ m^{(3n-k)}$$

indices diversi ex ipsorum $1, 2, \ldots 3n$ numero, supponere licet, ipsas $q_1, q_2, \ldots q_{3n-k}$ expressas esse per solas $3n-k$ quantitates

$$\xi_{m'},\ \xi_{m''},\ \ldots\ \xi_{m^{(3n-k)}};$$

tum autem Quotientis $\frac{M}{M_1}$ valor formam simpliciorem induit,

$$12.\qquad \frac{M}{M_1} = \left\{\Sigma \pm \frac{\partial q_1}{\partial \xi_{m'}}\cdot\frac{\partial q_2}{\partial \xi_{m''}}\cdots\frac{\partial q_{3n-k}}{\partial \xi_{m^{(3n-k)}}}\right\}^2$$
$$\times \left\{\Sigma \pm \frac{\partial \Pi}{\partial \xi_{m^{(3n-k+1)}}}\cdot\frac{\partial \Pi_1}{\partial \xi_{m^{(3n-k+2)}}}\cdots\frac{\partial \Pi_{k-1}}{\partial \xi_{m^{(3n)}}}\right\}^2,$$

siquidem $m^{(3n-k+1)}$, $m^{(3n-k+2)}$, ... $m^{(3n)}$ designant k reliquos indicum $1, 2, \ldots 3n$. Unde tandem per formulam notam (*Determ. Funct.* §. 3. (12.)) sequitur,

$$13.\qquad M\left\{\Sigma \pm \frac{\partial \xi_{m'}}{\partial q_1}\cdot\frac{\partial \xi_{m''}}{\partial q_2}\cdots\frac{\partial \xi_{m^{(3n-k)}}}{\partial q_{3n-k}}\right\}^2$$
$$= M_1\left\{\Sigma \pm \frac{\partial \Pi}{\partial \xi_{m^{(3n-k+1)}}}\cdot\frac{\partial \Pi_1}{\partial \xi_{m^{(3n-k+2)}}}\cdots\frac{\partial \Pi_{k-1}}{\partial \xi_{m^{(3n)}}}\right\}^2.$$

Quod antecedentibus suppositum est, novas variabiles $q_1, q_2, \dots q_{3n-k}$ per totidem quantitates $\xi_{m'}$, $\xi_{m''}$ etc. expressas esse, id fieri non potest, quoties ex aequationibus conditionalibus $\Pi = 0$ etc. aequatio inter easdem $3n-k$ quantitates $\xi_{m'}$ etc. sequitur; nam cum $3n-k$ quantitates q_1, q_2 etc. a se independentes sint, etiam $3n-k$ quantitates $\xi_{m'}$ etc., per quas exprimantur, a se independentes esse debent. Nihilo tamen minus pro eo quoque casu formula (13.) valet. Quoties enim ex aequationibus $\Pi = 0$ etc. fluit aequatio inter solas $3n-k$ quantitates $\xi_{m'}, \xi_{m''}, \dots, \xi_{m^{(3n-k)}}$, hae aequabuntur $3n-k$ functionibus quantitatum $q_1, q_2, \dots q_{3n-k}$ non a se independentibus, quarum functionum Determinans evanescere constat. (*Determ. Funct.* §. 6.) Porro si e k aequationibus $\Pi = 0$ etc. obtineri potest aequatio inter solas $3n-k$ quantitates $\xi_{m'}, \xi_{m''}, \dots \xi_{m^{(3n-k)}}$, fieri debet, ut ex iisdem reliquae k quantitates $\xi_{m^{(3n-k+1)}}$ etc. eliminari possint. At *si de k aequationibus $\Pi = 0$ etc. totidem quantitates eliminari possunt, functionum Π etc. Determinans earum quantitatum respectu formatum per ipsas aequationes evanescit* *). Unde casu de quo agitur, utroque Determinante ad dextram et laevam signi aequalitatis posito evanescente, aequatio (13.) iusta manet.

Si, quod secundum antecedentia licet, in aequatione (13.) pro systemate indicum $m', m'', \dots m^{(3n-k)}$ sumuntur quique $3n-k$ diversi indicum $1, 2, \dots 3n$, omnesque $\frac{3n.3n-1\dots 3n-k+1}{1\ .\ 3\ \dots\ k}$ aequationes provenientes consummantur, prodit aequatio

$$M\,S.\left\{\Sigma \pm \frac{\partial \xi_{m'}}{\partial q_1}\cdot\frac{\partial \xi_{m''}}{\partial q_2}\cdots\frac{\partial \xi_{m^{(3n-k)}}}{\partial q_{3n-k}}\right\}^2$$
$$= M_1 S.\left\{\Sigma \pm \frac{\partial \Pi}{\partial \xi_{m'}}\cdot\frac{\partial \Pi_1}{\partial \xi_{m''}}\cdots\frac{\partial \Pi_{n-1}}{\partial \xi_{m^{(k)}}}\right\}^2,$$

ubi in altera summa loco indicum $m^{(3n-k+1)}, m^{(3n-k+2)}, \dots m^{(3n)}$, quippe qui aliam non habent significationem quam quorumque k diversorum ex indicibus

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*) Ponamus enim, ex aequatione $\Pi = 0$ eliminari posse k quantitates ope reliquarum aequationum $\Pi_1 = 0$, $\Pi_2 = 0, \dots \Pi_{k-1} = 0$, per easdem induere debet Π formam producti μF, designante F functionem a k quantitatibus vacuam, ut ex aequationibus conditionalibus sequatur inter reliquas quantitates aequatio $F = 0$. Secundum §. 3. (12.) in Determinante functionum $\Pi, \Pi_1, \dots \Pi_{k-1}$ ipsum μF substituere licet functioni Π. Quoties autem $F = 0$, differentialia prima ipsius μF ita formare licet ac si factor μ constans esset, unde etiam in formando Determinante functionum $\mu F, \Pi_1, \Pi_2, \dots \Pi_{k-1}$ habere licet μ pro Constante. Quod igitur Determinans aequivalebit factori μ ducto in Determinans functionum $F, \Pi_1, \Pi_2, \dots \Pi_{k-1}$, ideoque evanescet, cum F ab ipsis quantitatibus vacua sit, quarum respectu Determinans functionale formatur.

1, 2, ... $3n$, scripsi m', m'', ... $m^{(k)}$. Aequatio antecedens perfecte congruit cum supra inventis. Nam secundum formulam (16.) §. 22. aequatur M summae ad dextram, secundum formulam (12.) §. 23. aequatur M_1 summae ad laevam signi aequalitatis positae.

Aequationum dynamicarum forma secunda in tertiam mutabatur introducendo variabilium $q'_1, q'_2, \dots q'_{3n-k}$ loco totidem alias $p_1, p_2, \dots p_{3n-k}$. Unde secundum §. 9. tertiae formae Multiplicator M_2 e secundae Multiplicatore M_1 obtinetur formula,

$$\frac{M_1}{M_2} = \Sigma \pm \frac{\partial p_1}{\partial q'_1} \cdot \frac{\partial p_2}{\partial q'_2} \cdots \frac{\partial p_{3n-k}}{\partial q'_{3n-k}}.$$

Dantur autem novae quantitates p_i aequationibus linearibus,

$$p_i = a_{i,1} q'_1 + a_{i,2} q'_2 \dots + a_{i,3n-k} q'_{3n-k}$$

posito secundum (11.) §. 23.

$$a_{i,i'} = \frac{\partial \xi_1}{\partial q_i} \cdot \frac{\partial \xi_1}{\partial q_{i'}} + \frac{\partial \xi_2}{\partial q_i} \cdot \frac{\partial \xi_2}{\partial q_{i'}} \dots + \frac{\partial \xi_{3n}}{\partial q_i} \cdot \frac{\partial \xi_{3n}}{\partial q_{i'}},$$

unde fit

$$\frac{M_1}{M_2} = \Sigma \pm a_{1,1} a_{2,2} \dots a_{3n-k, 3n-k}.$$

Quod rursus cum supra inventis congruit, cum secundum (9.) §. pr. aequetur M_1 Determinanti ad dextram, secundum (8.) autem M_2 *unitati*. Per considerationes antecedentes videmus, e valore $M_2 = 1$, qui sponte patet, inveniri potuisse M_2 et M, supra via diversissima inventos. Qua methodorum diversitate cum Multiplicatoris tum Determinantium functionalium theoria haud parum illustratur.

Principium ultimi multiplicatoris ad formam aequationum differentialium dynamicarum tertiam relatum sic enunciari potest.

„Punctorum materialium systema subiectum sit conditionibus et sollicitetur viribus quibuscunque, a sola positione systematis in spatio pendentibus; qua positione determinata per μ quantitates independentes q_i, semisumma virium vivarum T exprimatur per quantitates q_i et $q'_i = \frac{dq_i}{dt}$; ad motum systematis definiendum, eliminato tempore, integrandae erunt $2\mu - 1$ aequationes differentiales primi ordinis, quarum inventa sint $2\mu - 2$ Integralia, totidem Constantes Arbitrarias involventia, ita ut integranda restet unica aequatio differentialis primi ordinis inter duas variabiles u et v,

$$v' du - u' dv = 0,$$

designantibus in hac aequatione u' et v' ipsarum u et v functiones quibus quotientes differentiales $\frac{du}{dt}$ et $\frac{dv}{dt}$ ope Integralium inventorum aequantur; erit huius aequationis differentialis primi ordinis inter duas variabiles ultimo loco integrandae Multiplicator aequalis Determinanti functionali 2μ quantitatum q_i et $\frac{\partial T}{\partial q_i'}$, ipsarum u, v atque $2\mu-2$ Constantium Arbitrariarum respectu formato.

Iam novum principium generale mechanicum exemplis applicabo.

De motu puncti versus centrum fixum attracti.

§. 25.

Pro motu libero puncti in plano ex ultimi multiplicatoris principio generali fluit haec

Propositio.

Proponantur pro motu puncti in plano aequationes differentiales,

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y,$$

designantibus X et Y Coordinatarum puncti orthogonalium x et y functiones quascunque; si habentur aequationum differentialium propositarum duo Integralia

$$f(x, y, x', y') = \alpha, \qquad \varphi(x, y, x', y') = \beta,$$

ubi α et β sunt Constantes Arbitrariae, dabitur orbita puncti formula

$$\int\left(\frac{\partial x'}{\partial \alpha}\cdot\frac{\partial y'}{\partial \beta} - \frac{\partial x'}{\partial \beta}\cdot\frac{\partial y'}{\partial \alpha}\right)(y'dx - x'dy) = \gamma$$

sive etiam formula

$$\int\frac{y'dx - x'dy}{\frac{\partial f}{\partial x'}\cdot\frac{\partial \varphi}{\partial y'} - \frac{\partial f}{\partial y'}\cdot\frac{\partial \varphi}{\partial x'}} = \gamma,$$

ubi duorum Integralium inventorum ope exhibitis x' et y' per x, y, α, β quantitates sub integrationis signo differentialia completa fiunt atque γ tertiam Constantem Arbitrariam designat.

Aliam propositionem, qua puncti liberi in plano moti orbita Quadraturis definiri potest, si puncti velocitatis intensitas et directio per duo Integralia inventa determinatae sunt, iam ante multos annos cum illustri *Academia Parisiensi* communicavi, sed ea propositio tantum respiciebat casum quo vires Coordinatis parallelae X et Y eiusdem quantitatum x et y functionis aequantur differentialibus ipsarum x et y respectu sumtis, dum in propositione antecedente X et Y quantitatum x et y functiones quaecunque esse possunt.

Pro motu puncti in dato plano versus centrum fixum attracti duo constant Integralia principiis conservationis vis vivae et conservationis areae, quibus si principium ultimi multiplicatoris addis, per tria illa principia generalia a priori constat, eius motus determinationem solis Quadraturis absolvi. Quod facto calculo sic comprobatur.

Pro motu proposito habentur aequationes differentiales

$$\frac{d^2x}{dt^2} = -\frac{xF(r)}{r}, \quad \frac{d^2y}{dt^2} = -\frac{yF(r)}{r},$$

ubi x et y Coordinatae orthogonales sunt, quarum initium in centro attractionis est; porro $r = \sqrt{(xx+yy)}$ atque $F(r)$ intensitas vis attractivae pro distantia r. Posito

$$R = \int F(r)dr,$$

e principiis generalibus mechanicis conservationis vis vivae et areae statim habentur duo Integralia,

$$\begin{aligned} f &= \tfrac{1}{2}(x'x'+y'y')+R = \alpha, \\ \varphi &= xy'-yx' = \beta, \end{aligned}$$

designantibus α et β Constantes Arbitrarias. Unde fit,

$$\frac{\partial f}{\partial x'}\cdot\frac{\partial \varphi}{\partial y'} - \frac{\partial f}{\partial y'}\cdot\frac{\partial \varphi}{\partial x'} = xx'+yy'.$$

E duobus Integralibus appositis sequitur

$$xx'+yy' = \sqrt{\varrho},$$

posito

$$\varrho = 2r^2(\alpha-R)-\beta\beta.$$

Unde secundum principium ultimi Multiplicatoris dabitur puncti orbita per aequationem,

$$\int \frac{y'dx-x'dy}{\frac{\partial f}{\partial x'}\cdot\frac{\partial \varphi}{\partial y'} - \frac{\partial f}{\partial y'}\cdot\frac{\partial \varphi}{\partial x'}} = \int \frac{y'dx-x'dy}{\sqrt{\varrho}} = \gamma,$$

designante γ novam Constantem Arbitrariam. Ex aequationibus,

$$xy'-yx' = \beta, \quad xx'+yy' = \sqrt{\varrho},$$

sequitur

$$x' = \frac{x\sqrt{\varrho}-\beta y}{rr}, \quad y' = \frac{y\sqrt{\varrho}+\beta x}{rr};$$

unde substituendo $x\,dx+y\,dy = r\,dr$ fit

$$\frac{y'dx-x'dy}{\sqrt{\varrho}} = \frac{y\,dx-x\,dy}{rr}+\frac{\beta\,dr}{r\sqrt{\varrho}}.$$

Posito igitur $x = r\cos\vartheta$, $y = r\sin\vartheta$, unde $y\,dx-x\,dy = -rr\,d\vartheta$, dabitur orbita per formulam,

$$\vartheta+\gamma = \beta\int\frac{dr}{r\sqrt{(2r^2(\alpha-R)-\beta\beta)}}.$$

Si lex attractionis est *Neutoniana*, ponendum est $F(r) = \frac{k^2}{rr}$, $R = -\frac{k^2}{r}$, designante k^2 vim attractivam pro unitate distantiae, institutaque integratione prodit aequatio sectionis conicae inter Coordinatas polares r, $\vartheta+\gamma$.

Aequationum differentialium antecedentium dextrae parti addamus *Coordinatarum x et y functiones homogeneas* $(-3)^{tae}$ *dimensionis*, X et Y, aequationum differentialium provenientium,

$$\frac{d^2x}{dt^2} = -x\frac{F(r)}{r}+X,$$
$$\frac{d^2y}{dt^2} = -y\frac{F(r)}{r}+Y,$$

semper aliquod obtineri poterit Integrale. Nam ex his aequationibus eruitur,

$$\tfrac{1}{2}d.\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)^2 = (x\,dy-y\,dx)(xY-yX) = x^2(xY-yX)\,d.\frac{y}{x}.$$

At est $x^2(xY-yX)$ functio variabilium x et y homogenea *nullae* dimensionis ideoque functio ipsius $\frac{y}{x}$, unde aequationis antecedentis pars utraque est differentiale completum, factaque integratione prodit

$$\varphi = \tfrac{1}{2}(xy'-yx')^2-V = \tfrac{1}{2}\beta^2,$$

siquidem β Constans Arbitraria est atque

$$V = \int x^2(xY-yX)\,d\frac{y}{x}.$$

Si X et Y sunt differentialia partialia functionis homogeneae $(-2)^{tae}$ dimensionis U, ipsarum x et y respectu sumta, principium conservationis vis vivae alterum suppeditat Integrale

$$f = \tfrac{1}{2}(x'x'+y'y')+R-U = \alpha,$$

siquidem α est altera Constans Arbitraria atque rursus

$$R = \int F(r)\,dr.$$

Functiones f et φ inventas substituendo fit

$$\frac{\partial f}{\partial x'}\cdot\frac{\partial\varphi}{\partial y'}-\frac{\partial f}{\partial y'}\cdot\frac{\partial\varphi}{\partial x'} = (xx'+yy')(xy'-yx').$$

At ex Integralibus inventis eruitur

$$(xx'+yy')(xy'-yx') = \sqrt{\{2r^2(\alpha-R+U)-(2V+\beta^2)\}}.\sqrt{(2V+\beta^2)},$$

quippe ponendo

$$2r^2(\alpha-R+U)-(2V+\beta^2) = \varrho,$$

fit

$$xy'-yx' = \sqrt{(2V+\beta^2)}, \qquad xx'+yy' = \sqrt{\varrho}.$$

Hinc sequitur

$$\frac{\partial f}{\partial x'}\cdot\frac{\partial\varphi}{\partial y'}-\frac{\partial f}{\partial y'}\cdot\frac{\partial\varphi}{\partial x'} = \sqrt{\varrho}.\sqrt{(2V+\beta^2)};$$

$$x' = \frac{x\sqrt{\varrho}-\sqrt{(2V+\beta^2)}.y}{rr},$$

$$y' = \frac{y\sqrt{\varrho}+\sqrt{(2V+\beta^2)}.x}{rr}.$$

Quibus formulis substitutis in tertio Integrali, quod principio ultimi multiplicatoris suppeditatur,

$$\int\frac{y'dx-x'dy}{\frac{\partial f}{\partial x'}\cdot\frac{\partial\varphi}{\partial y'}-\frac{\partial f}{\partial y'}\cdot\frac{\partial\varphi}{\partial x'}} = \gamma,$$

obtinetur formula quae puncti orbitam determinat,

$$\int\left(\frac{y\,dx-x\,dy}{rr\sqrt{(2V+\beta^2)}}+\frac{dr}{r\sqrt{\varrho}}\right) = \gamma,$$

sive ponendo rursus $x = r\cos\vartheta$, $y = r\sin\vartheta$,

$$\int\left(\frac{dr}{r\sqrt{\varrho}}-\frac{d\vartheta}{\sqrt{(2V+\beta^2)}}\right) = \gamma,$$

semper designante γ tertiam Constantem Arbitrariam. Cum sit U functio homogenea $(-2)^{ti}$ ordinis, erit

$$2U = -\left\{x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}\right\} = -\{xX+yY\},$$

unde

$$\begin{aligned} d.r^2U &= -\{xX+yY\}(x\,dx+y\,dy)+\{xx+yy\}(X\,dx+Y\,dy) \\ &= (xY-yX)(x\,dy-y\,dx). \end{aligned}$$

Eadem quantitas aequabatur ipsi dV, unde in formulis antecedentibus statuere licet

$$V = rrU,$$
$$\varrho = 2r^2(\alpha-R)-\beta^2.$$

Secundum suppositionem factam fit $r^2 U = V$ ipsius $\frac{y}{x} = \operatorname{tang}\vartheta$ functio, unde in aequatione orbitae,

$$\int \frac{dr}{r\sqrt{(2r^2(\alpha - R) - \beta^2)}} = \int \frac{d\vartheta}{\sqrt{(2V + \beta^2)}} + \gamma,$$

alterum integrale solius r, alterum solius ϑ functio est. Temporis expressio habetur per formulam

$$t + \tau = \int \frac{r\,dr}{xx' + yy'} = \int \frac{r\,dr}{\sqrt{\varrho}} = \int \frac{r^2\,d\vartheta}{\sqrt{(2V + \beta^2)}},$$

in qua τ est nova Constans Arbitraria.

In motu antecedentibus considerato vis $F(r)$, qua punctum versus centrum fixum attrahitur, aucta est alia vi, quae secundum axes orthogonales disposita differentialibus partialibus $\frac{\partial U}{\partial x}$ et $\frac{\partial U}{\partial y}$ aequatur. Eadem vis secundum radii vectoris directionem eique perpendiculariter disposita evadit

$$P = \frac{1}{r}\left\{x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}\right\}, \qquad Q = \frac{1}{r}\left\{y\frac{\partial U}{\partial x} - x\frac{\partial U}{\partial y}\right\}.$$

Secundum suppositionem de functionis U indole factam statui potest

$$r^2 U = V = \Psi(\vartheta),$$

designante $\Psi(\vartheta)$ functionem anguli ϑ quem radius vector cum axe fixo format. Qua expressione substituta positoque $\frac{d\Psi(\vartheta)}{d\vartheta} = \Psi'(\vartheta)$, eruitur

$$P = -\frac{2}{r^3}\Psi(\vartheta), \qquad Q = -\frac{1}{r^3}\Psi'(\vartheta).$$

Si iam ponitur

$$\beta\int \frac{dr}{r\sqrt{\varrho}} = \beta\int \frac{dt}{r^2} = \beta\int \frac{d\vartheta}{\sqrt{(2\Psi(\vartheta) + \beta^2)}} = \Theta,$$

docent formulae antecedentibus inventae, illis viribus P et Q ad vim attractivam $F(r)$ accedentibus orbitae aequationem polarem eam mutationem subire ut angulus ϑ in angulum Θ mutetur. At simul videmus, ***illa virium P et Q accessione relationem inter radium vectorem et tempus omnino immutatam manere.*** Quae curiosa propositio valet etiam si non quod antecedentibus supposui motus in plano fit. Sit enim U ipsarum x, y, z functio homogenea $(-2)^{\text{tae}}$ dimensionis, ac proponantur aequationes differentiales,

$$\frac{d^2x}{dt^2} = -\frac{x}{r}F(r) + \frac{\partial U}{\partial x},$$
$$\frac{d^2y}{dt^2} = -\frac{y}{r}F(r) + \frac{\partial U}{\partial y},$$
$$\frac{d^2z}{dt^2} = -\frac{z}{r}F(r) + \frac{\partial U}{\partial z};$$

rursus $\int F(r)\,dr = R$ ponendo sequitur,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2(-R + U + \alpha),$$
$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} = -rF(r) - 2U.$$

Quibus additis fit

$$d\left\{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right\} = d.r\frac{dr}{dt} = \{2(\alpha - R) - rF(r)\}dt,$$

unde multiplicando per $2r\frac{dr}{dt}$ et integrando prodit,

$$r^2\left(\frac{dr}{dt}\right)^2 = 2r^2(\alpha - R) + \varepsilon,$$

ideoque

$$t + \tau = \int \frac{r\,dr}{\sqrt{\{2r^2(\alpha - R) + \varepsilon\}}},$$

qua in formula τ et ε Constantes Arbitrariae sunt. Patet autem quod demonstrandum erat, in hac formula nullum functionis U vestigium remansisse. Addo, si U gaudeat forma particulari,

$$U = \frac{1}{r^2}\left\{f\left(\frac{x}{r}\right) + \varphi\left(\frac{y}{r}\right)\right\},$$

designantibus f et φ functiones quascunque, eum ipsum motum, qui in plano non continetur, totum Quadraturis determinari posse.

Motus puncti in spatio pendet a *quinque* aequationibus differentialibus primi ordinis inter *sex* quantitates x, y, z, x', y', z'; unde *quatuor* Integralibus egemus ut problema ad aequationem differentialem primi ordinis inter duas variabiles revocetur, quae ope principii ultimi multiplicatoris per solas Quadraturas integrabitur. At quoties vires sollicitantes diriguntur versus axem fixum viriumque intensitates non pendent ab angulo quem planum per axem et mobile ductum cum plano fixo per eundem axem transeunte facit, problema ad motum puncti in plano revocari potest, et nonnisi *duobus* Integralibus opus erit ut totum absolvatur Quadraturis. Designantibus enim x, v, ζ puncti Coordinatas orthogonales positoque

$$vv+\zeta\zeta = yy,$$

sint aequationes differentiales, quibus motus puncti definitur,

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2v}{dt^2} = Y\frac{v}{y}, \quad \frac{d^2\zeta}{dt^2} = Y\frac{\zeta}{y},$$

ubi secundum suppositionem factam et X et Y solarum x et y functiones esse debent: erit

$$v\frac{d^2\zeta}{dt^2} - \zeta\frac{d^2v}{dt^2} = 0,$$

unde sequitur,

$$v\frac{d\zeta}{dt} - \zeta\frac{dv}{dt} = \alpha,$$

designante α Constantem Arbitrariam. Fit autem,

$$\frac{d^2y}{dt^2} = \frac{d^2\sqrt{(vv+\zeta\zeta)}}{dt^2} = \frac{(v\,d\zeta-\zeta\,dv)^2}{\sqrt{(vv+\zeta\zeta)^3}.dt^2} + \frac{v\,d^2v+\zeta\,d^2\zeta}{\sqrt{(vv+\zeta\zeta)}.dt^2},$$

ideoque

$$\frac{d^2y}{dt^2} = \frac{\alpha\alpha}{y^3} + Y.$$

Unde aequationes differentiales propositae evadunt sequentes,

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = \frac{\alpha\alpha}{y^3} + Y.$$

Cf. ***Diar. Crell. Vol. XXIV. pag. 16 sqq.*** Ponendo,

$$v = y\cos f, \qquad \zeta = y\sin f,$$

fit

$$v\frac{d\zeta}{dt} - \zeta\frac{dv}{dt} = yy\frac{df}{dt} = \alpha,$$

unde Constans α aequabitur plani per punctum mobile et axem fixum ducti velocitati rotatoriae initiali, multiplicatae per quadratum distantiae initialis puncti ab axe. Duobus Integralibus inter x, y, x', y' inventis, tertium integrale principio ultimi Multiplicatoris suppeditatur. Quorum Integralium ope si $y' = \frac{dy}{dt}$ per y exprimitur, cum rotationis angulus f tum tempus t Quadraturis determinantur ope formularum,

$$f = \alpha\int\frac{dt}{y^2} = \alpha\int\frac{dy}{y^2y'}, \qquad t = \int\frac{dy}{y'}.$$

Unde in casu proposito cognitis ***duobus*** Integralibus tria reliqua a solis Quadraturis pendent. Consideretur ex. gr. motus puncti versus centrum fixum attracti; posito $r = \sqrt{(xx+yy)}$, secundum antecedentia erit

$$\frac{d^2x}{dt^2} = -\frac{x}{r}F(r); \quad \frac{d^2y}{dt^2} = -\frac{y}{r}F(r)+\frac{\alpha\alpha}{y^3}.$$

Quae aequationes in eas redeunt, quas supra integravi, ponendo

$$Y = \frac{\alpha\alpha}{y^3}, \quad U = -\frac{\alpha\alpha}{2yy} = -\frac{\alpha\alpha}{2rr\sin\vartheta^2},$$

unde

$$V = \Psi(\vartheta) = -\frac{\alpha\alpha}{2\sin\vartheta^2},$$

$$\Theta = \int\frac{\beta.d\vartheta}{\sqrt{(2\Psi(\vartheta)+\beta^2)}} = \int\frac{\sin\vartheta\, d\vartheta}{\sqrt{(\beta^2\sin\vartheta^2-\alpha^2)}}.$$

ideoque,

$$\cos\Theta = \frac{\beta}{\sqrt{(\beta^2-\alpha^2)}}\cos\vartheta.$$

Si r et ϑ sunt puncti attracti Coordinatae polares in plano fixo in quo illud revera movetur, in aequatione orbitae, quam in hoc plano describit, angulus Θ loco ipsius ϑ substitui debet ut eruatur orbita descripta in plano mobili per axem ipsarum x ducto. Relationem inter r et t pro motu in utroque plano eandem manere, ex ipsa natura rei patet. Plani angulus rotatorius f datur per formulam,

$$df = \frac{\alpha\, dt}{yy} = \frac{\alpha\, dt}{rr\sin^2\vartheta} = \frac{d\Theta}{\sin^2\vartheta} = \frac{\alpha\beta.d\Theta}{\alpha^2\cos\Theta^2+\beta^2\sin\Theta^2},$$

unde, designante ε Constantem Arbitrariam,

$$\operatorname{tang}(f+\varepsilon) = \frac{\beta}{\alpha}\operatorname{tang}\Theta.$$

Si per centrum attractionis ex arbitrio axis fixus ducitur, in formulis antecedentibus axem Coordinatarum x pro axe fixo sumendo motus puncti attracti componitur e motu puncti in plano per ipsum et axem fixum ducto eiusque plani rotatione circa axem fixum. Statuatur $\alpha = \beta\sin\delta$, erit

$$\cos\vartheta = \cos\delta\cos\Theta, \quad \operatorname{tang}\Theta = \sin\delta\operatorname{tang}(f+\varepsilon), \quad \sin\vartheta\sin(f+\varepsilon) = \sin\Theta.$$

E centro attractionis describatur superficies sphaerica, cuius intersectio cum axe fixo, cum radio vectore et cum plano orbitae puncti attracti sit A, P et circulus maximus PQ; porro in sphaera e A ad circulum maximum PQ demittatur perpendicularis AO: in triangulo rectangulo sphaerico AOP erit

$$AO = \delta, \quad AP = \vartheta, \quad PO = \Theta, \quad OAP = f+\varepsilon.$$

Cuius constructionis ope formulae antecedentes geometrice comprobari possunt.

Si punctum versus centra fixa quotcunque in eadem recta disposita secundum ***Neutonianam*** sive aliam quamcunque legem attrahitur, quibus attractionis viribus accedere potest vis constans rectae parallela, e duobus Integralibus, quae antecedentibus poscebantur ut reliquae integrationes omnes Quadraturis absolverentur, alterum conservationis vis vivae principio suppeditatur. Si abest vis constans atque duo tantum sunt centra attrahentia lexque attractionis est ***Neutoniana,*** alterum Integrale ***Eulerus*** invenit. Eo igitur casu motus ille principio conservationis areae certi cuiusdam axis respectu valentis, principio conservationis vis vivae, Integrali ***Euleriano,*** tandem principio ultimi multiplicatoris ad Quadraturas revocatur. Quod iam accuratius exponam.

(Cont. fasc. seq.)