16.

Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi.

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Argumentum.

§. 1.

Propositurus sum sequentibus ***Euleriani*** Multiplicatoris extensionem, per totum calculum integralem uberrimi usus et frequentissimae applicationis, eamque ab amplificationibus ab ipso ***Eulero*** et ***Lagrange*** factis diversissimam. Quae amplificatio maxime nititur analogia, quam in alia Commentatione pluribus prosecutus sum, inter quotientes differentiales et Determinantia functionalia. Efficit ***Eulerianus*** Multiplicator ut duae ***duarum*** variabilium functiones datae producant eiusdem functionis differentialia partialia. Respondent autem differentialibus partialibus Determinantia functionalia partialia, quae formari possunt quoties variabilium numerus numerum functionum superat, variis eligendo modis variabiles quarum respectu Determinans formetur Ita datis n functionibus $n+1$ variabilium earum functionum dabuntur $n+1$ Determinantia partialia; veluti si f et φ trium variabilium x, y, z functiones sunt, tria earum functionum Determinantia partialia erunt

$$\frac{\partial f}{\partial x}\cdot\frac{\partial\varphi}{\partial z}-\frac{\partial f}{\partial z}\cdot\frac{\partial\varphi}{\partial y},\quad \frac{\partial f}{\partial z}\cdot\frac{\partial\varphi}{\partial x}-\frac{\partial f}{\partial x}\cdot\frac{\partial\varphi}{\partial z},\quad \frac{\partial f}{\partial x}\cdot\frac{\partial\varphi}{\partial y}-\frac{\partial f}{\partial y}\cdot\frac{\partial\varphi}{\partial x}.$$

Quibus considerationibus motus, ut ***Eulerianam*** theoriam amplificarem, generaliter Multiplicatorem examinavi, in quem ducendae essent $n+1$ functiones $n+1$ variabilium ut producta haberi possent pro earundem n functionum Determinantibus functionalibus partialibus. Quemadmodum autem, proposita functione duarum variabilium, inter bina eius differentialia partialia intercedit aliqua conditio ex elementis nota, scilicet ut alterius differentiale secundum alteram variabilem sumtum alterius differentiali secundum alteram variabilem sumto aequale sit: ita inter illa $n+1$ Determinantia functionalia partialia inveni locum habere conditionem analogam. Singulis enim Determinantibus functionalibus partialibus respective secundum singulas variabiles differentiatis, aggregatum $n+1$ quantitatium provenientium videbimus identice evanescere. Quod suppeditat aequationem differentialem partialem, cui Multiplicator ille satisfacere debeat, ei analogam qua ***Eulerianus*** Multiplicator definitur. Et vice versa, sicuti in theoria ***Euleriana,*** quamcunque quantitatem, aequationi illi differentiali partiali satisfacientem, videbimus pro Multiplicatore haberi posse. Unde ad Multiplicatorem aliquem obtinendum non necessarium erit ut illae n functiones ipsae innotescant.

Investigatio ipsius functionis duarum variabilium, cuius differentialia partialia datis functionibus proportionalia sint, pendet ab integratione completa aequationis differentialis vulgaris primi ordinis inter duas variabiles; quippe quae ea erit functio, quae Constanti Arbitrariae aequalis evadit. Multiplicator autem, qui functiones datas ***aequales*** efficit binis differentialibus eius functionis partialibus, ipsius ***aequationis differentialis*** Multiplicator appellatur. Qui aequationis differentialis integratione completa sponte suppeditatur, et vice versa eius cognitione ipsa integratio maxime expeditur, videlicet ad solas revocatur Quadraturas. Similiter datis $n+1$ variabilium $n+1$ functionibus, ut obtineantur n functiones quarum Determinantia partialia rationes easdem atque illae inter se habeant: facile patebit, integrandum esse systema n aequationum differentialium vulgarium primi ordinis, quo scilicet statuitur illarum $n+1$ variabilium differentialia esse in ratione ipsarum $n+1$ quantitatum propositarum. Quo complete integrato functiones, quae Constantibus Arbitrariis a se independentibus aequales evadunt, ipsae erunt n functiones quaesitae. Atque Multiplicatorem, qui $n+1$ quantitates datas Determinantibus earum functionum partialibus aequales efficit, per analogiam illius *systematis aequationum differentialium vulgarium Multiplicatorem* appello. Iam quidem complete integrato systemate aequationum differentialium vulgarium, eius facile innotescit Multiplicator; quippe ad quem inveniendum tantum opus est ut functionum Constantibus Arbitrariis aequalium, quae per integrationem completam constant, unum aliquod formetur Determinans partiale. At vice versa, cognito aliquo systematis aequationum differentialium Multiplicatore, sive quod idem est, cognita aliqua solutione aequationis differentialis partialis qua Multiplicator definitur, non ita patebat, utrum et quodnam inde commodum vel auxilium ad integrandum systema peti posset, ita ut nostri Multiplicatoris analogia cum ***Euleriano*** videretur in ea ipsa re deficere, qua propter olim ***Eulerus*** sui Multiplicatoris theoriam condidit. Contigit tandem usum introspicere plane singularem quem in integrando aequationum differentialium systemate e Multiplicatoris cognitione percipere liceat, quod scilicet eius ope non prima aliqua, sed omnium ultima integratio ad Quadraturas revocetur.

Hinc in theoria integrationis aequationum differentialium vulgarium novus disquisitionum aperitur campus, videlicet ultimas investigandi integrationes, dum primae non innotescunt. Quippe in vastis et luculentissimis problematis per theoriam hic propositam fit ut ultima generaliter absolvatur integratio, dum in casibus tantum particularibus Integralia prima invenire licet.

Capite primo examinabo Multiplicatoris nostri varias formas insignioresque proprietates. In altero Capite eius monstrabo usum in integrando aequationum differentialium vulgarium systemate. In Capite tertio theoriam Multiplicatoris extendam ad systemata aequationum differentialium vulgarium cuiuslibet ordinis. In Commentationibus deinde subsequentibus mihi propositum est praecepta hic tradita variis illustrare applicationibus; e quibus est principium novum mechanicum latissime patens, nuper a me sine demonstratione divulgatum.

Caput primum.

Novi Multiplicatoris definitio et varii proprietates.

Lemma Fundamentale eiusque varii usus; de Determinantibus functionalibus partialibus.

§. 2.

Aequatione inter variabiles x et y proposita,

$$f(x, y) = \text{const.},$$

obtinetur differentialium dx et dy ratio,

$$dx : dy = \frac{\partial f}{\partial y} : -\frac{\partial f}{\partial x} \;*).$$

Si de hac ratione differentialium dx et dy sola agitur, in dextra parte aequationis antecedentis omittere licet differentialium partialium $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ factorem vel denominatorem, si quo afficiuntur communem. Ubi vero pro quantitatibus, quae differentialibus dx et dy proportionales evadunt ipsa sumere placet $\frac{\partial f}{\partial y}$ et $-\frac{\partial f}{\partial x}$ vel $-\frac{\partial f}{\partial y}$ et $\frac{\partial f}{\partial x}$, qualia differentiatione partiali prodeunt, nullo

*) Differentialia vulgaria ut in aliis Commentationibus charactere — d —, partialia charactere — ∂ — denoto.

factore aut denominatore communi rejecto, eam conditionem formula analytica exprimi posse constat.

Videlicet si quantitas ipsi dx proportionalis differentiatur ipsius x respectu, quantitas ipsi dy proportionalis differentiatur ipsius y respectu, quantitatum differentiatione provenientium summa identice evanescere debet. Theorema simile ad plures variabiles valet.

Aequationibus enim inter x, y, z propositis,

$$f(x, y, z) = \text{Const.}, \quad \varphi(x, y, z) = \text{Const.},$$

obtinetur differentiando,

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0,$$

$$\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz = 0.$$

E quibus aequationibus eruuntur differentialium dx, dy, dz rationes,

$$dx : dy : dz = A : B : C,$$

siquidem ponitur

$$A = \frac{\partial f}{\partial y} \cdot \frac{\partial \varphi}{\partial z} - \frac{\partial f}{\partial z} \cdot \frac{\partial \varphi}{\partial y},$$

$$B = \frac{\partial f}{\partial z} \cdot \frac{\partial \varphi}{\partial x} - \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial z},$$

$$C = \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial y} - \frac{\partial f}{\partial y} \cdot \frac{\partial \varphi}{\partial x}.$$

Si tantum de rationibus differentialium dx, dy, dz agitur, factorem vel denominatorem communem quantitatum $A, B, C,$ si quo afficiuntur, omittere licet. Ubi vero pro quantitatibus, quae differentialibus dx, dy, dz proportionales evadunt, ipsa sumere placet $A, B, C,$ nullo factore vel denominatore communi rejecto, eam conditionem aliqua formula analytica exprimi posse videbimus. Fit enim

$$\frac{\partial A}{\partial x} = \frac{\partial \varphi}{\partial z} \cdot \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial \varphi}{\partial y} \cdot \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial f}{\partial z} \cdot \frac{\partial^2 \varphi}{\partial y \partial x},$$

$$\frac{\partial B}{\partial y} = \frac{\partial \varphi}{\partial x} \cdot \frac{\partial^2 f}{\partial z \partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial \varphi}{\partial z} \cdot \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial x} \cdot \frac{\partial^2 \varphi}{\partial z \partial y},$$

$$\frac{\partial C}{\partial z} = \frac{\partial \varphi}{\partial y} \cdot \frac{\partial^2 f}{\partial x \partial z} + \frac{\partial f}{\partial x} \cdot \frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial \varphi}{\partial x} \cdot \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial f}{\partial y} \cdot \frac{\partial^2 \varphi}{\partial x \partial z}.$$

Quae expressiones additae sese mutuo destruunt, unde eruitur,

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0,$$

hoc est, si quantitatem ipsi dx proportionalem ipsius x respectu, quantitatem ipsi dy porportionalem ipsius y respectu, quantitatem ipsi dz proportionalem ipsius z respectu differentiamus, trium quantitatum differentiatione provenientium summa identice evanescere debet. Quae conditio prorsus analoga est ei, quae antecedentibus de duabus variabilibus tradita est atque e primis elementis constat. Antecedentia ad numerum variabilium quemcunque extendere licet, siquidem advocantur propositiones quas in *Diario Crell. Vol. XXIII.* de Determinantibus algebraicis et functionalibus tradidi et quarum per totam hanc Commentationem usum frequentissimum faciam. Habetur enim sequens

Lemma fundamentale.

„Sint A, A_1, A_2, ... A_n quantites quae in Determinante Functionali

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}$$

respective per $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, ... $\frac{\partial f}{\partial x_n}$ *multiplicatae reprehenduntur, erit*

$$\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} \cdots + \frac{\partial A_n}{\partial x_n} = 0."$$

Demonstratio.

Secundum definitionem quantitatum A, A_1 etc. fit

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots + \frac{\partial f_n}{\partial x_n} = \frac{\partial f}{\partial x} A + \frac{\partial f}{\partial x_1} A_1 + \frac{\partial f}{\partial x_2} A_2 \cdots + \frac{\partial f}{\partial x_n} A_n.$$

Unde Lemma demonstratu propositum sic quoque exhibere licet:

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = \frac{\partial . fA}{\partial x} + \frac{\partial . fA_1}{\partial x_1} + \frac{\partial . fA_2}{\partial x_2} \cdots + \frac{\partial . fA_n}{\partial x_n}.$$

Facio hanc formulam iam demonstratam esse pro $n-1$ functionibus n variabilium, probabo Lemma ad n functiones $n+1$ variabilium valere.

Designo per (i, k) quantitatem quae in Determinante Functionali $\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$ multiplicata reprehenditur per factorem

$$\frac{\partial f}{\partial x_i} \cdot \frac{\partial f_1}{\partial x_k}.$$

Constat autem per Determinantium proprietates iam olim ab ill. *Laplace* adnotatas, *bina Aggregata, in Determinante functionali proposito resp. per* $\frac{\partial f}{\partial x_i} \cdot \frac{\partial f_m}{\partial x_k}$ *et per* $\frac{\partial f}{\partial x_k} \cdot \frac{\partial f_m}{\partial x_i}$ *multiplicata, valoribus oppositis gaudere.* Unde sequitur

$$(i, k) = -(k, i) \quad \text{sive} \quad (i, k) + (k, i) = 0.$$

Est A_i complexus terminorum eius Determinantis qui per $\frac{\partial f}{\partial x_i}$ multiplicantur, unde fit

$$A_i = \frac{\partial f_1}{\partial x}(i, 0) + \frac{\partial f_1}{\partial x_1}(i, 1) + \frac{\partial f_1}{\partial x_2}(i, 2) \dots + \frac{\partial f_1}{\partial x_n}(i, n),$$

qua in formula ipsum (i, i) aut omittendum aut $= 0$ ponendum est. Est porro A_i Determinans functionum $f_1, f_2, \dots f_n$ formatum respectu variabilium $x, x_1, x_2, \dots x_{i-1}, x_{i+1}, \dots x_n$ atque sunt $(i, 0)$, $(i, 1)$ etc. quantitates quae in Determinante Functionali A_i multiplicatae reprehenduntur per $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_1}{\partial x_1}$ etc. Unde si Lemma propositum ad $n-1$ functiones n variabilium valet, erit pro indicis i valoribus $0, 1, 2, \dots n$,

$$\frac{\partial(i, 0)}{\partial x} + \frac{\partial(i, 1)}{\partial x_1} \dots + \frac{\partial(i, n)}{\partial x_n} = 0,$$

ideoque etiam

$$A_i = \frac{\partial . f_1(i, 0)}{\partial x} + \frac{\partial . f_1(i, 1)}{\partial x_1} \dots + \frac{\partial . f_1(i, n)}{\partial x_n}.$$

Quae formula pro quolibet ipsius i valore $0, 1, 2, \dots n$ valet. Iam generaliter observo, *quoties ponatur*

$$H_i = \frac{\partial . a_{i,0}}{\partial x} + \frac{\partial . a_{i,1}}{\partial x_1} \dots + \frac{\partial . a_{i,n}}{\partial x_n},$$

designantibus $a_{i,k}$ ***quantitates quascunque pro quibus sit***

$$a_{i,k} + a_{k,i} = 0, \quad a_{i,i} = 0,$$

fieri

$$\frac{\partial H}{\partial x} + \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} \dots + \frac{\partial H_n}{\partial x_n} = 0.$$

Bina enim differentialia inter se juncta,

$$\frac{\partial . \frac{\partial . a_{i,k}}{\partial x_k}}{\partial x_i} + \frac{\partial . \frac{\partial . a_{k,i}}{\partial x_i}}{\partial x_k},$$

mutuo destruuntur, unde totam expressionem $\frac{\partial H}{\partial x} + \frac{\partial H_1}{\partial x_1} \dots + \frac{\partial H_n}{\partial x_n}$ identice evanescere invenis. Ponendo autem $f_1.(i, k) = a_{i,k}$, satisfit conditioni $a_{i,k} = -a_{k,i}$, porro fit $H_i = A_i$; ideoque

$$\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} \dots + \frac{\partial A_n}{\partial x_n} = 0,$$

sive Lemma de n functionibus $n+1$ variabilium justum erit, dummodo de $n-1$ functionibus n variabilium locum habet. Unde tantum necesse est ut Lemma pro una functione duarum variabilium constet. Pro una autem functione f_1

duarum variabilium x et y abeunt quantitates A etc. in differentialia partialia $\frac{\partial f_1}{\partial y}$ et $-\frac{\partial f_1}{\partial x}$, ideoque Lemma redit in formulam

$$\frac{\partial . \frac{\partial f_1}{\partial x}}{\partial y} - \frac{\partial . \frac{\partial f_1}{\partial y}}{\partial x} = 0,$$

quae est differentialium partialium proprietas fundamentalis supra commemorata.

Lemma generale etiam directe demonstrari potest absque illa reductione numeri n ad numerum $n-1$. Nam cum A_i vacet differentialibus, ipsius x_i respectu sumtis, e quantitatibus $\frac{\partial A_i}{\partial x_i}$, nulla implicare potest differentialia bis secundum eandem variabilem sumta. Differentialia autem secunda, secundum variabiles diversas x_i et x_k sumta, non provenire possunt nisi e solis duobus terminis

$$\frac{\partial A_i}{\partial x_i} + \frac{\partial A_k}{\partial x_k}.$$

Unde ad probandum Lemma propositum sufficit ut demonstretur, in Aggregato $\frac{\partial A_i}{\partial x_i} + \frac{\partial A_k}{\partial x_k}$ se mutuo destruere terminos per quantitates $\frac{\partial^2 f_m}{\partial x_i \partial x_k}$ multiplicatos. Quod facile patet. Ponamus enim

$$A_i = \alpha_1 \frac{\partial f_1}{\partial x_k} + \alpha_2 \frac{\partial f_2}{\partial x_k} \dots + \alpha_n \frac{\partial f_n}{\partial x_k},$$

fit secundum Determinantium proprietatem, in priore demonstratione in usum vocatam,

$$A_k = -\left\{\alpha_1 \frac{\partial f_1}{\partial x_i} + \alpha_2 \frac{\partial f_2}{\partial x_i} \dots + \alpha_n \frac{\partial f_n}{\partial x_i}\right\}.$$

Quantitates α_1, α_2 etc. neque differentialibus secundum x_i sumtis, neque differentialibus secundum x_k sumtis afficiuntur. Unde substituendo ipsarum A_i et A_k expressiones antecedentes, de Aggregato

$$\frac{\partial A_i}{\partial x_i} + \frac{\partial A_k}{\partial x_k},$$

prorsus exulant differentialia secunda, secundum variabiles x_i et x_k sumta, terminis binis,

$$+ \alpha_m \frac{\partial^2 f_m}{\partial x_k \partial x_i} - \alpha_m \frac{\partial^2 f_m}{\partial x_i \partial x_k},$$

se mutuo destruentibus. Erant autem inter omnes terminos Aggregati propositi

$$\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} \dots + \frac{\partial A_n}{\partial x_n}$$

soli termini $\frac{\partial A_i}{\partial x_i} + \frac{\partial A_k}{\partial x_k}$ qui affici possint differentialibus $\frac{\partial^2 f_m}{\partial x_i \partial x_k}$, unde in Aggregato proposito termini differentialibus secundis secundum x_i et x_k sumtis affecti se mutuo destruunt. Unde cum x_i et x_k binae quaecunque variabiles esse possint a se diversae, illud Aggregatum totum evanescit. Q. d. e.

Quoties numerus variabilium, quas datae functiones $f_1, f_2, \dots f_n$ implicant, ipsum functionum numerum n superat, proponi potest, earum functionum Determinantia respectu quarumque n variabilium formare. Quae vocabo functionum $f_1, f_2, \dots f_n$ ***Determinantia Partialia*** secundum analogiam denominationis de differentialibus usitatae.

Si numerus variabilium est $n+1$ sicuti antecedentibus, erit numerus Determinantium Functionalium Partialium $n+1$; si numerus variabilium est $n+2$, dabuntur $\frac{1}{2}(n+2)(n+1)$ Determinantia Functionalia Partialia, et ita porro. Eorum Determinantium Functionalium Partialium signa cum in arbitrio posita sint, casu quo variabilium numerus numerum functionum tantum unitate superat, supponam, signa omnium Determinantium ab eorum uno ita pendere, ut binorum Determinantium partialium alterum de altero deducatur, in signis differentialibus binarum variabilium independentium commutatione facta, omnium simul terminorum mutatis signis. Quem invenis esse habitum quantitatum $A, A_1, \dots \dots A_n$, quae sunt functionum $f_1, f_2, \dots f_n$ Determinantia partialia. Videlicet de uno

$$A = \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n},$$

deducitur $-A_i$, loco ipsorum

$$\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \dots \frac{\partial f_n}{\partial x_i}$$

respective scribendo

$$\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial x}, \dots \frac{\partial f_n}{\partial x}.$$

Pro una duarum variabilium x et y functione f_1 abibunt Determinantia partialia in differentialia partialia functionis f_1, alterum positivo alterum negativo signo sumtum,

$$\frac{\partial f_1}{\partial y}, -\frac{\partial f_1}{\partial x} \quad \text{vel} \quad -\frac{\partial f_1}{\partial y}, \frac{\partial f_1}{\partial x}.$$

Et quemadmodum inter differentialia partialia $\frac{\partial f_1}{\partial x}$ et $\frac{\partial f_1}{\partial y}$, locum habet formula fundamentalis,

$$\frac{\partial . \frac{\partial f_1}{\partial y}}{\partial x} - \frac{\partial . \frac{\partial f_1}{\partial x}}{\partial y} = 0,$$

ita $n+1$ variabilium $x, x_1, x_2, \ldots x_n$ propositis n functionibus $f_1, f_2, \ldots f_n$ Lemmate antecedente constituitur inter Determinantia Partialia $A, A_1, A_2, \ldots \ldots A_n$ aequatio conditionalis fundamentalis,

$$\frac{\partial A}{\partial x}+\frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2}\ldots+\frac{\partial A_n}{\partial x_n}=0.$$

Quod igitur Lemma gravissimam manifestat analogiam Determinantium Functionalium et quotientium differentialium partialium.

Lemma traditum dedi olim in Commentatione, *Vol. VI. Diar. Crell. pg. 263 sqq. inserta, „De resolutione aequationum per series infinitas."* Quod eo loco adhibui ad demonstrandam Propositionem quae et ipsa luculentam analogiam Determinantium Functionalium cum differentialibus constituit. Nam cum pateat seriei e solis variabilis x potestatibus conflatae quotientem differentialem vacare termino $\frac{1}{x}$, demonstravi, *serierum $f, f_1, \ldots f_n$, conflatarum e solis variabilium $x, x_1, \ldots x_n$ potestatibus, Determinans Functionale*

$$\Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}$$

vacare termino $\frac{1}{x x_1 x_2 \ldots x_n}$. Quippe Determinans antecedens per Lemma nostrum aequatur quantitati

$$\frac{\partial . fA}{\partial x}+\frac{\partial . fA_1}{\partial x_1}\ldots+\frac{\partial . fA_n}{\partial x_n},$$

cuius terminus primus evolutus vacare debet termino in $\frac{1}{x}$ ducto, secundus termino in $\frac{1}{x_1}$ ducto, et ita porro, ita ut in tota quantitate evoluta non obvenire possit terminus $\frac{1}{x x_1 x_2 \ldots x_n}$.

Quae propositio adhiberi potest ad amplificandum theoriam *Cauchyanam* residuorum dictam, eiusque ope radices systematis simultanei aequationum in series infinitas evolvi, quod in Commentatione citata videas.

Data occasione breviter adhuc innuam usum Lemmatis propositi in integralibus multiplicibus inter datos limites determinandis. Proponatur integrale multiplex,

$$\int U df\, df_1 \ldots df_n,$$

ponamusque limites, inter quos integratio afficienda sit, eo definiri, quod introducendo certas alias variabiles $x, x_1, \ldots x_n$ pro variabilibus independentibus, harum novarum variabilium limites a se invicem independentes sive constantes sint. Constat, novis variabilibus exhibitum integrale propositum fore,

$$\int U df\, df_1 \ldots\ldots df_n = \int U \Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \ldots\ldots \frac{\partial f_n}{\partial x_n}\, dx\, dx_1 \ldots\ldots dx_n.$$

Variabilibus propositis $f, f_1, \ldots\ldots f_n$ expressa U integrataque ipsius f respectu, prodeat Π ita ut sit

$$\Pi = \int U \partial f, \quad U = \frac{\partial \Pi}{\partial f},$$

erit

$$U \Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \ldots\ldots \frac{\partial f_n}{\partial x_n} = \Sigma \pm \frac{\partial \Pi}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \ldots\ldots \frac{\partial f_n}{\partial x_n}.$$

Quod patet substituendo valores

$$\frac{\partial \Pi}{\partial x_i} = \frac{\partial \Pi}{\partial f} \cdot \frac{\partial f}{\partial x_i} + \frac{\partial \Pi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_i} \ldots\ldots + \frac{\partial \Pi}{\partial f_n} \cdot \frac{\partial f_n}{\partial x_i},$$

et observando, post substitutionem factam evanescere quantitates omnes in

$$\frac{\partial \Pi}{\partial f_1}, \frac{\partial \Pi}{\partial f_2}, \ldots\ldots \frac{\partial \Pi}{\partial f_n}$$

ductas. Fit autem e Lemmate proposito

$$\Sigma \pm \frac{\partial \Pi}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \ldots\ldots \frac{\partial f_n}{\partial x_n} = \frac{\partial . \Pi A}{\partial x} + \frac{\partial . \Pi A_1}{\partial x_1} \ldots\ldots + \frac{\partial . \Pi A_n}{\partial x_n}.$$

Unde eruitur formula reductionis

$$\int U df\, df_1 \ldots\ldots df_n =$$
$$\int (\Pi A)\, dx_1 dx_2 \ldots\ldots dx_n + \int (\Pi A_1)\, dx\, dx_2 \ldots\ldots dx_n \ldots\ldots + \int (\Pi A_n)\, dx\, dx_1 \ldots\ldots dx_{n-1};$$

Hic signo (ΠA_i) denoto, in functionibus $f, f_1, \ldots\ldots f_n$ ipsi x_i substituendos esse binos eius limites constantes, binasque expressiones ipsius ΠA_i provenientes alteram de altera detrahendas esse. Hinc integrale $n+1$tuplex propositum videmus revocari ad $2n+2$ integralia $n$tuplicia. Quae singula eadem quidem formula exhiberi possunt

$$\int \Pi df_1 df_2 \ldots\ldots df_n \text{ *)},$$

sed pro singulis erit Π diversa ipsarum $f_1, f_2, \ldots\ldots f_n$ functio, limitesque ipsarum $f_1, f_2, \ldots\ldots f_n$ diversi erunt. Singula deinde integralia $n$tuplicia eadem methodo ad $2n$ integralia $(n-1)$tuplicia revocari possunt, eaque ratione pergere licet, usque dum tota integratio inter limites propositos perfecta sit.

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*) Habendo enim x pro Constante, fit

$$\int \Pi A\, dx_1 dx_2 \ldots\ldots d_n = \int \Pi df_1 df_2 \ldots\ldots df_n,$$

cum sit

$$A = \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \ldots\ldots \frac{\partial f_n}{\partial x_n},$$

et similis formula pro reliquis integralibus valet.

Lemma traditum sub alia quoque forma proponi potest memoratu digna. Habeamus enim $x, x_1, \dots x_n$ pro ipsarum $f, f_1, \dots f_n$ functionibus, earumque quaeramus differentialia partialia, ipsius f respectu sumta. Quae per regulas notas inveniuntur,

$$\frac{\partial x}{\partial f} = \frac{A}{R}, \quad \frac{\partial x_1}{\partial f} = \frac{A_1}{R}, \quad \dots \quad \frac{\partial x_n}{\partial f} = \frac{A_n}{R},$$

siquidem R est Determinans propositum,

$$R = \Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \dots \frac{\partial f_n}{\partial x_n}.$$

Hinc formula nostra

$$\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} \dots + \frac{\partial A_n}{\partial x_n} = 0,$$

si repulamus esse

$$\frac{\partial R}{\partial f} = \frac{\partial R}{\partial x} \cdot \frac{\partial x}{\partial f} + \frac{\partial R}{\partial x_1} \cdot \frac{\partial x_1}{\partial f} \dots + \frac{\partial R}{\partial x_n} \cdot \frac{\partial x_n}{\partial f},$$

formam induit sequentem,

$$0 = \frac{\partial R}{\partial f} + R \left\{ \frac{\partial \cdot \frac{\partial x}{\partial f}}{\partial x} + \frac{\partial \cdot \frac{\partial x_1}{\partial f}}{\partial x_1} \dots + \frac{\partial \cdot \frac{\partial x_n}{\partial f}}{\partial x_n} \right\}$$

sive

$$0 = \frac{\partial \log R}{\partial f} + \frac{\partial \cdot \frac{\partial x}{\partial f}}{\partial x} + \frac{\partial \cdot \frac{\partial x_1}{\partial f}}{\partial x_1} \dots + \frac{\partial \cdot \frac{\partial x_n}{\partial f}}{\partial x_n}.$$

In his formulis supponitur, ipsas $R, x, x_1, \dots x_n$ primum pro quantitatum $f, f_1, \dots f_n$ functionibus haberi omnesque secundum f differentiari; deinde differentialia partialia $\frac{\partial x}{\partial f}$, $\frac{\partial x_1}{\partial f}$ etc. rursus per ipsas $x, x_1, \dots x_n$ exprimi, et respective secundum $x, x_1, \dots x_n$ differentiari. Commutando quantitates x, x_1 etc. cum quantitatibus f, f_1 etc. formula antecedens in aliam abit, quam in ***Diar. Crell.*** Vol. XXII. pag. 336 demonstravi.

Novi Multiplicatoris definitio. Aequatio differentialis partialis cui satisfacit. Variae formae quas Multiplicatoris valor induere potest.

§. 3.

Sint $X, X_1, \dots X_n$ variabilium $x, x_1, \dots x_n$ functiones quaecunque non simul omnes identice evanescentes; proposita aequatione differentiali partiali lineari primi ordinis,

$$0 = X\frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} \dots + X_n \frac{\partial f}{\partial x_n},$$

solutiones ejus exstant n a se invicem independentes. Quarum Determinantia partialia erunt inter se ut Coëfficientes aequationis differentialis partialis propositae X, X_1, ... X_n. Solutionibus enim illis a se independentibus vocatis

$$f_1,\ f_2,\ \dots\ f_n,$$

habentur aequationes identicae,

$$0 = X\frac{\partial f_1}{\partial x} + X_1\frac{\partial f_1}{\partial x_1} \dots + X_n\frac{\partial f_1}{\partial f_n},$$

$$0 = X\frac{\partial f_2}{\partial x} + X_1\frac{\partial f_2}{\partial x_1} \dots + X_n\frac{\partial f_2}{\partial x_n},$$

$$\dots\dots\dots\dots\dots$$

$$0 = X\frac{\partial f_n}{\partial x} + X_1\frac{\partial f_n}{\partial x_1} \dots + X_n\frac{\partial f_n}{\partial x_n},$$

quae sunt n aequationes lineares inter $n+1$ quantitates X, X_1, ... X_n, terminis carentes constantibus. Quibus aequationibus determinantur rationes quas ipsae X, X_1, etc. inter se tenent. Videlicet per regulas notas algebraicas invenitur, ipsas X, X_1, ... X_n esse inter se ut quantitates A, A_1, ... A_n, §. pr. consideratas, quae erant complexus terminorum, in Determinante functionali

$$\Sigma \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\dots\frac{\partial f_n}{\partial x_n}$$

respective per $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x_1}$, ... $\frac{\partial f}{\partial x_n}$ multiplicatorum, sive functionum f_1, f_2, ... f_n Determinantia partialia. Sit M factor per quem Coëfficientes X, X_1, ... X_n multiplicati ipsa producant Determinantia partialia A, A_1, ... A_n, ita ut fiat:

$$1.\quad MX = A,\quad MX_1 = A_1,\quad \dots\quad MX_n = A_n.$$

Posito

$$R = \Sigma \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\dots\frac{\partial f_n}{\partial x_n},$$

cum habeatur

$$R = A\frac{\partial f}{\partial x} + A_1\frac{\partial f}{\partial x_1} + \dots + A_n\frac{\partial f}{\partial x_n},$$

sequitur

$$2.\quad R = M\left(X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} \dots + X_n\frac{\partial f}{\partial x_n}\right).$$

Iisdem substitutis formulis (1.) Lemma §. pr. demonstratum in hanc formulam abit:

$$3.\quad 0 = \frac{\partial . MX}{\partial x} + \frac{\partial . MX_1}{\partial x_1} \dots + \frac{\partial . MX_n}{\partial x_n}.$$

Habemus igitur Propositionem sequentem, qua Multiplicatoris M continetur definitio.

Propositio.

„Proponatur expressio

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\dots+X_n\frac{\partial f}{\partial x_n},$$

in qua sint X, X_1, ... X_n datae variabilium x, x_1, ... x_n functiones: functionibus f_1, f_2, ... f_n rite determinatis, ipsa f autem indeterminata manente, semper exstabit factor M, per quem multiplicata expressio proposita formam induat Determinantis functionalis

$$M\left(X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\dots+X_n\frac{\partial f}{\partial x_n}\right)=\Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\dots\frac{\partial f_n}{\partial x_n},$$

isque Multiplicator satisfaciet aequationi differentiali partiali,

$$0=\frac{\partial.MX}{\partial x}+\frac{\partial.MX_1}{\partial x_1}\dots+\frac{\partial.MX_n}{\partial x_n}.“$$

E valoribus ipsius M in sequentibus perpetuo excludo valorem $M=0$. Quem patet satisfacere aequationi (2.), qua Multiplicator definitur, dummodo statuatur functionum f_1, f_2, ... f_n unam reliquarum functionem esse; constat enim Determinans Functionale evanescere si functiones propositae non a se invicem sint independentes. Illo autem ipsius M valore excluso, Propositio antecedens inverti potest. Videlicet *si Multiplicator M definitur conditione ut pro functione indefinita f expressio,*

$$M\left(X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\dots+X_n\frac{\partial f}{\partial x_n}\right)$$

evadat Determinans functionale,

$$R=\Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial x_1}\dots\frac{\partial f_n}{\partial x_n},$$

functiones f_1, f_2, ... f_n necessario erunt solutiones a se independentes aequationis differentialis partialis linearis,

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\dots+X_n\frac{\partial f}{\partial x_n}=0.$$

Nam pro ipsa f, quae erat functio indefinita, sumendo aliquam functionum f_1, f_2, ... f_n, identice evanescit Determinans R. Quod cum supponatur aequale expressioni,

$$M\left(X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\dots+X_n\frac{\partial f}{\partial x_n}\right),$$

atque factor M a nihilo diversus statuatur, fieri debet ut substituendo ipsi f functiones f_1, f_2, ... f_n identice habeatur,

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\dots+X_n\frac{\partial f}{\partial x_n}=0,$$

sive ut $f_1, f_2, \ldots. f_n$ ipsae sint aequationis differentialis partialis propositae solutiones. Eruntque solutiones illae $f_1, f_2, \ldots. f_n$ a se invicem independentes; si enim una reliquarum functio esset, Determinans R identice evanesceret pro functione f indefinita; unde etiam pro functione indefinita f evanescere deberet expressio

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\ldots.+X_n\frac{\partial f}{\partial x_n},$$

quod fieri non potest nisi omnes X, X_1 etc. simul identice evanescunt.

Datis functionibus $f_1, f_2, \ldots. f_n$ una quaelibet ex aequationum (1.) numero ad definiendum Multiplicatorem sufficit, veluti aequatio,

$$MX = A = \Sigma\pm\frac{\partial f_1}{\partial x_1}.\frac{\partial f_2}{\partial x_2}\ldots.\frac{\partial f_n}{\partial x_n},$$

e qua sequitur

$$4.\quad M = \frac{1}{X}\Sigma\pm\frac{\partial f_1}{\partial x_1}.\frac{\partial f_2}{\partial x_2}\ldots.\frac{\partial f_n}{\partial x_n}.$$

Qua tamen formula ut definiatur Multiplicator aequationis differentialis partialis propositae, addenda conditio est ut X et A non evanescant.

Pro duabus variabilibus x et x_1 Multiplicator antecedentibus definitus cum ***Euleriano*** convenit. Sint enim X, X_1 datae variabilium x et x_1 functiones, atque proponatur aequatio differentialis primi ordinis inter x et x_1,

$$Xdx_1 - X_1 dx = 0.$$

Est Multiplicator ***Eulerianus*** eiusmodi factor M per quem multiplicata pars laeva aequationis antecedentis abit in differentiale completum functionis alicuius f_1, ita ut sit

$$df_1 = \frac{\partial f_1}{\partial x}dx+\frac{\partial f_1}{\partial x_1}dx_1 = M(Xdx_1 - X_1 dx),$$

sive,

$$MX = \frac{\partial f_1}{\partial x_1},\qquad MX_1 = -\frac{\partial f_1}{\partial x}.$$

E quibus formulis sequitur, pro functione indefinita f induere expressionem,

$$M\left(X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\right),$$

formam Determinantis functionalis

$$\frac{\partial f}{\partial x}.\frac{\partial f_1}{\partial x_1}-\frac{\partial f}{\partial x_1}.\frac{\partial f_1}{\partial x},$$

et Multiplicatorem M satisfacere aequationi differentiali partiali,

$$\frac{\partial.MX}{\partial x}+\frac{\partial.MX_1}{\partial x_1} = 0.$$

Quae pro duobus variabilibus independentibus sunt eaedem proprietates characteristicae, quae Multiplicatori generali assignavi.

Problema solvendi aequationem differentialem partialem propositam,

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\ldots\ldots+X_n\frac{\partial f}{\partial x_n}=0,$$

cum duobus aliis problematis arctissime coniunctum est. Designante enim Π quamcunque aequationis praecedentis solutionem, ex aequatione

$$\Pi=0,$$

petatur ipsius x expressio per reliquas variabiles x_1, x_2, ... x_n: notum est eam fieri solutionem alterius aequationis differentialis partialis,

$$X=X_1\frac{\partial x}{\partial x_1}+X_2\frac{\partial x}{\partial x_2}\ldots\ldots+X_n\frac{\partial x}{\partial x_n}.$$

Unde haec aequatio differentialis partialis ad aequationem differentialem partialem propositam revocari potest. Porro ad aequationis differentialis partialis propositae solutionem constat revocari posse integrationem completam systematis aequationum differentialium vulgarium primi ordinis inter $n+1$ variabiles x, x_1, ... x_n, quod repraesentemus proportionibus,

$$dx:dx_1\ldots\ldots:dx_n=X:X_1:\ldots\ldots:X_n.$$

Videlicet si aequationis differentialis partialis propositae solutiones, a se independentes, sunt f_1, f_2, ... f_n, obtinentur aequationes, quibus illud aequationum differentialium vulgarium systema complete integratur, aequando solutiones illas Constantibus Arbitrariis. Et vice versa, si ex aequationibus integralibus completis petuntur variabilium functiones Constantibus Arbitrariis a se independentibus aequales, ab iisdemque Constantibus Arbitrariis ipsae vacuae: hae functiones erunt aequationis differentialis partialis propositae solutiones a se independentes. Propter hunc trium problematum consensum Multiplicatorem M ad tria illa problemata perinde refero. Qua de re *ipsum M perinde appellabo Multiplicatorem huius aequationis differentialis partialis,*

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\ldots\ldots+X_n\frac{\partial f}{\partial x_n}=0,$$

vel huius,

$$0=X-X_1\frac{\partial x}{\partial x_1}-X_2\frac{\partial x}{\partial x_2}\ldots\ldots-X_n\frac{\partial x}{\partial x_n},$$

vel etiam systematis aequationum differentialium vulgarium,

$$dx:dx_1:dx_2\ldots\ldots:dx_n=X:X_1:X_2\ldots\ldots:X_n.$$

Ubi ad has refertur Multiplicator, quod plerumque usu venit, pro variis formis, quibus earum aequationes integrales completae proponuntur, variae obtinentur Multiplicatoris repraesentationes. Quas sequentibus exponam.

Si aequationes integrales proponuntur ipsa forma cuius modo mentionem iniecimus,

$$5.\quad f_1 = \alpha_1,\quad f_2 = \alpha_2,\quad \ldots\ldots\quad f_n = \alpha_n,$$

designantibus α_1 etc. Constantes Arbitrarias, functiones f_1 etc. non afficientes, ideoque $f_1, f_2, \ldots\ldots f_n$ solutiones a se independentes aequationis,

$$X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1}\ldots\ldots + X_n\frac{\partial f}{\partial x_n} = 0,$$

erat Multiplicator,

$$6.\quad M = \frac{1}{X}\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\ldots\ldots\frac{\partial f_n}{\partial x_n}.$$

Iam vero proponantur aequationes integrales completae hac forma maxime usitata, ut variabiles omnes per earum unam veluti x, et Constantes Arbitrarias exprimantur,

$$7.\quad x_1 = \varphi_1(x),\quad x_2 = \varphi_2(x),\quad \ldots\ldots\quad x_n = \varphi_n(x),$$

functionibus φ_1, φ_2 etc. involventibus praeter variabilem x Constantes Arbitrarias α_1 etc., erit

$$8.\quad \Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\ldots\ldots\frac{\partial f_n}{\partial x_n} = \frac{1}{\Sigma\pm\frac{\partial \varphi_1}{\partial \alpha_1}\cdot\frac{\partial \varphi_2}{\partial \alpha_2}\ldots\ldots\frac{\partial \varphi_n}{\partial \alpha_n}},$$

D. F. §. 9. (3.) *). Unde fit,

$$9.\quad M = \frac{1}{X\Sigma\pm\frac{\partial \varphi_1}{\partial \alpha_1}\cdot\frac{\partial \varphi_2}{\partial \alpha_2}\ldots\ldots\frac{\partial \varphi_n}{\partial \alpha_n}} = \frac{1}{X\Sigma\pm\frac{\partial x_1}{\partial \alpha_1}\cdot\frac{\partial x_2}{\partial \alpha_2}\ldots\ldots\frac{\partial x_n}{\partial \alpha_n}}.$$

Si vero generalius inter omnes $2n+1$ quantitates, $x, x_1, \ldots\ldots x_n$ $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$, proponuntur n aequationes integrales,

$$\Pi_1 = 0,\quad \Pi_2 = 0,\quad \ldots\ldots\quad \Pi_n = 0,$$

fit (D. F. §. 10. (5.)),

$$10.\quad \Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\ldots\ldots\frac{\partial f_n}{\partial x_n} = \frac{(-1)^n\Sigma\pm\frac{\partial \Pi_1}{\partial x_1}\cdot\frac{\partial \Pi_2}{\partial x_2}\ldots\ldots\frac{\partial \Pi_n}{\partial x_n}}{\Sigma\pm\frac{\partial \Pi_1}{\partial \alpha_1}\cdot\frac{\partial \Pi_2}{\partial \alpha_2}\ldots\ldots\frac{\partial \Pi_n}{\partial \alpha_n}}.$$

Unde obtinetur, rejecto quod licet signo ancipiti,

$$11.\quad M = \frac{1}{X}\cdot\frac{\Sigma\pm\frac{\partial \Pi_1}{\partial x_1}\cdot\frac{\partial \Pi_2}{\partial x_2}\ldots\ldots\frac{\partial \Pi_n}{\partial x_n}}{\Sigma\pm\frac{\partial \Pi_1}{\partial \alpha_1}\cdot\frac{\partial \Pi_2}{\partial \alpha_2}\ldots\ldots\frac{\partial \Pi_n}{\partial \alpha_n}},$$

quae est Multiplicatoris expressio maxime generalis.

Formula (10.) ope investigatio valoris Determinantis functionalis haud raro egregie expeditur. Transponamus ex. gr. Constantes Arbitrarias in alte-

*) Commentationem de Determinantibus Functionalibus Vol. XXII Diarii Crelliani insertam designabo per D. F.

ram partem aequationum (1.), atque pro quolibet ipsius i valore statuamus unctionem Π_i aequalem functioni $f_i - \alpha_i$, quocunque modo per aequationes,

$$f_{i+1} = \alpha_{i+1}, \quad f_{i+2} = \alpha_{i+2}, \quad \dots\dots \quad f_n = \alpha_n,$$

transformatae. Poterit in locum cuiusque aequationis $f_i = \alpha_i$ adhiberi aequatio $\Pi_i = 0$, unde systema aequationum sequentium,

$$\Pi_1 = 0, \quad \Pi_2 = 0, \quad \dots\dots \quad \Pi_n = 0,$$

haberi poterit pro aequationum integralium completarum systemate. Quae ita sunt comparatae aequationes, ut quaelibet functio Π_i non involvat quantitates $\alpha_1, \alpha_2, \dots\dots \alpha_{i-1}$, quantitatem α_i autem in unico termino addito $-\alpha_i$. Unde erit

$$\frac{\partial \Pi_i}{\partial \alpha_1} = \frac{\partial \Pi_i}{\partial \alpha_2} \dots\dots = \frac{\partial \Pi_i}{\partial \alpha_{i-1}} = 0, \quad \frac{\partial \Pi_i}{\partial \alpha_i} = -1,$$

sive quantitatibus $\frac{\partial \Pi_i}{\partial \alpha_k}$ in figuram quadratam dispositis hunc in modum,

$$\begin{array}{llll}
\frac{\partial \Pi_1}{\partial \alpha_1}, & \frac{\partial \Pi_1}{\partial \alpha_2}, & \dots\dots & \frac{\partial \Pi_1}{\partial \alpha_n}, \\
\frac{\partial \Pi_2}{\partial \alpha_1}, & \frac{\partial \Pi_2}{\partial \alpha_2}, & \dots\dots & \frac{\partial \Pi_2}{\partial \alpha_n}, \\
\dots & \dots & \dots & \dots \\
\frac{\partial \Pi_n}{\partial \alpha_1}, & \frac{\partial \Pi_n}{\partial \alpha_2}, & \dots\dots & \frac{\partial \Pi_n}{\partial \alpha_n},
\end{array}$$

quadratoque per diagonalem, a laeva ad dextram partem ductam, in duas partes diviso, termini in laeva parte positi omnes evanescunt. Quod ubi fit, abit Determinans in productum terminorum in ipsa diagonali positorum. Qui termini cum singuli fiant -1, eruitur

$$\Sigma \pm \frac{\partial \Pi_1}{\partial \alpha_1} \cdot \frac{\partial \Pi_2}{\partial \alpha_2} \dots\dots \frac{\partial \Pi_n}{\partial \alpha_n} = \pm 1,$$

ideoque

$$12. \quad XM = \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \dots\dots \frac{\partial f_n}{\partial x_n}$$
$$= \Sigma \pm \frac{\partial \Pi_1}{\partial x_1} \cdot \frac{\partial \Pi_2}{\partial x_2} \dots\dots \frac{\partial \Pi_n}{\partial x_n}.$$

Quae docet formula propositionem frequentissimae applicationis, ***valentibus aequationibus*** $f_1 = \alpha_1$, $f_2 = \alpha_2$, ... $f_n = \alpha_n$, ***Determinans functionale,***

$$\Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \dots\dots \frac{\partial f_n}{\partial x_n},$$

valorem non mutare, si ante differentiationes partiales transigendas quaeque functio f_i ***per aequationes***

$$f_{i+1} = \alpha_{i+1}, \quad f_{i+2} = \alpha_{i+2}, \quad \dots\dots \quad f_n = \alpha_n,$$

quascunque subeat mutationes. In hac propositione sunt $\alpha_1, \alpha_2, \ldots \alpha_n$ Constantes; quae si iunguntur functionibus $f_1, f_2, \ldots f_n$, ita ut ipsius $f_i - \alpha_i$ loco scribatur f_i, refertur propositio ad valorem quem induit Determinans functionale, functionibus ipsis evanescentibus. In applicatione huius propositionis facienda functiones $f_1, f_2, \ldots f_n$ sive aequationes, $f_1 = 0$, $f_2 = 0$, $\ldots f_n = 0$, certo disponendae sunt ordine tali, ut quaeque aequatio $f_i = 0$ insequentium ope formam induere possit concinnam, simulque differentialia partialia functionis f_i evadant simplicissima. Quin adeo eandem operationem indefinite repetere licet, siquidem post idoneas mutationes, pro certo functionum et aequationum ordine factas, eaedem functiones alio semperque alio ordine disponuntur et pro quaque nova dispositione mutationes vel eliminationes convenientes operantur. Quantascunque autem mutationes per varias istas dispositiones et eliminationes subire possunt functiones propositae f_1 etc., non tamen inde nascuntur functionum mutationis quae obtineri possunt, si ***eodem tempore*** ad unamquamque transformandam, nullo ordinis functionum respectu habito, omnes adhibentur n aequationes, quae reliquas omnes functiones nihilo aequando proveniunt. Nam in propositione tradita unica tantum erat e $n+1$ functionibus, ad quam transformandam adhiberi poterant n aequationes; praeter hanc una tantum erat ad quam transformandam $n-1$ aequationes adhiberi poterat, et ita porro. Functionibus in alium aliumque ordinem dispositis et pro quaque nova dispositione propositionis traditae applicatione facta, effici quidem potest ut unaquaeque functio sua vice adiumento n aequationum transmutetur; sed differentia in eo constituitur, quod hac ratione aequationes ad transmutationes adhibendae non amplius proveniant nihilo aequando functiones propositas sed functiones et ipsas iam transmutatas. Veluti si f per aequationem $f_1 = 0$ mutatur in φ, ac deinde f_1 per aequationem $\varphi = 0$ in φ_1: ipsa φ_1 non easdem induere potest formas, in quas mutari potest f_1 nihilo aequando ipsam functionem propositam f. Nam si valorem generalem functionis, in quam f per aequationem $f_1 = 0$ mutari potest, designamus quod licet per

$$\varphi = f + \lambda f_1,$$

atque similiter valorem generalem functionis, in quam f_1 per aequationem $\varphi = 0$ mutatur, per

$$\varphi_1 = f_1 + \mu\varphi = (1+\lambda\mu)f_1 + \mu f:$$

haec functio diversa erit a functione $f_1 + \mu f$, in quam f_1 per aequationem $f = 0$ mutatur. Atque Determinans functionum φ et φ_1 idem quidem erit atque functionum propositarum; functionum vero $f + \lambda f_1$, $f_1 + \mu f$ ab illo discrepabit, scilicet aequabitur Determinanti functionum f et f_1, per factorem $1+\lambda\mu$ multiplicato. Quod pluribus illustrare placuit, ut emendarem errorem quem in Commentatione *de Determinantibus functionalibus* commisi proponendo, Determinantis functionalis valorem quem induat ipsis functionibus evanescentibus, immutatum manere, si unaquaeque functio mutationes subeat, quascunque nihilo aequando reliquas omnes subire possit. Generaliter si ponitur

$$\varphi_i = \lambda^i f + \lambda_1^i f_1 \ldots + \lambda_n^i f_n,$$

demonstrabitur per Determinantium proprietates, valentibus aequationibus

$$f = 0, \quad f_1 = 0, \quad \ldots \quad f_n = 0,$$

fieri

$$\Sigma \pm \frac{\partial \varphi}{\partial x} \cdot \frac{\partial \varphi_1}{\partial x_1} \ldots \frac{\partial \varphi_n}{\partial x_n} = \Sigma + \lambda \lambda_1' \lambda_2'' \ldots \lambda_n^n . \Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \ldots \frac{\partial f_n}{\partial x_n}.$$

Unde ut Determinantia functionum $f, f_1, \ldots f_n$ et $\varphi, \varphi_1, \ldots \varphi_n$ inter se aequalia existant, habetur conditio generalis,

$$\Sigma \pm \lambda \lambda_1' \ldots \lambda_n^n = 1.$$

E Propositione supra tradita, identidem pro aliis aliisque functionum dispositionibus repetita, innumera deducuntur quantitatum λ_k^i systemata quae conditioni illi satisfaciunt.

Inter mutationes, quas functio variabilium x, x_1 etc. per aequationes inter easdem variabiles positas subire potest, referri potest eliminatio variabilium numeri numero aequationum aequalis. Unde in formula (12.) definire licet Π_i ut functionem variabilium $x, x_1, \ldots x_i$, in quam abeat $f_i - \alpha_i$, si ope aequationum $f_{i+1} = \alpha_{i+1}, f_{i+2} = \alpha_{i+2}, \ldots f_n = \alpha_n$ variabiles $x_{i+1}, x_{i+2}, \ldots x_n$ eliminantur. Quo statuto, omnia evanescunt differentialia partialia $\frac{\partial \Pi_i}{\partial x_k}$, in quibus $k > i$; unde figura quadrata, quae a quantitatibus $\frac{\partial \Pi_i}{\partial x_k}$ formatur, ita comparata erit, ut in ea per diagonalem divisa, rursus termini in altera parte positi evanescant, ideoque fiat,

$$\Sigma \pm \frac{\partial \Pi_1}{\partial x_1} \cdot \frac{\partial \Pi_2}{\partial x_2} \ldots \frac{\partial \Pi_n}{\partial x_n} = \frac{\partial \Pi_1}{\partial x_1} \cdot \frac{\partial \Pi_2}{\partial x_2} \ldots \frac{\partial \Pi_n}{\partial x_n}.$$

Hinc formula (12.) abit in hanc,

$$13. \quad XM = \frac{\partial \Pi_1}{\partial x_1} \cdot \frac{\partial \Pi_2}{\partial x_2} \ldots \frac{\partial \Pi_n}{\partial x_n},$$

sive Determinans functionale quo Multiplicator definitur in simplex productum redit. Forma autem aequationum integralium

$$\Pi_1 = 0, \quad \Pi_2 = 0, \ldots \Pi_n = 0,$$

quae illam simplicem Determinantis functionalis expressionem suppeditat, eadem est atque per integrationem *successivam* proveniens, post quodque Integrale inventum una variabilium eliminata. Servata enim functionum Π_1, Π_2, ... Π_n significatione antecedente, si eliminatur x_n per Integrale,

$$\Pi_n = f_n - \alpha_n = 0,$$

erit $\Pi_{n-1} = 0$ Integrale aequationum differentialium,

$$dx : dx_1 \ldots : dx_{n-1} = X : X_1 \ldots : X_{n-1},$$

cuius Integralis ope eliminata x_{n-1} erit $\Pi_{n-2} = 0$ Integrale aequationum differentialium,

$$dx : dx_1 \ldots : dx_{n-2} = X : X_1 \ldots : X_{n-2},$$

et ita porro. Si e functione Π_i Constantes arbitrarias α_{i+1}, α_{i+2}, ... α_n, quas implicat, ope aequationum,

$$\Pi_{i+1} = 0, \quad \Pi_{i+2} = 0, \ldots \Pi_n = 0,$$

eliminamus, redit aequatio $\Pi_i = 0$ in aequationum differentialium propositarum Integrale $f_i - \alpha_i = 0$. Voco autem, ut in aliis Commentationibus, ***Integrale*** systematis aquationum differentialium vulgarium huiusmodi aequationem integralem, quae differentiata identica evadat per solas aequationes differentiales propositas, neque ipsa illa aequatione integrali neque ulla alia in auxilium advocata.

Multiplicatoris expressio generalis. Bini Multiplicatores suppeditant Integrale. Expressio generalis functionum quarum detur Determinans datum.

§. 4.

Iam varias quae de Multiplicatore nostro tradi possunt proprietates exponam. Ac primum inquiram quomodo uno cognito Multiplicatore eruantur alii innumeri, sive Multiplicatoris investigabo formam generalem. Sit M datus Multiplicator aequationis,

$$1. \quad X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} \ldots + X_n\frac{\partial f}{\partial x_n} = 0,$$

satisfacere debet M secundum §. pr. huiusmodi aequationi,

$$2. \quad MX = \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n},$$

designantibus f_1, f_2, ... f_n solutiones aequationis (1.) a se invicem indepentes. Sit μ alius quicunque Multiplicator, satisfaciens aequationi,

$$3. \quad \mu X = \Sigma \pm \frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdots \frac{\partial F_n}{\partial x_n},$$

designantibus F_1, F_2, ... F_n aliud systema solutionum eiusdem aequationis (1.) a se invicem independentium. Functiones F_1, F_2, etc. esse debent solarum $f_1, f_2, \ldots . f_n$ functiones; cognitis enim aequationis (1.) solutionibus n a se invicem independentibus, quaevis alia eiusdem aequationis solutio harum n solutionum functio est. Fit autem per formulam notam (D. F. §. 11. Prop. II.),

$$4. \quad \Sigma \pm \frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdots \frac{\partial F_n}{\partial x_n}$$
$$= \Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots \frac{\partial F_n}{\partial f_n} \cdot \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n},$$

siquidem habentur $F_1, F_2, \ldots . F_n$ in laeva formulae parte pro variabilium $x, x_1, \ldots . x_n$ functionibus, in dextra parte pro functionibus ipsarum $f_1, f_2, \ldots . f_n$. E (2.) — (4.) autem obtinetur haec formula,

$$5. \quad \mu = M \Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots \frac{\partial F_n}{\partial f_n}.$$

Unde sequitur vice versa, ipsarum $f_1, f_2, \ldots . f_n$ quibuscunque sumtis functionibus a se independentibus $F_1, F_2, \ldots . F_n$, Multiplicatorem M ductum in harum functionum Determinans,

$$\Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots \frac{\partial F_n}{\partial f_n},$$

alterum suppeditare Multiplicatorem μ. Quaecunque enim sint $F_1, F_2, \ldots . F_n$ ipsarum $f_1, f_2, \ldots . f_n$ functiones a se independentes, ex aequatiouibus (2.), (4.), (5.) sequitur formula (3.), in qua $F_1, F_2, \ldots . F_n$ erunt aequationis (1.) solutiones a se invicem independentes, unde secundum §. pr. tradito quantitas μ, formula (3.) determinata, aequationis (1.) erit Multiplicator.

Videmus ex antecedentibus, binorum quorumque Multiplicatorum Quotientem $\frac{\mu}{M}$ aequari functioni ipsarum $f_1, f_2, \ldots . f_n$, videlicet Determinanti ipsarum $F_1, F_2, \ldots . F_n$, pro functionibus quantitatum $f_1, f_2, \ldots . f_n$ habitarum, et vice versa, Multiplicatore M ducto in Determinans quarumcunque n functionum a se independentium quantitatum $f_1, f_2, \ldots . f_k$, alterum obtineri Multiplicatorem. Semper autem quantitatum $f_1, f_2, \ldots . f_n$ functiones $F_1, F_2, \ldots . F_n$ invenire licet, quarum Determinans sit earundem quantitatum data quaecunque functio. Unde non modo binorum Multiplicatorum M et μ Quotiens functioni aequatur ipsarum $f_1, f_1, \ldots . f_n$, sed etiam vice versa, Multiplicatore M in quamcunque functionem ipsarum $f_1, f_2, \ldots . f_n$ ducto, rursus prodit Multiplicator. Et eum ipsarum, $f_1, f_2, \ldots . f_n$, quaelibet functio aequationis (1.) solutio sit, neque aliae aequationis (1.) solutiones extare possint, nisi quae ipsarum $f_1, f_2, \ldots . f_n$ functiones sint, sequitur ex antecedentibus haec Propositio.

Propositio.

„Designante M Multiplicatorem aequationis differentialis partialis,

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\ldots\ldots+X_n\frac{\partial f}{\partial x_n}=0,$$

erit Multiplicatoris forma generalis,

$$\Pi M,$$

designante Π quamcunque aequationis propositae solutionem."

Cognita aequationis (1.) solutione Π ac designante α Constantem Arbitrariam, aequatione $\Pi=\alpha$ determinatur variabilium $x_1,\ x_2,\ \ldots.\ x_n$ functio x, satisfaciens aequationi differentiali partiali,

$$6.\quad 0=X-X_1\frac{\partial x}{\partial x_1}-X_2\frac{\partial x}{\partial x_2}\ldots\ldots-X_n\frac{\partial x}{\partial x_n},$$

nec non erit $\Pi=\alpha$ Integrale aequationum differentialium vulgarium simultanearum,

$$7.\quad dx:dx_1\ldots\ldots:dx_n=X:X_1\ldots\ldots:X_n.$$

Unde Propositio antecedens docet, *cognitis aequationis differentialis partialis* (6.) *vel aequationum* (7.) *differentialium vulgarium binis Multiplicatoribus M et M_1, non solo factore constante inter se diversis, aequationem*

$$\frac{M_1}{M}=\text{Const.}$$

fore aequationis differentialis partialis (6.) *solutionem vel systematis aequationum differentialium* (7.) *Integrale.*

Pluribus datis Multiplicatoribus $M,\ M_1,\ \ldots.\ M_k$, haec quoque quantitas,

$$MF\left(\frac{M_1}{M},\ \frac{M_2}{M},\ \ldots.\ \frac{M_k}{M}\right)$$

erit multiplicator. Designante enim F ipsarum $\frac{M_1}{M}$ etc. functionem arbitrariam, non tantum fractiones $\frac{M_1}{M},\ \frac{M_2}{M}$ etc., sed ipsa F quoque aequationis (1.) solutio fit. Unde etiam aequatione $F=0$ sive quod idem est *quacunque aequatione homogenea inter datos Multiplicatores posita determinatur aequationis* (6.) *solutio.* Nec non designantibus $\alpha_1,\ \alpha_2,\ \ldots.\ \alpha_n$ Constantes Arbitrarias, erunt

$$\frac{M_1}{M}=\alpha_1,\quad \frac{M_2}{M}=\alpha_2,\ \ldots.\ \frac{M_k}{M}=\alpha_k,$$

Integralia aequationum differentialium vulgarium (7.).

Restat, ut paucis exponam quomodo inveniantur fnnctiones quarum Determinans datae variabilium functioni aequetur, quod semper fieri posse supra innui. Immo videbimus idem innumeris modis succedere, videlicet functiones praeter unam omnes ex arbitrio sumi posse, una reliqua per solam Quadraturam determinata.

Designante Π datam quamcunque quantitatum $f_1, f_2, \ldots. f_n$ functionem, simplicissima habetur solutio aequationis,

$$8. \qquad \Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots\cdot \frac{\partial F_n}{\partial f_n} = \Pi,$$

ponendo,

$$F_2 = f_2, \quad F_3 = f_3, \quad \ldots. \quad F_n = f_n,$$

unde Determinans propositum in simplex differentiale abit,

$$\frac{\partial F_1}{\partial f_1} = \Pi.$$

Quo igitur casu fit,

$$F_1 = \int \Pi df_1,$$

cui integrali functionem ipsarum $f_2, f_3, \ldots. f_n$ arbitrariam addere licet, quippe quae inter integrationem pro Constantibus habentur. Aequationis (8.) solutio generalis obtinetur sequenti modo. Pro ipsis $F_2, F_3, \ldots. F_n$ ex arbitrio sumantur ipsarum $f_1, f_2, \ldots. f_n$ functiones a se independentes, atque fingatur, reliquam functionem F_1 exhiberi per quantitates,

$$f_1, \quad F_2, \quad F_3, \quad \ldots. \quad F_n.$$

Functionis F_1 hoc modo repraesentatae differentialia partialia uncis includam, quo distinguantur a differentialibus eiusdem functionis per $f_1, f_2, \ldots. f_n$ exhibitae, ita ut sit,

$$\frac{\partial F_1}{\partial f_1} = \left(\frac{\partial F_1}{\partial f_1}\right) + \left(\frac{\partial F_1}{\partial F_2}\right)\frac{\partial F_2}{\partial f_1} + \left(\frac{\partial F_1}{\partial F_3}\right)\frac{\partial F_3}{\partial f_1} \cdots + \left(\frac{\partial F_1}{\partial F_n}\right)\frac{\partial F_n}{\partial f_1},$$

et quoties index i ab unitate diversus est,

$$\frac{\partial F_1}{\partial f_i} = \left(\frac{\partial F_1}{\partial F_2}\right)\frac{\partial F_2}{\partial f_i} + \left(\frac{\partial F_1}{\partial f_3}\right)\frac{\partial F_3}{\partial f_i} \cdots + \left(\frac{\partial F_1}{\partial F_n}\right)\frac{\partial F_n}{\partial f_i}.$$

Quae ipsarum

$$\frac{\partial F_1}{df_1}, \quad \frac{\partial F_1}{\partial f_2}, \quad \ldots. \quad \frac{\partial F_1}{\partial f_n}$$

expressiones si substituuntur in Determinante,

$$\Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots\cdot \frac{\partial F_n}{\partial f_n},$$

identice evanescunt singula aggregata per singula differentialia partialia

$$\left(\frac{\partial F_1}{\partial F_2}\right), \quad \left(\frac{\partial F_1}{\partial F_3}\right), \quad \ldots. \quad \left(\frac{\partial F_1}{\partial F_n}\right)$$

multiplicatae, unde simplex formula obtinetur,

$$9.\quad \Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots \frac{\partial F_n}{\partial f_n} = \left(\frac{\partial F_1}{\partial f_1}\right) \Sigma \pm \frac{\partial F_2}{\partial f_2} \cdot \frac{\partial F_3}{\partial f_3} \cdots \frac{\partial F_n}{\partial f_n}$$

(D. F. §. 12. (4.)). E (8. et 9.) sequitur

$$\left(\frac{\partial F_1}{\partial f_1}\right) = \frac{\Pi}{\Sigma \pm \frac{\partial F_2}{\partial f_2} \cdot \frac{\partial F_3}{\partial f_3} \cdots \frac{\partial F_n}{\partial f_n}},$$

qua formula exprimendo $f_2, f_3, \ldots f_n$ per $f_1, F_2, F_3, \ldots F_n$, sic quoque exhiberi potest,

$$10.\quad \left(\frac{\partial F_1}{\partial f_1}\right) = \Pi \Sigma \pm \frac{\partial f_2}{\partial F_2} \cdot \frac{\partial f_3}{\partial F_3} \cdots \frac{\partial f_n}{\partial F_n}$$

(D. F. §. 9. (3.)). Secundum hanc formulam, ut modo maxime generali variabilium $f_1, f_2, \ldots f_n$ inveniantur functiones, quarum Determinans datae earundem variabilium functioni Π aequatur, ex arbitrio exprimantur $f_2, f_3, \ldots f_n$ per f_1 aliasque $n-1$ quantitates $F_2, F_3, \ldots F_n$, determinataque F_1 per formulam,

$$11.\quad F_1 = \int \Pi \Sigma \pm \frac{\partial f_2}{\partial F_2} \cdot \frac{\partial f_3}{\partial F_3} \cdots \frac{\partial f_n}{\partial F_n}\, \partial f_1,$$

ipsae $F_1, F_2, \ldots F_n$, vice versa per $f_1, f_2, \ldots f_n$ expressae erunt functiones quaesitae.

Ponendo $\Pi = 1$ antecedentibus innumera obtinentur systemata functionum quantitatum $f_1, f_2, \ldots f_n$, quarum Determinans unitati aequatur. Quibus omnibus idem respondet Multiplicator. Quoties enim

$$\Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots \frac{\partial F_n}{\partial f_n} = 1,$$

sequitur e (5.)

$$\mu = M.$$

Vice versa, si idem Multiplicator respondet binis systematis n solutionum a se independentium aequationis differentialis partialis (1.), $f_1, f_2, \ldots f_n$ atque $F_1, F_2, \ldots F_n$, ita ut sit,

$$MX = \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}$$
$$= \Sigma \pm \frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdots \frac{\partial F_n}{\partial x_n}:$$

erunt $F_1, F_2, \ldots F_n$ quantitatum $f_1, f_2, \ldots f_n$ functiones, quarum Determinans unitati aequatur.

Multiplicatoris definitio per aequationem differentialem partialem. Conditio, ut Multiplicator aequari possit unitati.

§. 5.

Vidimus §. 3. aequationis differentialis partialis,

$$1.\quad X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\dots+X_n\frac{\partial f}{\partial x_n}=0,$$

Multiplicatorem quemcunque M alii satisfacere aequationi differentiali partiali,

$$2.\quad \frac{\partial.MX}{\partial x}+\frac{\partial.MX_1}{\partial x_1}\dots+\frac{\partial.MX_n}{\partial x_n}.$$

Vice versa *quaecunque habetur solutio* μ *aequationis differentialis partialis,*

$$3.\quad \frac{\partial.\mu X}{\partial x}+\frac{\partial.\mu X_1}{\partial x_1}\dots+\frac{\partial.\mu X_n}{\partial x_n}=0,$$

erit illa aequationis (1.) *Multiplicator.*

Ponamus enim $\mu=\Pi.M$, abit aequatio (3.) in sequentem,

$$0=\Pi\left(\frac{\partial.MX}{\partial x}+\frac{\partial.MX_1}{\partial x_1}\dots+\frac{\partial.MX_n}{\partial x_n}\right)$$
$$+M\left(X\frac{\partial\Pi}{\partial x}+X_1\frac{\partial\Pi}{\partial x_1}\dots+X_n\frac{\partial\Pi}{\partial x_n}\right).$$

Partis dextrae Aggregatum in Π ductum secundum (2.) evanescit; unde, cum supponamus ipsum M non evanescere, sequitur,

$$0=X\frac{\partial\Pi}{\partial x}+X_1\frac{\partial\Pi}{\partial x_1}\dots+X_n\frac{\partial\Pi}{\partial x_n}.$$

Erit igitur Π aequationis (1.) solutio ideoque secundum Propositionem §. pr. traditam, Multiplicatorem in solutionem aequationis (1.) quamcunque ductum reproducere Multiplicatorem, erit $\Pi.M=\mu$ Multiplicator, q. d. e.

Cum quilibet Multiplicator sit solutio aequationis (3.) et secundum antecedentia quaelibet aequationis (3.) solutio sit Multiplicator, poterit aequatio (3.) adhiberi ad Multiplicatorem definiendum. Habemus igitur Propositionem sequentem.

Propositio I.

„Designante M solutionem quamcunque aequationis differentialis partialis,

$$\frac{\partial.MX}{\partial x}+\frac{\partial.MX_1}{\partial x_1}\dots+\frac{\partial.MX_n}{\partial x_n}=0,$$

semper dantur functiones f_1, f_2, ... f_n, quae pro functione f indefinita efficiant aequationem,

$$M\left(X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\dots+X_n\frac{\partial f}{\partial x_n}\right)=\Sigma\pm\frac{\partial f}{\partial x}.\frac{\partial f_1}{\partial x_1}\dots\frac{\partial f_n}{\partial x_n}.“$$

Videri possit parum lucri percipi e nova Multiplicatoris determinatione per aequationem differentialem partialem (3.). Aequationis (3.) enim solutio generalis non habetur nisi aequationis (1.) data sit solutio generalis sive eius innotescant n solutiones particulares a se invicem independentes. His autem cognitis habetur Multiplicator per formulam (2.) §. pr. At observo ad Multiplicatorem eruendum tantum nos indigere una aliqua solutione particulari aequationis (3.) et quamquam aequationis (3.) solutio generalis a solutione aequationis (1.) pendet et pro complicatiore habenda est, fieri tamen potest ut aequationis (3.) innotescat solutio particularis, dum aequationis (1.) solutiones adhuc omnes ignoramus.

Inter solutiones aequationis differentialis partialis (1.) non referri solet, quae sponte se offert, $f =$ Const. Sed e solutionibus aequationis (3.) quae Multiplicatorem suggerunt quantitates constantes non excluduntur. Fit autem Multiplicator Constanti vel si placet unitati aequalis, si inter ipsas X, X_1 etc. locum habet aequatio,

$$4. \qquad \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} \dots + \frac{\partial X_n}{\partial x_n} = 0.$$

Eo casu ipsa expressio proposita,

$$X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} \dots + X_n\frac{\partial f}{\partial x_n},$$

pro functione f indefinita aequivalet alicui Determinanti functionali,

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n},$$

sive adhibendo notationes §. 3. usitatas statuere licet.

$$X = A, \quad X_1 = A_1, \quad \dots \quad X_n = A_n.$$

Quod, si ea tenes quae §. 2. de Determinantibus functionalibus partialibus monui, sic quoque proponi potest.

Propositio II.

„Si $n+1$ variabilium $x, x_1, \dots x_n$ functiones $X, X_1, \dots X_n$ satisfaciant conditioni,

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \dots + \frac{\partial X_n}{\partial x_n} = 0,$$

ipsae $n+1$ quantitates $X, X_1, \dots X_n$ haberi possunt pro certarum n functionum Determinantibus partialibus."

Haec Propositio analoga est notae elementari, si variabilium x et y functiones X et Y satisfaciant conditioni, $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0$, ipsas Y et $-X$

respective haberi posse pro eiusdem functionis differentialibus partialibus, variabilium x et y respectu sumtis.

Si inter quantitates X, X_1 etc. conditio (4.) locum habet, aequatio differentialis partialis (3.), qua Multiplicator definitur, in ipsam (1.) redit. Eo igitur casu quaecunque aequationis (1.) solutio eiusdem aequationis Multiplicator erit, siquidem iam unitatem vel numeros constantes inter solutiones referimus. Unde etiam patet, eo casu aequationum differentialium vulgarium,

$$dx : dx_1 \ldots\ldots : dx_n = X : X_1 \ldots\ldots : X_n,$$

Multiplicatorem fore quantitatem quamcunque, aut per se constantem, aut quae per aequationes integrales completas Constanti aequetur.

Cognito systematis aequationum differentialium vulgarium Multiplicatore quocunque eruuntur Determinantia functionum quae per aequationes integrales completas valoribus variabilium initialibus aequivalent.

§. 6.

Vidimus §. 3. designantibus f_1, f_2, ... f_n solutiones a se independentes aequationis,

$$1.\quad X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \ldots\ldots + X_n\frac{\partial f}{\partial x_n} = 0,$$

harum functionum Determinantia partialia A_1, A_2, ... A_n esse inter se ut aequationis (1.) Coëfficientes, sive fieri,

$$2.\quad A : A_1 \ldots\ldots : A_n = X : X_1 \ldots\ldots : X_n.$$

Unde omnia A_1, A_2, ... A_n uno determinantur A. Antecedentibus autem demonstravi, designante μ Multiplicatorem aequationis (1.) quemcunque sive quamcunque solutionem aequationis

$$3.\quad \frac{\partial . X\mu}{\partial x} + \frac{\partial . X_1\mu}{\partial x_1} \ldots\ldots + \frac{\partial . X_n\mu}{\partial x_n} = 0,$$

fieri $\mu = \Pi M$, ideoque

$$4.\quad \mu X = \Pi . A = \Pi . \Sigma \pm \frac{\partial f_1}{\partial x_1} . \frac{\partial f_2}{\partial x_2}, \ldots\ldots \frac{\partial f_n}{\partial x_n},$$

ubi Π certa quaedam est ipsarum f_1, f_2 .... f_n functio sive aequationis (1.) solutio. Hinc e data quacunque aequationis (3.) solutione μ cognoscitur valor Determinantis A, dummodo determinata erit functio Π. ***Eruitur autem functio Π, dummodo Determinantis A innotescat valor quem pro $x = 0$ induit.*** Generaliter enim, ut functio f aequationi differentiali partiali (1.) satisfaciens omnino determinata sit, poscitur et sufficit ut aliqua cognoscatur functio, cui illa aequalis evadat ubi inter variabiles x, x_1, ... x_n data aliqua aequatio locum habet, veluti si ipsius f datur valor quem pro $x=0$ induit. Hinc si ponimus pro $x=0$ abire μ, X, A in variabilium x_1, x_2, ... x_n functiones μ^0, X^0, A^0; functio Π eo determinabitur quod esse debeat aequationis (1.) solutio atque pro $x=0$ aequalis evadet variabilium x_1, x_2, ... x_n functioni

$$\frac{\mu^0 X^0}{A^0}.$$

Eiusmodi solutio autem ut inveniatur sint f_1^0, f_2^0, ... f_n^0 variabilium x_1, x_2, ... x_n functiones, in quas pro $x=0$ abeunt f_1, f_2, ... f_n; exprimatur porro variabilium x_1, x_2, ... x_n functio $\frac{\mu^0 X^0}{A^0}$ per f_1^0, f_2^0, ... f_n^0; in qua expressione ponendo ipsarum f_1^0, f_2^0, ... f_n^0 loco ipsas f_1, f_2, ... f_n, prodibit functio quaesita Π. Quippe functio sic inventa erit aequationis (1.) solutio et pro $x=0$ abibit in variabilium x_1, x_2, ... x_n functionem $\frac{\mu^0 X^0}{A^0}$.

Functionem A^0 casu prae ceteris notando a priori assignare licet, videlicet quoties f_1, f_2, ... f_n tales sunt aequationis (1.) solutiones *quae pro $x=0$ in ipsas variabiles x_1, x_2, ... x_n abeunt.* Tunc enim habetur

$$f_1^0 = x_1, \quad f_2^0 = x_2, \quad .... \quad f_n^0 = x_n,$$

ideoque

$$A^0 = \Sigma \pm \frac{\partial f_1^0}{\partial x_1} \cdot \frac{\partial f_2^0}{\partial x_2} \dots \frac{\partial f_n^0}{\partial x_n} = 1.$$

Hinc secundum regulam traditam functio Π e functione $\mu^0 X^0$ eruitur substituendo variabilibus x_1, x_2, ... x_n functiones f_1, f_2, ... f_n, sive quod idem est, substituendo in ipsa μX variabilibus x, x_1, x_2, ... x_n quantitates 0, f_1, f_2, ... f_n. Id quod sequentem suppeditat Propositionem.

Propositio I.

„Sint f_1, f_2, ... f_n solutiones aequationis

$$X\frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} \dots + X_n \frac{\partial f}{\partial x_n} = 0,$$

quae pro $x=0$ in ipsas variabiles x_1, x_2, ... x_n abeunt; sit μ quantitas quaecunque satisfaciens aequationi

$$\frac{\partial . X\mu}{\partial x} + \frac{\partial . X_1 \mu}{\partial x_1} \dots + \frac{\partial . X_n \mu}{\partial x_n} = 0,$$

atque sit Π ipsarum f_1, f_2, ... f_n functio quae e producto μX provenit substituendo variabilibus x, x_1, x_2, ... x_n quantitates 0, f_1, f_2, ... f_n: erit

$$\Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n} = \frac{\mu X}{\Pi},$$

sive generalius, designante f functionem indefinitam, erit

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = \frac{\mu}{\Pi} \left\{ X \frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} \cdots + X_n \frac{\partial f}{\partial x_n} \right\}."$$

Observo hac occasione generaliter, datis aequationis (1.) solutionibus f_1, f_2, ... f_n, quae pro $x = 0$ ipsas x_1, x_2, ... x_n abeant, quamvis aliam eiusdem aequationis solutionem Π per ipsas f_1, f_2, ... f_n absque omni eliminationis negotio exhiberi. Scilicet sufficit in functione Π variabilibus x, x_1, x_2, ... x_n substituere quantitates 0, f_1, f_2, ... f_n.

Casu speciali, quem sub finem §. pr. consideravi, posita insuper $X = 1$, e Propositione praecedente emergit haec:

Propositio II.

„Sint f_1, f_2, ... f_n tales solutiones aequationis,

$$\frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} \cdots + X_n \frac{\partial f}{\partial x_n} = 0,$$

quae pro $x = 0$ respective in x_1, x_2, ... x_n abeant, sitque identice,

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \cdots + \frac{\partial X_n}{\partial x_n} = 0,$$

erit,

$$\Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = 1,$$

atque reliqua functionum f_1, f_2, ... f_n Determinantia partialia A_1, A_2, ... A_n in ipsas redeunt quantitates X_1, X_2, ... X_n."

Convenit Propositiones antecedentibus inventas ad systemata aequationum differentialium vulgarium referre. Proponatur enim systema aequationum differentialium vulgarium,

$$dx : dx_1 : dx_2 \ldots : dx_n = X : X_1 : X_2 \ldots : X_n,$$

eiusque integratione completa facta, pro Constantibus Arbitrariis adhibeantur valores quos x_1, x_2, ... x_n pro $x = 0$ induunt; resolutione deinde aequationum integralium erui poterunt variabilium x, x_1, ... x_n functiones illis Constantibus Arbitrariis aequales, quae ipsae erunt functiones f_1, f_2, ... f_n, in Propp. I. et II. consideratae. Generaliter Integralia completa sint,

$$f_1 = \alpha_1, \quad f_2 = \alpha_2, \quad \ldots \quad f_n = \alpha_n,$$

designantibus α_1, α_2 etc. Constantes Arbitrarias quascunque, a quibus ipsae f_1, f_2 etc. vacuae supponuntur. Quorum Integralium ope expressis x_1, x_2, ... x_n per x et Constantes Arbitrarias α_1, α_2, ... α_n, fit secundum formulas de Determinantibus functionalibus traditas,

$$A = \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = \left\{ \Sigma \pm \frac{\partial x_1}{\partial \alpha_1} \cdot \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n} \right\}^{-1}.$$

Unde formula (4.) docet, cognito aequationum differentialium vulgarium propositarum Multiplicatore aliquo μ, sive aequationis (3.) solutione, fieri

$$\Sigma \pm \frac{\partial x_1}{\partial \alpha_1} \cdot \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n} = \frac{C}{\mu X},$$

designante C functionem Constantium Arbitrariarum. Quoties sunt $\alpha_1, \alpha_2, \ldots \alpha_n$ valores initiales variabilium $x_1, x_2, \ldots x_n$, ipsi $x = 0$ respondentes, Determinans functionale, in laeva parte aequationis antecedentis collocatum, ponendo $x = 0$ in *unitatem* abit. Quo igitur casu Constans C ex ipsa μX eruitur ponendo variabilium $x, x_1, x_2, \ldots x_n$ loco valores $0, \alpha_1, \alpha_2, \ldots \alpha_n$. Casu speciali quo Multiplicator unitatem aequat, e Propositione II. eruitur sequens prae ceteris simplex Propositio.

Propositio III.

„Proponantur aequationes differentiales vulgares simultaneae,

$$\frac{\partial x_1}{\partial x} = X_1, \quad \frac{\partial x_2}{\partial x} = X_2, \quad \ldots \quad \frac{\partial x_n}{\partial x} = X_n,$$

in quibus sint $X_1, X_2, \ldots X_n$ tales variabilium $x, x_1, x_2, \ldots x_n$, functiones quae satisfaciant aequationi,

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \cdots + \frac{\partial X_n}{\partial x_n} = 0;$$

integratione completa expressis $x_1, x_2, \ldots x_n$ per x earumque valores initiales $\alpha_1, \alpha_2, \ldots \alpha_n$, erit non tantum pro $x = 0$, sed pro valore ipsius x indefinito,

$$\Sigma \pm \frac{\partial x_1}{\partial \alpha_1} \cdot \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n} = 1."$$

Quae licet a proposito meo aliena utile videbatur obiter adnotare.

Quo rectius intelligantur quae supra monui de definienda solutione f aequationis differentialis partialis (1.), sequentia adiicio. Sit φ functio in quam abire debet f pro aequatione aliqua inter variabiles $x, x_1, \ldots x_n$ data. Si φ et ipsa aequationis (1.) solutio est, erit $f = \varphi$ functio quaesita, quaecunque sit illa aequatio. Si φ non est aequationis (1.) solutio, fieri non debet ut aequatio illa ad aliam inter quantitates $f_1, f_2, \ldots f_n$ revocari possit, sive ut ex aequatione illa peti possit solutio aequationis differentialis partialis,

$$X = X_1 \frac{\partial x}{\partial x_1} + X_2 \frac{\partial x}{\partial x_2} \cdots + X_n \frac{\partial x}{\partial x_n}.$$

Nisi forte eiusmodi solutio sit *singularis* seu non redeat in aequationem inter quantitates $f_1, f_2, \ldots f_n$, quo casu nihil impedit quo minus functio f definiatur ope valoris quem pro data illa aequatione induit. Infra autem videbimus pro aequationis differentialis partialis antecedentis solutione singulari fieri,

$$\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\dots+\frac{\partial X_n}{\partial x_n}=\infty,$$

ubi ipsae X, X_1 etc. cum a factoribus communibus tum a denominatoribus purgatae supponuntur. Ita non definiri poterit f ope valoris quem pro $x=0$ induit, ubi pro $x=0$ habetur $X=0$ nec simul $\frac{\partial X}{\partial x}=\infty$. Quod obiter observo.

Multiplicatoris definitio per aequationem differentialem vulgarem.

§. 7.

Multiplicatorem, quem antecedentibus per aequationem differentialem partialem definivi, etiam per formulam differentialem vulgarem definire licet. Quae nova forma aequationis praeceteris indagando Multiplicatori apta est.

Primum aequationem differentialem partialem, qua Multiplicator μ definitur, sic exhibeo,

$$1.\quad 0=X\frac{\partial\mu}{\partial x}+X_1\frac{\partial\mu}{\partial x_1}\dots+X_n\frac{\partial\mu}{\partial x_n}+\mu\left\{\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\dots+\frac{\partial X_n}{\partial x_n}\right\},$$

vel dividendo per μ,

$$2.\quad 0=X\frac{\partial\log\mu}{\partial x}+X_1\frac{\partial\log\mu}{\partial x_1}\dots+X_n\frac{\partial\log\mu}{\partial x_n}+\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\dots+\frac{\partial X_n}{\partial x_n}.$$

Per aequationes autem differentiales vulgares quarum μ est Multiplicator,

$$3.\quad dx:dx_1:dx_2\dots:dx_n=X:X_1:X_2\dots:X_n,$$

aequationem praecedentem brevius sic repraesentare licet,

$$4.\quad 0=X\frac{d\log\mu}{dx}+\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\dots+\frac{\partial X_n}{\partial x_n}.$$

Hinc poterit aequationum differentialium vulgarium (3.) Multiplicator μ definiri ut *functio quae solarum aequationum differentialium propositarum* (3.) *ope, nulla in auxilium vocata aequatione integrali, aequationi* (4.) *satisfaciat.* Quippe quod fieri non potest nisi μ *identice* satisfaciat aequationi (2.) qua Multiplicator definiebatur.

Sequitur ex antecedentibus, ad investigandum Multiplicatorem circumspiciendum esse, an aequationum differentialium (3.) ope contingat, expressioni

$$\left\{\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\dots+\frac{\partial X_n}{\partial x_n}\right\}\frac{dx}{X}$$

formam conciliare alicuius differentialis completi dU. Quippe hoc patrato fit e (4.) Multiplicator,

$$5.\quad \mu = e^{-\int\left(\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\cdots\cdot+\frac{\partial X_n}{\partial x_n}\right)\frac{dx}{X}} = e^{-U}.$$

Hanc indagandi Multiplicatoris methodum in aliis Commentationibus per varia exempla illustrabo, in quibus integrationem quae Multiplicatorem suggerit videbimus praestari posse, aequationum differentialium vulgarium propositarum nullo Integrali cognito. Esse tamen poterit formulae (4.) usus etiam si aequationes differentiales complete integratae sunt. Tum enim formula (4.) docet, formationi Determinantis functionalis, quam determinatio Multiplicatoris requirebat, substitui posse Quadraturam, minus interdum molestam. Etenim ope integrationis completae quantitas ipsi $\frac{d\log\mu}{dx}$ aequalis per solam x et Constantes Arbitrarias exhiberi potest, unde ipsum $\log\mu$ per Quadraturam obtines,

$$6.\quad \log\mu = -\int\frac{dx}{X}\left(\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\cdots\cdot+\frac{\partial X_n}{\partial x_n}\right).$$

Post integrationem factam substituendo Constantibus Arbitrariis variabilium x, x_1, x_2, ... x_n functiones aequivalentes, prodibit ipsius $\log\mu$ expressio, aequationi differentiali partiali (2.) satisfaciens.

Post aequationum (3.) integrationem completam expressis x_1, x_1, ... x_n per x et Constantes Arbitrarias α_1, α_2, ... α_n fit secundum §. pr.

$$7.\quad \log\Sigma\pm\frac{\partial x_1}{\partial\alpha_1}\cdot\frac{\partial x_2}{\partial\alpha_2}\cdots\cdot\frac{\partial x_n}{\partial\alpha_n} = \log\frac{C}{\mu X},$$

designante C Constantium Arbitrariarum functionem. Unde, omissa quod licet Constante, e formula (6.) eruitur

$$8.\quad \log\Sigma\pm\frac{\partial x_1}{\partial\alpha_1}\cdot\frac{\partial x_2}{\partial\alpha_2}\cdots\cdot\frac{\partial x_n}{\partial\alpha_n} = \log\frac{1}{X}+\int\frac{dx}{X}\left(\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\cdots\cdot+\frac{\partial X_n}{\partial x_n}\right).$$

Quae formula immutata manere debet, omnibus X, X_1, ... X_n per factorem quemcunque communem multiplicatis. Quod ut pateat observo, per aequationes differentiales vulgares propositas aequationem (4.) aucta symmetria sic proponi posse:

$$9.\quad 0 = d\log\mu+\frac{\partial\log X}{\partial x}dx+\frac{\partial\log X_1}{\partial x_1}dx_1\cdots\cdot+\frac{\partial\log X_n}{\partial x_n}dx_n.$$

Unde e formula (7.) eruitur:

$$\log\Sigma\pm\frac{\partial x_1}{\partial\alpha_1}\cdot\frac{\partial x_2}{\partial\alpha_2}\cdots\cdot\frac{\partial x_n}{\partial\alpha_n} = \log\frac{C}{\mu X}$$

$$= \log\frac{1}{X}+\int\left(\frac{\partial\log X}{\partial x}dx+\frac{\partial\log X_1}{\partial x_1}dx_1\cdots\cdot+\frac{\partial\log X_n}{\partial x_n}dx_n\right).$$

Si in hac formula simul omnes X, X_1 etc. in factorem communem ν ducuntur, augetur integrale quantitate,

$$\int\left(\frac{\partial \log \nu}{\partial x}\,dx + \frac{\partial \log \nu}{\partial x_1}\,dx_1 \ldots + \frac{\partial \log \nu}{\partial x_n}\,dx_n\right) = \int d\log \nu = \log \nu.$$

Eadem autem quantitate minuitur $\log\frac{1}{X}$, unde tota expressio immutata manet, q. d. e.

Si in formula (8.) ponimus $X = 1$, prodit Propositio sequens.

Propositio.

„Facta integratione completa aequationum differentialium vulgarium,

$$\frac{dx_1}{dx} = X_1, \quad \frac{dx_2}{dx} = X_2, \quad \ldots\ldots \quad \frac{dx_n}{dx} = X_n,$$

exhibeantur $x_1, x_2, \ldots\ldots x_n$ per x et Constantes Arbitrarias, $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$, erit,

$$\log \Sigma \pm \frac{\partial x_1}{\partial \alpha_1}\cdot\frac{\partial x_2}{\partial \alpha_2}\ldots\frac{\partial x_n}{\partial \alpha_n} = \int\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}\ldots + \frac{\partial X_n}{\partial x_u}\right)dx,$$

quantitate sub signo et ipsa per x et Constantes Arbitrarias expressa."

Si in Propositione antecedente ipsae $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ designant variabilium valores initiales, valori $x = 0$ respondentes, integrationem inde a valore $x = 0$ fieri oportet. Ope huius Propositionis vel formulae generalioris (8.) fieri potest ut Quadratura alias satis abscondita eruatur; sicuti vice versa si Quadratura in promtu est, valor inde eruitur Determinantis functionalis.

Propositio antecedens primum a cl. *Liouville* tradita est in Commentatione *„sur la variation des constantes arbitraires,"* ipsius Diario Mathematico (Vol. III. pg. 342) inserta. Eadem sequitur e formula iam supra citata D. F. §. 9. (1.), loco f, f_1 etc. scribendo $x_1, x_2, \ldots\ldots x_n$ atque x loco α, loco x_1, x_2 etc. autem $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$. Scilicet est ea consequentia lemmatis quod circa variationem logarithmi Determinantis loco citato dedi. Habeantur enim n systemata aequationum linearium inter n incognitas $u_1, u_2, \ldots\ldots u_n$, quae systemata iisdem gaudeant Coëfficientibus incognitarum et tantum terminis prorsus constantibus inter se discrepent, unde etiam omnibus idem erit Determinans. Denotentur in $k$to aequationum linearium systemate termini constantes, in altera parte aequationum positi, respective per variationes Coëfficientium quibus in singulis aequationibus incognita u_k afficitur, atque e primo systemate aequationum petatur valor ipsius u_1, e secundo valor ipsius u_2, et ita porro: omnium horum valorum summa aequivalebit variationi logarithmi Determinantis.

In signis: sit $(u_k)_k$ valor ipsius u_k petitus e systemate aequationum,

$$10.\quad \begin{cases} a'_1 u_1 + a'_2 u_2 \,\ldots\, + a'_n u_n = \delta a'_k, \\ a''_1 u_1 + a''_2 u_2 \,\ldots\, + a''_n u_n = \delta a''_k, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_1^{(n)} u_1 + a_2^{(n)} u_2 \ldots + a_n^{(n)} u_n = \delta a_k^{(n)}, \end{cases}$$

erit

$$11.\quad (u_1)_1 + (u_2)_2 \ldots + (u_n)_n = \delta \log \Sigma \pm a'_1 a''_2 \ldots a_n^{(n)}.$$

Faciamus iam, in aequationibus differentialibus

$$\frac{dx_1}{dx} = X_1, \quad \frac{dx_2}{dx} = X_2, \quad \ldots \quad \frac{dx_n}{dx} = X_n,$$

substitui variabilium $x_1, x_2, \ldots x_n$ valores per x et Constantes Arbitrarias $\alpha_1, \alpha_2, \ldots \alpha_n$ exhibitas, qua substitutione prodire debent aequationes identicae Quas si ipsarum α_i respectu differentiamus, obtinemus nn huiusmodi aequationes,

$$12.\quad d\frac{\partial x_k}{\partial \alpha_i} = \left\{\frac{\partial X_k}{\partial x_1}\cdot\frac{\partial x_1}{\partial \alpha_i} + \frac{\partial X_k}{\partial x_2}\cdot\frac{\partial x_2}{\partial \alpha_i} \ldots + \frac{\partial X_k}{\partial x_n}\cdot\frac{\partial x_n}{\partial \alpha_i}\right\} dx.$$

Ipsi i tribuendo valores $1, 2, \ldots n$, ex aequatione antecedente prodeunt n aequationes lineares, in quibus habentur pro incognitis quantitates $\frac{\partial X_k}{\partial x_1}dx$, $\frac{\partial X_k}{\partial x_2}dx$ etc. Prodeunt n eiusmodi systemata aequationum linearum tribuendo ipsi quoque k valores $1, 2, \ldots n$; in omnibusque illis aequationum linearium systematis incognitae iisdem gaudebunt Coëfficientibus. Hinc si in aequationibus (10.) ponimus

$$a_k^{(i)} = \frac{\partial x_k}{\partial \alpha_i},$$

atque variationibus substituimus differentialia, aequationes (10.) abeunt in aequationes (12.), unde eruitur

$$(u_k)_k = \frac{\partial X_k}{\partial x_k}dx.$$

Unde e (11.) sequitur

$$\left\{\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \ldots + \frac{\partial X_n}{\partial x_n}\right\} dx = d\log \Sigma \pm \frac{\partial x_1}{\partial \alpha_1}\cdot\frac{\partial x_2}{\partial \alpha_2} \ldots \frac{\partial x_n}{\partial \alpha_n}$$

qua formula integrata Propositio supra tradita obtinetur.

Aequationis $X - X_1 \frac{\partial x}{\partial x_1} - X_2 \frac{\partial x}{\partial x_2} \dots - X_n \frac{\partial x}{\partial x_n}$ pars laeva Multiplicatore suo efficitur Determinans functionale completum. Pro solutione singulari Multiplicator fit infinitus. Multiplicatorem nihilo aut infinito aequando obtinetur aequatio integralis.

§. 8.

Quemadmodum, proposito una plurium variabilium functione, destinguimus inter differentialia eius partialia, in quibus variabiles omnes pro independentibus habentur, et differentiale completum, in quo omnes ab earum una *indefinite* pendent, ita, propositis n functionibus $n+m$ variabilium, praeter earum Determinantia partialia, de quibus supra dixi, in quibus variabiles omnes pro independentibus habentur, in considerationem venire potest ***Determinans completum,*** quod formatur habendo numerum m variabilium pro reliquarum n functionibus *indefinitis*. Designantibus A et B ipsarum x et y functiones, aequationem differentialem,

$$A + B \frac{dy}{\partial x} = 0,$$

docuit ***Eulerus,*** semper in talem duci posse Multiplicatorem, ut altera aequationis pars evadat differentiale completum sive differentiale certae functionis variabilium x et y, in qua y pro functione ipsius x habetur *indefinita*. Similiter *aequatio differentialis partialis,*

$$1. \quad X - X_1 \frac{\partial x}{\partial x_1} - X_2 \frac{\partial x}{\partial x_2} \dots - X_n \frac{\partial x}{\partial x_n} = 0,$$

in qua X, X_1, ... X_n designant variabilium x, x_1, ... x_n functiones, semper in talem duci potest Multiplicatorem ut altera aequationis pars evadat Determinans functionale completum sive Determinans certarum n functionum variabilium x, x_1, x_2, ... x_n, in quibus habetur x pro variabilium x_1, x_2, ... x_n functione indefinita. Functio in aequationem (1.) ducenda ipse est aequationis (1.) ***Multiplicator*** supra appellatus et antecedentibus fusius explicatus. Unde nova nostri et ***Euleriani*** Multiplicatoris similitudo emergit novaque inter Determinantia functionalia et differentialia analogia.

Demonstratio Propositionis antecedentis sic patet. Designantibus rursus f_1, f_2, ... f_n solutiones a se independentes aequationis,

$$X \frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} \dots + X_n \frac{\partial f}{\partial x_n} = 0,$$

supra vidimus, semper dari Multiplicatorem M, in quem ductae ipsae X, X_1, ... X_n evadant functionum f_1, f_2, ... f_n Determinantia partialia, ita ut ponendo pro functione f indefinita,

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = A \frac{\partial f}{\partial x} + A_1 \frac{\partial f}{\partial x_1} \cdots + A_n \frac{\partial f}{\partial x_n},$$

identice sit,

$$MX = A, \quad MX_1 = A_1, \quad \ldots \quad MX_n = A_n.$$

Hinc eruitur

$$2. \quad M \left\{ X - X_1 \frac{\partial x}{\partial x_1} - X_2 \frac{\partial x}{\partial x_2} \cdots - X_n \frac{\partial x}{\partial x_n} \right\}$$
$$= A - A_1 \frac{\partial x}{\partial x_1} - A_2 \frac{\partial x}{\partial x_2} \cdots - A_n \frac{\partial x}{\partial x_n}.$$

At in Commentatione de Det. F. §. 17. (6.) demonstravi, siquidem in functionibus f_1, f_2, ... f_n habeatur x pro variabilium x_1, x_2, ... x_n functione indefinita, fieri,

$$3. \quad \Sigma \pm \left(\frac{\partial f_1}{\partial x_1}\right) \left(\frac{\partial f_2}{\partial x_2}\right) \cdots \left(\frac{\partial f_n}{\partial x_n}\right) = A - A_1 \frac{\partial x}{\partial x_1} - A_2 \frac{\partial x}{\partial x_2} \cdots - A_n \frac{\partial x}{\partial x_n}.$$

Qua in formula uncis innui haberi x pro reliquarum variabilium x_1, x_2, ... x_n functione. Scilicet in Determinante Functionali (3.) substituendo ipsorum $\left(\frac{\partial f_i}{\partial x_k}\right)$ expressiones

$$\left(\frac{\partial f_i}{\partial x_k}\right) = \frac{\partial f_i}{\partial x_k} + \frac{\partial f_i}{\partial x} \cdot \frac{\partial x}{\partial x_k},$$

mutuo destruuntur termini omnes, in quibus inter se multiplicata inveniuntur differentialia partialia $\frac{\partial x}{\partial x_1}$, $\frac{\partial x}{\partial x_2}$ etc., ita ut horum differentialium non nisi ipsa expressio *linearis* remaneat, quae dextram partem aequationis (3.) constituit. E (2.) et (3.) sequitur formula,

$$4. \quad M \left\{ X - X_1 \frac{\partial x}{\partial x_1} - X_2 \frac{\partial x}{\partial x_2} \cdots - X_n \frac{\partial x}{\partial x_n} \right\}$$
$$= \Sigma \pm \left(\frac{\partial f_1}{\partial x_1}\right) \left(\frac{\partial f_2}{\partial x_2}\right) \cdots \left(\frac{\partial f_n}{\partial x_n}\right).$$

Unde ducta aequatione (1.) in Multiplicatorem eius M, altera eius pars identice aequatur Determinanti functionum f_1, f_2, ... f_n, in quibus x pro variabilium x_1, x_2, ... x_n functione habetur indefinita. Q. d. e.

Formula (4.) methodum suppeditat, ut *Lagrangii* appellatione utar, syntheticam ad eruendam aequationis (1.) solutionem generalem. Nam secundum (4.) aequatio (1.) identice convenit cum sequente,

$$5. \quad \Sigma \pm \left(\frac{\partial f_1}{\partial x_1}\right) \left(\frac{\partial f_2}{\partial x_2}\right) \cdots \left(\frac{\partial f_n}{\partial x_n}\right) = 0.$$

Quoties autem f_1, f_2, ... f_n sunt variabilium x_1, x_2, ... x_n functiones earumque Determinans identice evanescit, semper et sine ulla exceptione inter functiones f_1, f_2, ... f_n aliqua locum habere debet aequatio, et vice versa, si qua inter functiones f_1, f_2, ... f_n locum habet aequatio, earum Determinans evanescit (D. F. §. 7.). Hinc docet formula (5.), ut ipsius x expressio per x_1, x_2, ... x_n sit aequationis (1.) solutio, sufficere et posci, post eius substitutionem ipsas f_1, f_2, ... f_n abire in tales variabilium x_1, x_2, ... x_n functiones, inter quas una quaecunque locum habeat aequatio. Unde vice versa dabitur solutio generalis petendo functionis quaesitae valorem ex aequatione arbitraria inter f_1, f_2, ... f_n posita,

$$\Pi(f_1, f_2, \dots f_n) = 0;$$

sive quod idem est, obtinetur aequationis (1.) solutio nihilo aequando solutionem quamcunque aequationis,

$$6. \quad X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} \dots + X_n\frac{\partial f}{\partial x_n} = 0,$$

Haec egregia methodus aequationem differentialem partialem (1.) ad (6.) revocandi cum ea convenit quam olim ill. *Lagrange* tradidit (Hist. Ac. Ber. ad a. 1779 pag. 154), ubi primum hanc quaestionem aggressus est. Quae prolixior quidem videri possit methodus quam aliae quibus ipse *Lagrange* aliique postea usi sunt; qua de re ipse auctor eam ad exemplum tantum trium variabilium applicuit. Sane supponendo aequationem inter x, x_1, ... x_n quaesitam certe unam involvere Constantem Arbitrariam α, eamque aequationem ipsius α respectu resolutam fieri $f = \alpha$, aequatio proposita (1.) extemplo ad (6.) reducitur. Sed eadem ratione omnes quoque inveniri solutiones a Constantibus Arbitrariis prorsus vacuas, non ita bene per alias methodos constat atque illam *Lagrangianam*. Scilicet aequatio identica (4.) docet, nullam dari exceptionem solutionis traditae, nisi forte exstet solutio pro qua Multiplicator M evadat infinitus. Quodsi igitur more consueto solutionem eiusmodi exceptionalem seu quae generali se subducit appellamus *singularem*, methodus hic tradita rigorose demonstrat, *si qua extet aequationis* (1.) *solutio singularis, semper eam reddere Multiplicatorem aequationis infinitum.* Quod novam nostri Multiplicatoris similitudinem cum *Euleriano* manifestat.

Loco aequationis differentialis partialis (1.) consideremus systema aequationum differentialium vulgarium cum ea connexum, atque systema aequationum integralium *singulare* appellemus quod e completo provenit tribuendo uni pluribusve Constantibus Arbitrariis valores particulares seu unam pluresve relationes inter Constantes Arbitrarias statuendo: quo facto ex antecedentibus haec eruitur

Propositio I.

„*Proponantur aequationes differentiales*

$$dx : dx_1 \ldots . : dx_n = X : X_1 \ldots . : X_n,$$

earumque extet systema aequationum integralium singulare, $n-1$ Constantes Arbitrarias involvens: eliminatis Constantibus Arbitrariis e n aequationibus integralibus, prodit aequatio quae Multiplicatorem systematis aequationum differentialium propositarum reddit infinitum."

Ut Propositio haec demonstretur, primum generaliter ponamus aequationes integrales datas $n-1$ Constantibus Arbitrariis affici. Quarum aequationum ubi $n-1$ resolvuntur Constantium Arbitrarium respectu, quod semper fieri posse suppono, harumque valores provenientes in $n$ta aequatione integrali substituuntur, obtinebitur aequatio a Constantibus Arbitrariis vacua. E qua petatur unius variabilium veluti x valor per reliquas variabiles x_1, x_2 etc. expressus, atque in differentiali eius,

$$dx = \frac{\partial x}{\partial x_1} dx_1 + \frac{\partial x}{\partial x_2} dx_2 \ldots . + \frac{\partial x}{\partial x_n} dx_n,$$

substituantur aequationes differentiales propositae,

$$7. \quad dx : dx_1 \ldots . : dx_n = X : X_1 \ldots . : X_n;$$

eruitur

$$X = \frac{\partial x}{\partial x_1} X_1 + \frac{\partial x}{\partial x_2} X_2 \ldots . + \frac{\partial x}{\partial x_n} X_n,$$

sive ille ipsius x valor suppeditabit aequationis differentialis partialis (1.) solutionem. Scilicet non fit ut aequatio antecedens ex aliis $n-1$ aequationibus integralibus datis fluat, quippe e quibus supponitur non deduci posse alteram aequationem a Constantibus Arbitrariis liberam. Eritque solutio illa aut particularis aut singularis, prout aequatio a Constantibus Arbitrariis libera, cuius ope ipsa x per reliquas variabiles exprimebatur, in aequationem inter quantitates $f_1, f_2, \ldots . f_n$ redit aut non redit. Iam demonstrabo, etiam systema aequationum integralium propositum iisdem casibus aut particulare aut singulare fore. Substituamus enim eum ipsius x valorem in $n-1$ aequationibus integralibus, quarum ope Constantes Arbitrariae eliminabantur, simulque in functionibus X_1, $X_2, \ldots . X_n$ aequationibus illis, ut $n-1$ Constantes Arbitrarias involventibus, *complete* integrantur aequationes differentiales

$$8. \quad dx_1 : dx_2 \ldots . : dx_n = X_1 : X_2 \ldots . : X_n.$$

Unde quibuscunque aequationibus integralibus, $n-1$ Constantes Arbitrarias involventibus, semper haec forma conciliari potest, ut earum una exhibeatur una variabilium x per reliquas variabiles x_1, x_2 etc., reliquae $n-1$ aequationes autem sint Integralia completa aequationum differentialium (8.), in quibus ille ipsius x valor in functionibus X_1, X_2, ... X_n substitutus est. Ponamus aequationem illam a Constantibus Arbitrariis vacuam, e qua valor ipsius x petitus est, redire in aequationem aliquam $F = 0$, designante F quantitatum f_1, f_2, ... f_n functionem. Designantibus F, F_1, ... F_{n-1} earundem f_1, f_2, ... f_n functiones a se invicem independentes, dabitur aequationum differentialium propositarum (7.) integratio completa per formulas

$$9. \quad F = \alpha, \quad F_1 = \alpha_1, \quad \dots . \quad F_{n-1} = \alpha_{n-1},$$

designantibus α, α_1 etc. Constantes Arbitrarias. Ex aequatione $F = \alpha$ petito ipsius x valore eoque in functionibus F_1, F_2, ... F_{n-1}, X_1, X_2, ... X_n substituto, evadunt

$$F_1 = \alpha_1, \quad F_2 = \alpha_2, \quad \dots . \quad F_{n-1} = \alpha_{n-1}$$

Integralia completa aequationum differentialium,

$$dx_1 : dx_2 \dots . : dx_n = X_1 : X_2 \dots . : X_n,$$

quae cum aequationibus differentialibus (8.) supra consideratis conveniunt ponendo $\alpha = 0$. Unde ponendo $\alpha = 0$ in aequationum differentialium propositarum Integralibus completis (9.), prodit systema aequationum integralium propositarum. Quippe quae redibant in aequationem qua ipsa x exprimitur per reliquas variabiles et quae cum aequatione $F = 0$ conveniebat, atque in aequationum differentialium (8.) Integralia completa, quae ex aequationibus $F_1 = \alpha_1$, $F_2 = \alpha_2$, ... $F_{n-1} = \alpha_{n-1}$ obtinentur, eliminata x ope aequationis $F = 0$. Unde aequationibus differentialibus (7.) integratis systemate aequationum, $n-1$ Constantes Arbitrarias involventium, quoties aequatio eliminatione Constantium Arbitrariarum proveniens redit in aequationem inter ipsas f_1, f_2, ... f_n, illud aequationum integralium systema erit particulare, utpote e completo proveniens tribuendo Constanti Arbitrariae valorem particularem. Hinc vice versa, si illud aequationum integralium systema non est particulare, aequatio eliminatione $n-1$ Constantium Arbitrariarum proveniens non redit in aequationem inter quantitates f_1, f_2, ... f_n, ideoque solutio quam suppeditat aequationis differentialis partialis (1.) erit singularis. Cuiusmodi solutione, cum secundum antecedentibus probata efficiatur $M = \infty$, demonstratum est quod propositum erat, ***quoties systema aequationum differentialium vulgarium integretur systemate aequationum singulari, numerum Constantium Arbitrariarum involvente unitate minorem quam completum involvit, Constantium Arbitrariarum eliminatione provenire aequationem, qua Multiplicator systematis aequationum differentialium abeat in infinitum.*** Et in hac propositione supponitur, quantitates X, X_1 etc. ita a denominatoribus purgatas esse, ut earum nulla pro illa aequatione integrali seu solutione singulari infinita evadat.

Propositionis antecedentis alia haec est demonstratio. Integratione completa exprimantur $x_1, x_2, \dots. x_n$ per x et Constantes Arbitrarias $\beta_1, \beta_2, \dots. \beta_n$. Ponamus aequationibus differentialibus satisfieri posse statuendo $\beta_1, \beta_2, \dots. \beta_n$ esse ipsius x functiones; sequitur e formula,

$$dx_i = \frac{\partial x_i}{\partial x} dx + \frac{\partial x_i}{\partial \beta_1} d\beta_1 + \frac{\partial x_i}{\partial \beta_2} d\beta_2 \dots. + \frac{\partial x_i}{\partial \beta_n} d\beta_n;$$

haec

$$\frac{X_i}{X} dx = \frac{\partial x_i}{\partial x} dx + \frac{\partial x_i}{\partial \beta_1} d\beta_1 + \frac{\partial x_i}{\partial \beta_2} d\beta_2 \dots. + \frac{\partial x_i}{\partial \beta_n} d\beta_n$$

At eliminando quantitates $\beta_1, \beta_2, \dots. \beta_n$ sequitur ex aequationibus integralibus positis,

$$\frac{X_i}{X} = \frac{\partial x_i}{\partial x},$$

quippe quod prodire debebat ponendo $\beta_1, \beta_2, \dots. \beta_n$ esse Constantes; illis autem eliminatis quantitatibus perinde est sive constantes sive variabiles fuerint. Substituendo aequationem antecedentem eruitur pro singulis ipsius i valoribus,

$$10. \quad \frac{\partial x_i}{\partial \beta_1} d\beta_1 + \frac{\partial x_i}{\partial \beta_2} d\beta_2 \dots. + \frac{\partial x_i}{\partial \beta_n} d\beta_n = 0.$$

Ut satisfiat n aequationibus quae ponendo $i = 1, 2, \dots. n$ ex antecedente fluunt, neque simul sit $d\beta_1 = d\beta_2 \dots. = d\beta_n = 0$ sive $\beta_1, \beta_2, \dots. \beta_n$ Constantes sint, evadere debet

$$11. \quad \Sigma \pm \frac{\partial x_1}{\partial \beta_1} \cdot \frac{\partial x_2}{\partial \beta_2} \dots. \frac{\partial x_n}{\partial \beta_n} = 0.$$

Quoties poscitur ut functiones $\beta_1, \beta_2, \dots. \beta_n$ involvant $n-1$ Constantes Arbitrarias, non fieri potest ut aequatio (11.) in relationem inter solas variabiles $\beta_1, \beta_2, \dots. \beta_n$ redeat, sed fieri debet ut e (11.) peti possit ipsius x valor per $\beta_1, \beta_2, \dots. \beta_n$ expressus; quo substituto in quantitatibus $\frac{\partial x_i}{\partial \beta_k}$, habebuntur e (10.) $n-1$ aequationes differentiales primi ordinis inter quantitates β_1, $\beta_2, \dots. \beta_n$, quibus complete integratis prodibunt $n-1$ aequationes inter quantitates $\beta_1, \beta_2, \dots. \beta_n$, $n-1$ Constantibus Arbitrariis affectae. Quibus $n-1$ aequationibus iuncta aequatione qua x per $\beta_1, \beta_2, \dots. \beta_n$ exprimebatur, ipsarumque β_1, β_2 etc. loco substitutis variabilium $x, x_1, \dots. x_n$ functionibus, quibus per integrationem completam aequivalent, obtinetur systema aequationum integralium singularium, $n-1$ Constantibus Arbitrariis affectum. Fit autem secundum §. 6.,

$$\Sigma \pm \frac{\partial x_1}{\partial \beta_1} \cdot \frac{\partial x_2}{\partial \beta_2} \cdots \frac{\partial x_n}{\partial \beta_n} = \frac{C}{X_\mu},$$

designante C quantitatum β_1, β_2, ... β_n functionem atque μ aequationum differentialium propositarum Multiplicatorem. Unde, cum supponatur aequationem (10.) non redire in relationem inter quantitates β_1, β_2, ... β_n, porro ipsam X non infinitam evadere, sequitur e (10.) $\mu = \infty$, q. d. e.

Secundum ea quae §. 7. tradidi, Multiplicator M systematis aequationum differentialium post earum integrationem completam factam sic erui potest. Sint rursus Integralia completa,

$$f_1 = \alpha_1, \quad f_2 = \alpha_2, \quad \dots \quad f_n = \alpha_n,$$

eorum ope exprimatur

$$-\frac{1}{X}\left\{\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} \cdots + \frac{\partial X_n}{\partial x_n}\right\}$$

per x, α_1, α_2, ... α_n. Qua expressione integrata ipsius x respectu, prodeat

$$\varphi(x, \alpha_1, \alpha_2, \dots \alpha_n),$$

secundum §. 7. erit Multiplicator

$$e^{\varphi(x, f_1, f_2, \dots f_n)}.$$

Haec quantitas ut infinita evadat per solutionem seu aequationem integralem singularem, hoc est per solutionem seu aequationem integralem quae non redeat in aequationem inter solas quantitates f_1, f_2, ... f_n (quod semper fieri vidimus quoties omnino eiusmodi aequatio singularis extat) ex ea aequatione talis provenire debet valor ipsius x per quantitates f_1, f_2, ... f_n expressus, quae quantitatem $\varphi(x, f_1, f_2, \dots f_n)$ reddat infinitam. A *fortiori* igitur pro ea ipsius x valore infinita evadere debet quantitas

$$\frac{\partial \varphi}{\partial x} = -\frac{1}{X}\left\{\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} \cdots + \frac{\partial X_n}{\partial x_n}\right\},$$

cum generaliter *quoties pro certo ipsius x valore infinita evadat functio aliqua $\varphi(x)$, pro eadem etiam infinita evadit functio $\frac{\partial \varphi}{\partial x}$ vel adeo $\frac{\partial \varphi}{\varphi \partial x}$* *). Supponimus autem, aequatione singulari non in infinitum abire quantitatem X, unde haec emergit

Propositio II.

„*Quoties extat solutio singularis aequationis differentialis partialis,*

$$X = X_1 \frac{\partial x}{\partial x_1} + X_2 \frac{\partial x}{\partial x_2} \cdots + X_n \frac{\partial x}{\partial x_n},$$

*) Demonstrationem huius propositionis quivis sibi supplere potest.

pro eadem fit

$$\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\cdots\cdots+\frac{\partial X_n}{\partial x_n}=\infty.\text{"}$$

Difficilius videtur solidis argumentis evincere propositionem inversam, videlicet quoties aequatio

$$\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\cdots\cdots+\frac{\partial X_n}{\partial x_n}=\infty$$

suppeditet aequationis differentialis partialis (1.) solutionem, eam fore singularem. Neque video solidam dari demonstrationem in casu elementari aequationis differentialis primi ordinis inter duas variabiles, cum in demonstrationibus passim traditis minus recte supponatur, functionem quae pro $\alpha=0$ evanescat semper evolvi posse secundum ipsius α dignitates positivas.

Sub finem demonstretur de Multiplicatore nostro haec gravissima

Propositio III.

„*Quoties aequatio $M=0$ aut $M=\infty$ est aequatio legitima, semper ea suppeditat solutionem aequationis differentialis partialis, seu aequationem integralem systematis aequationum differentialium vulgarium, cuius M est Multiplicator.*"

Sit M aut $\frac{1}{M}$ aequale functioni u, ita ut aequatio $u=\infty$ alterutram significet aequationum $M=0$ aut $\frac{1}{M}=0$. Eam aequationem legitimam dico si eius ope quaeque variabilium quas continet determinatur ut functio reliquarum, eiusque differentialia quoque prorsus definiantur differentialibus reliquarum variabilium. Statim patet non esse legitimam aequationem $u=\infty$, si est $u=1$; sed eo dicendi modo etiam non erit legitima huiusmodi aequatio $\frac{1}{x+y}=0$, quippe qua non definitur, ut ipsius x functio, sed enunciatur tantum $x+y$ esse functionem quamcunque per Constantem infinite magnam multiplicatam; neque definitur ipsius y incrementum quod capit, ubi x in $x+dx$ abit, cum aequatio $x+y=\infty$ salva maneat si x et y incrementa quaecunque a se independentia capiunt. Addo, si ex aequatione $u=\infty$ fluat variabilis x valor per $x_1, x_2, \ldots. x_n$ expressus, fractiones $\frac{\partial u}{\partial x_i}:\frac{\partial u}{\partial x}$ per aequationem $u=\infty$ infinitas evadere non posse, cum negative sumtae aequentur differentialibus partialibus functionis variabilium $x_1, x_2, \ldots. x_n$, cui x aequalis invenitur. His praeparatis propositio tradita sic patet. Secundum aequationem differentialem partialem qua M definitur, sequitur ex aequatione $u = \infty$,

$$12. \quad X - X_1 \frac{\partial x}{\partial x_1} - X_2 \frac{\partial x}{\partial x_2} \dots - X_n \frac{\partial x}{\partial x_n}. \\ = \pm \frac{1}{\frac{\partial \log u}{\partial x}} \left\{ \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} \dots + \frac{\partial X_n}{\partial x_n} \right\}.$$

Iam si supponitur, uti supra, aequatione $u = \infty$ nullam quantitatem $X, X_1, \dots X_n$ infinitam reddi, quaelibet quantitatum ad dextram, $\frac{\partial X_i}{\partial x_i} : \frac{\partial \log u}{\partial x}$, pro $u = \infty$ evanescit, etsi $\frac{\partial X_i}{\partial x_i}$ pro $u = \infty$ infinitum fiat. Quod sufficit probare de quantitate $\frac{\partial X_i}{\partial x_i} : \frac{\partial \log u}{\partial x_i}$, cum fractio $\frac{\partial u}{\partial x_i} : \frac{\partial u}{\partial x}$ valorem finitum habeat. Generale autem habetur lemma cuius demonstrationi difficultatibus non obnoxiae hic brevitatis causa supersedeo, *si binae functiones pro certo variabilis valore altera infinita fiat, altera finita maneat, prioris differentiale pro eodem variabilis valore infinite maius fore quam posterioris differentiale.* Petendo autem ex aequatione $u = \infty$ valorem ipsius x_i, pro eo ipsius x_i valore secundum suppositionem factam X_i finita manet dum $\log u$ infinitus evadit, unde fractiones $\frac{\partial X_i}{\partial x_i} : \frac{\partial \log u}{\partial x_i}$ ideoque etiam fractiones $\frac{\partial X_i}{\partial x_i} : \frac{\partial \log u}{\partial x}$ pro $u = \infty$ evanescunt. Unde evanescente aequationis (12.) parte dextra, aequatio $u = \infty$ suppeditat aequationis differentialis partialis (1.) solutionem, ideoque etiam aequationem integralem systematis aequationum differentialium vulgarium (7.)

Notione aequationis legitimae supra propositae solvitur paradoxon quod in theoria integrationum singularium obvenit. Constat enim rarissime aequationes differentiales gaudere integrationibus singularibus. At methodus *Lagrangiana* quandam prae se fert generalitatis speciem, quae in errorem inducere possit, ac si de quavis integratione completa deducere liceat singularem. Scilicet ill. *Lagrange*, de aequationibus $y = f(x, \alpha)$, $\frac{\partial f}{\partial \alpha} = 0$, ipsum α eliminare iubet; at in rarissimis casibus quando $y = f(x, \alpha)$ est aequatio integralis completa, Constante Arbitraria α affecta, fit $\frac{\partial f}{\partial \alpha} = 0$ aequatio legitima, qua sola hic uti licet. Idem ad methodum valet, qua supra de systemate aequationum integralium completarum deduxi aequationum integralium singularium systema, quod numerum Constantium Arbitrariarum unitate minorem implicat.

Caput secundum.

De usu novi Multiplicatoris in aequationibus differentialibus integrandis. Principium ultimi Multiplicatoris.

De Multiplicatore aequationum differentialium transformatarum e propositarum derivando.

§. 9.

In aequationibus differentialibus propositis,

$$1. \quad dx : dx_1 \ldots : dx_n = X : X_1 \ldots : X_n,$$

loco variabilium $x, x_1, \ldots x_n$ aliae introducantur $w, w_1, \ldots w_n$, quae supponuntur datae variabilium $x, x_1, \ldots x_n$ functiones a se independentes, unde etiam $x, x_1, \ldots x_n$ erunt quantitatum $w, w_1, \ldots w_n$ functiones independentes. Cum fiat,

$$dw_i = \frac{\partial w_i}{\partial x} dx + \frac{\partial w_i}{\partial x_1} dx_1 \ldots + \frac{\partial w_i}{\partial x_n} dx_n,$$

sequitur ex aequationibus (1.):

$$2. \quad dw : dw_1 \ldots : dw_n = W : W_1 \ldots : W_n,$$

ponendo,

$$3. \quad W_i = \Delta\left\{\frac{\partial w_i}{\partial x} X + \frac{\partial w_i}{\partial x_1} X_1 \ldots + \frac{\partial w_i}{\partial x_n} X_n\right\},$$

ubi Δ factor adhuc indeterminatus sit. Porro fit,

$$\frac{\partial f}{\partial x_i} = \left(\frac{\partial f}{\partial w}\right)\frac{\partial w}{\partial x_i} + \left(\frac{\partial f}{\partial w_1}\right)\frac{\partial w_1}{\partial x_i} \ldots + \left(\frac{\partial f}{\partial w_n}\right)\frac{\partial w_n}{\partial x_i},$$

siquidem uncis, quibus includimus differentialia partialia, innuimus functiones differentiandas per novas variabiles $w, w_1, \ldots w_n$ exhibitas esse. Antecedente formula substituta et advocata (3.) sequitur *pro quacunque functione f:*

$$4. \quad \Delta\left\{X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} \ldots + X_n\frac{\partial f}{\partial x_n}\right\}$$
$$= W\left(\frac{\partial f}{\partial w}\right) + W_1\left(\frac{\partial f}{\partial w_1}\right) \ldots + W_n\left(\frac{\partial f}{\partial w_n}\right).$$

Aequationum (1.) Multiplicator M definiebatur aequatione,

$$5. \quad M\left\{X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} \ldots + X_n\frac{\partial f}{\partial x_n}\right\} = \Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \ldots \frac{\partial f_n}{\partial x_n}.$$

Similiter datur aequationum (2.) Multiplicator N per formulam,

$$6. \quad N\left\{W\left(\frac{\partial f}{\partial w}\right) + W_1\left(\frac{\partial f}{\partial w_1}\right) \ldots + W_n\left(\frac{\partial f}{\partial w_n}\right)\right\}$$
$$= \Sigma \pm \left(\frac{\partial f}{\partial w}\right)\left(\frac{\partial f_1}{\partial w_1}\right) \ldots \left(\frac{\partial f_n}{\partial w_n}\right).$$

At secundum propositionem notam (*De Determ. Funct.* §. 11. *Prop.* II. §. 9. (3.)) fit,

$$7.\quad \Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\cdot\frac{\partial f_n}{\partial x_n}$$

$$=\Sigma\pm\left(\frac{\partial f}{\partial w}\right)\left(\frac{\partial f_1}{\partial w_1}\right)\cdots\cdot\left(\frac{\partial f_n}{\partial w_n}\right)\cdot\Sigma\pm\frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\cdot\frac{\partial w_n}{\partial x_n}.$$

Unde e (4.), (5.) obtinetur pro quacunque functione f:

$$8.\quad \frac{M}{\Delta}\left\{W\left(\frac{\partial f}{\partial w}\right)+W_1\left(\frac{\partial f_1}{\partial w_1}\right)\cdots\cdot+W_n\left(\frac{\partial f_n}{\partial w_n}\right)\right\}$$

$$=\Sigma\pm\frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\cdot\frac{\partial w_n}{\partial x_n}\cdot\Sigma\pm\left(\frac{\partial f}{\partial w}\right)\left(\frac{\partial f_1}{\partial w_1}\right)\cdots\cdot\left(\frac{\partial f_n}{\partial w_n}\right).$$

Quam formulam comparando cum (6.) sequitur, *posito in formula* (3.),

$$9.\quad \Delta=\frac{1}{\Sigma\pm\frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\cdot\frac{\partial w_n}{\partial x_n}}=\Sigma\pm\left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right)\cdots\cdot\left(\frac{\partial x_n}{\partial w_n}\right),$$

fieri $N=M$ *sive aequationum differentialium propositarum* (1.) *atque transformatarum* (2.) *eundem fore Multiplicatorem.*

Servando factori Δ valorem (9.), cum sit idem M aequationum (1.) et (2) Multiplicator, fit e proprietate Multiplicatoris fundamentali,

$$10.\quad 0=X\frac{\partial M}{\partial x}+X_1\frac{\partial M}{\partial x_1}\cdots\cdot+X_n\frac{\partial M}{\partial x_n}$$
$$+M\left\{\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\cdots\cdot+\frac{\partial X_n}{\partial x_n}\right\},$$

$$11.\quad 0=W\left(\frac{\partial M}{\partial w}\right)+W_1\left(\frac{\partial M}{\partial w_1}\right)\cdots\cdot+W_n\left(\frac{\partial M}{\partial w_n}\right)$$
$$+M\left\{\left(\frac{\partial W}{\partial w}\right)+\left(\frac{\partial W_1}{\partial w_1}\right)\cdots\cdot+\left(\frac{\partial W_n}{\partial w_n}\right)\right\}.$$

At ponendo M pro functione indefinita f in formula (4.) fit,

$$X\frac{\partial M}{\partial x}+X_1\frac{\partial M}{\partial x_1}\cdots\cdot+X_n\frac{\partial M}{\partial x_n}=\frac{1}{\Delta}\left\{W\frac{\partial M}{\partial w}+W_1\frac{\partial M}{\partial w_1}\cdots\cdot+W_n\frac{\partial M}{\partial w_n}\right\}.$$

Unde de aequatione (11.) per Δ divisa detrahendo aequationem (10.) et dividendo per M eruitur:

$$12.\quad \frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\cdots\cdot+\frac{\partial X_n}{\partial x_n}=\frac{1}{\Delta}\left\{\left(\frac{\partial W}{\partial w}\right)+\left(\frac{\partial W_1}{\partial w_1}\right)\cdots\cdot+\left(\frac{\partial W_n}{\partial w_n}\right)\right\}.$$

Quae est formula memoratu digna, in qua $X, X_1, \ldots. X_n$ sunt functiones quaecunque, ipsae autem Δ, W, W_1, ... W_n formulis (9.) et (3.) definiuntur.

Si quantitates W, W_1 etc. per factorem communem Δ dividimus, per eundem multiplicandus erit aequationum (2.) Multiplicator. Unde si definimus quantitates W_i formula

$$W_i = \frac{\partial w_i}{\partial x} X + \frac{\partial w_i}{\partial x_1} X_1 \dots + \frac{\partial w_i}{\partial x_n} X_n,$$

aequationum differentialium,

$$dw : dw_1 \dots dw_n = W : W_1 \dots : W_n,$$

erit Multiplicator $\Delta . M.$ Ponamus

$$t = \int \frac{dx}{X},$$

poterunt aequationes differentiales (1.) sic proponi:

$$13. \quad \frac{dx}{dt} = X, \quad \frac{dx_1}{dt} = X_1, \quad \dots \quad \frac{dx_n}{dt} = X_n;$$

unde sequitur,

$$\frac{dw_i}{dt} = \frac{\partial w_i}{\partial x} X + \frac{\partial w_i}{\partial x_1} X_1 \dots + \frac{\partial w_i}{\partial x_n} X_n,$$

sive,

$$\frac{dw_i}{dt} = W_i.$$

Aequationum (1.) *Multiplicatorem in sequentibus etiam appellabo Multiplicatorem aequationum* (13.). Unde antecedentibus inventa sic poterunt enunciari:

Propositio I.

„Designantibus $X, X_1, \dots X_n$ variabilium $x, x_1, \dots x_n$ functiones quaslibet, proponantur aequationes differentiales,

$$\frac{dx}{dt} = X, \quad \frac{dx_1}{dt} = X_1, \quad \dots \quad \frac{dx_n}{dt} = X_n,$$

quarum sit M Multiplicator; in quibus aequationibus ipsarum x, x_1 etc. loco aliae introducantur variabiles $w, w_1, \dots w_n$; quo facto si obtinentur aequationes differentiales,

$$14. \quad \frac{dw}{dt} = W, \quad \frac{dw_1}{dt} = W_1, \quad \dots \quad \frac{dw_n}{dt} = W_n,$$

harum aequationum Multiplicator erit $\Delta . M$, posito

$$\Delta = \frac{1}{\Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \dots \frac{\partial w_n}{\partial x_n}} = \Sigma \pm \left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \dots \left(\frac{\partial x_n}{\partial w_n}\right)."$$

Ubi rursus quantitates W_i formula (3.) definimus, formulam (12.) sic proponere licet.

Propositio II.

„Ipsarum $x, x_1, \ldots. x_n$ loco introducendo $w, w_1, \ldots. w_n$, ponendoque

$$dt = \Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \ldots. \frac{\partial w_n}{\partial x_n} . dt,$$

ex aequationibus differentialibus

$$\frac{dx}{dt} = X, \quad \frac{dx_1}{dt} = X_1, \quad \ldots. \quad \frac{dx_n}{dt} = X_n,$$

proveniant sequentes,

$$\frac{dw}{dt} = W, \quad \frac{dw_1}{dt} = W_1, \quad \ldots. \quad \frac{dw_n}{dt} = W_n,$$

erit

$$\left\{\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} \ldots. + \frac{\partial X_n}{\partial x_n}\right\} dt = \left\{\left(\frac{\partial W}{\partial w}\right) + \left(\frac{\partial W_1}{\partial w_1}\right) \ldots. + \left(\frac{\partial W_n}{\partial w_n}\right)\right\} dt."$$

In antecedentibus suppositum est, neque ipsas X, X_1 etc. implicare variabilem t neque eam variabilem afficere relationes quae inter variabiles propositas $x, x_1, \ldots. x_n$ atque novas $w, w_1, \ldots. w_n$ intercedunt. *Si quantitates X, X_1 etc. praeter variabiles x, x_1 etc. ipsa quoque t afficiuntur, aequationum* (13.) *Multiplicatorem eundem dicere placet atque aequationum,*

$$15. \quad dt : dx : dx_1 \ldots. dx_n = 1 : X : X_1 \ldots. : X_n.$$

Designantibus x, x_1 etc. ipsarum $t, w, w_1, \ldots. w_n$, sive w, w_1 etc. ipsarum $t, x, x_1, \ldots. x_n$ functiones, ponamus rursus ex aequationibus differentialibus (13.) vel (15.) sequi aequationes (14.) sive aequationes,

$$16. \quad dt : dw : dw_1 \ldots. dw_n = 1 : W : W_1 \ldots. : W_n,$$

atque aequationum (15.) Multiplicatorem esse M, aequationum (16.) Multiplicatorem $\Delta . M$. Quibus statutis, secundum antecedentia ad $n+2$ variabiles amplificata erit,

$$\Delta = \Sigma \pm \left(\frac{\partial t}{\partial t}\right)\left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \ldots. \left(\frac{\partial x_n}{\partial w_n}\right).$$

Sed habetur $\left(\frac{\partial t}{\partial t}\right) = 1$, $\left(\frac{\partial t}{\partial w_i}\right) = 0$, unde,

$$\Sigma \pm \left(\frac{\partial t}{\partial t}\right)\left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \ldots. \left(\frac{\partial x_n}{\partial w_n}\right) = \Sigma \pm \left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \ldots. \left(\frac{\partial x_n}{\partial w_n}\right).$$

Hinc sequitur, ***Propositionem I. ad eum quoque casum valere, quo quantitates X, X_1 etc. atque functiones novis variabilibus aequandae w, w_1 etc. praeter ipsas x, x_1 etc. variabili t afficiuntur.***

Si tantum pro parte variabilium aliae introducuntur, ipsius Δ expressio simplicior evadit. Propositis enim aequationibus (13.)

$$\frac{dx}{dt} = X, \quad \frac{dx_1}{dt} = X_1, \quad \dots \quad \frac{dx_n}{dt} = X_n,$$

quarum est M Multiplicator, si tantum loco variabilium x, x_1, ... x_μ aliae introducuntur w, w_1, ... w_μ, ita ut aequationes differentiales transformatae fiant,

$$\frac{dw}{dt} = W, \quad \frac{dw_1}{dt} = W_1, \quad \dots \quad \frac{dw_\mu}{dt} = W_\mu,$$

$$\frac{dx_{\mu+1}}{dt} = X_{\mu+1}, \quad \frac{dx_{\mu+2}}{dt} = X_{\mu+2}, \quad \dots \quad \frac{dx_n}{dt} = X_n,$$

fit harum Multiplicator $\Delta . M$, posito,

$$\Delta = \Sigma \pm \left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \dots \left(\frac{\partial x_\mu}{\partial w_\mu}\right) = \frac{1}{\Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \dots \frac{\partial w_\mu}{\partial x_\mu}},$$

sicuti ex expressione generali ipsius Δ patet ponendo $w_{\mu+1} = x_{\mu+1}$, $w_{\mu+2} = x_{\mu+2}$ etc. Quae formulae variis applicationibus idoneae sunt.

Multiplicator aequationum differentialium ope Integralium completorum reductarum e Multiplicatore propositarum eruitur. Pro reductionibus diversis Multiplicatores alii de aliis deducuntur.

§. 10.

Per formulas §. pr. traditas facile solvitur quaestio, si aequationum differentialium

$$1. \quad dx : dx_1 \dots : dx_n = X : X_1 \dots : X_n,$$

inventa sint m Integralia,

$$2. \quad w = \alpha, \quad w_1 = \alpha_1, \quad \dots \quad w_{m-1} = \alpha_{m-1},$$

designantibus α, α_1, ... α_{m-1} Constantes Arbitrarias, aequationum differentialium ope illorum Integralium reductarum Multiplicatorem e Multiplicatore propositarum investigandi. Sint enim w_m, w_{m+1}, ... w_n aliae variabilium x, x_1, ... x_n functiones a se ipsis et ab ipsis w, w_1, ... w_{m-1} independentes, inter quas propositum sit aequationes differentiales exhibere reductas. Poterunt w, w_1, ... w_n ipsarum x, x_1, ... x_n loco pro variabilibus in Calculum introduci. Quo facto secundum §. pr. abeunt aequationes differentiales vulgares (1.) in sequentes:

$$3. \quad dw : dw_1 : dw_2 \dots : dw_n = W : W_1 : W_2 \dots : W_n,$$

siquidem statuitur

$$4. \quad W_i = \Delta \left\{ \frac{\partial w_i}{\partial x} + X_1 \frac{\partial w_i}{\partial x_1} \dots + X_n \frac{\partial w_i}{\partial x_n} \right\}.$$

Ponendo factorem Δ, quem ex arbitrio determinare licet, fieri,

$$5.\quad \Delta = \Sigma \pm \left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right)\cdots\left(\frac{\partial x_n}{\partial w_n}\right) = \frac{1}{\Sigma \pm \frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}},$$

vidimus §. pr. Multiplicatorem aequationum differentialium propositarum (1.) eundem evadere Multiplicatorem aequationum transformatarum (3.). Unde designante M aequationum (1.) Multiplicatorem, identice erit

$$6.\quad \left(\frac{\partial.MW}{\partial w}\right)+\left(\frac{\partial.MW_1}{\partial w_1}\right)\cdots+\left(\frac{\partial.MW_n}{\partial w_n}\right) = 0,$$

qua in formula $M, W, W_1, \ldots W_n$ per variabiles $w, w_1, \ldots w_n$ expressae finguntur. At cum sint (2.) aequationum differentialium (1.) Integralia, sequitur esse $w, w_1, \ldots w_{m-1}$ solutiones aequationis differentialis partialis

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}\cdots+X_n\frac{\partial f}{\partial x_n} = 0,$$

unde patet e formula (4.), identice fieri,

$$7.\quad W = 0,\quad W_1 = 0,\quad \ldots\quad W_{m-1} = 0.$$

Unde aequatio (6.) in hanc reducitur,

$$8.\quad \left(\frac{\partial.MW_m}{\partial w_m}\right)+\left(\frac{\partial.MW_{m+1}}{\partial w_{m+1}}\right)\cdots+\left(\frac{\partial.MW_n}{\partial w_n}\right) = 0.$$

In aequatione antecedente expressae sunt MW_m, MW_{m+1} etc. per $w, w_1, \ldots \ldots w_n$, sed differentiationes partiales solarum $w_m, w_{m+1}, \ldots w_n$ respectu transiguntur. Unde in aequatione praecedente ipsis $w, w_1, \ldots w_{m-1}$ substituere licet Constantes Arbitrarias aequivalentes $\alpha, \alpha_1, \ldots \alpha_{m-1}$. Idem si facimus in aequationibus differentialibus (3.), obtinemus aequationes differentiales per inventa Integralia (2.) reductas,

$$9.\quad dw_m : dw_{m+1} \ldots : dw_n = W_m : W_{m+1} \ldots : W_n,$$

in quibus sunt $W_m, W_{m+1}, \ldots W_n$ ipsarum $w_m, w_{m+1}, \ldots w_n$ et Constantium Arbitrariarum $\alpha, \alpha_1, \ldots \alpha_{m-1}$ functiones, in quas quantitates (4.) per inventa Integralia (2.) abeunt. Simulque docet aequatio identica (8.) ipsum M, per $w_m, w_{m+1}, \ldots w_n$ atque $\alpha, \alpha_1, \ldots \alpha_{m-1}$ expressum, fore aequationum quoque reductarum (9.) Multiplicatorem.

Antecedentibus valores quantitatum W_i per talem factorem Δ multiplicavi, ut aequationum differentialium (1.) atque (3.) Multiplicator M idem fiat. Si in formulis (4.) hunc factorem omittimus sive omnes quantitates W_i per factorem Δ dividimus, ipsum M per eundem multiplicari debebat, sive aequationum (3.) vel (9.) Multiplicator poni debebat $\Delta.M$ (§. 9.). Quod si facimus, antecedentibus inventa sic proponere licet.

Propositio I.

„Aequationum differentialium

$$dx : dx_1 \ldots . : dx_n = X : X_1 \ldots . : X_n,$$

quarum sit M Multiplicator, inventa sint m Integralia,

$$w = \alpha, \quad w_1 = \alpha_1, \ \ldots . \ w_{m-1} = \alpha_{m-1},$$

quorum ope variabiles $x, x_1, \ldots . x_n$ omnes exprimantur per Constantes Arbitrarias $\alpha, \alpha_1, \ldots . \alpha_{m-1}$ atque variabilium $x, x_1, \ldots . x_n$ functiones

$$w_m, \quad w_{m+1}, \ \ldots . \ w_n,$$

ponendo

$$W_i = X\frac{\partial w_i}{\partial x} + X_1\frac{\partial w_i}{\partial x_1} \ldots . + X_n\frac{\partial w_i}{\partial x_n},$$

dabuntur inter variabiles $w_m, w_{m+1}, \ldots . w_n$ aequationes differentiales,

$$dw_m : dw_{m+1} \ldots . : dw_n = W_m : W_{m+1} \ldots . : W_n,$$

harumque Multiplicator erit

$$\varDelta . M,$$

siquidem ponitur

$$\varDelta = \Sigma \pm \left(\frac{\partial x}{\partial w_m}\right)\left(\frac{\partial x_1}{\partial w_{m+1}}\right) \ldots . \left(\frac{\partial x_{n-m}}{\partial w_n}\right)\left(\frac{\partial x_{n-m+1}}{\partial \alpha}\right)\left(\frac{\partial x_{n-m+2}}{\partial \alpha_1}\right) \ldots . \left(\frac{\partial x_n}{\partial \alpha_{m-1}}\right)$$

$$= \left\{\Sigma \pm \frac{\partial w_m}{\partial x} \cdot \frac{\partial w_{m+1}}{\partial x_1} \ldots . \frac{\partial w_n}{\partial x_{n-m}} \cdot \frac{\partial w}{\partial x_{n-m+1}} \cdot \frac{\partial w_1}{\partial x_{n-m+2}} \ldots . \frac{\partial w_{m-1}}{\partial x_n}\right\}^{-1}."$$

Quae est Propositio in theoria Multiplicatoris fundamentalis. Determinans inversum, quo $\varDelta$ exprimitur, sic quoque scribi potest,

$$\left\{\Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \ldots . \frac{\partial w_n}{\partial x_n}\right\}^{-1},$$

cum permutatione functionum w, w_1 etc. valor Determinantis tantum signum mutare queat, quod hic non curamus.

Pro ipsis $w_m, w_{m+1}, \ldots . w_n$ etiam $n-m+1$ quantitates e numero ipsarum $x, x_1, \ldots . x_n$ sumere licet. Si statuimus

$$w_m = x, \quad w_{m+1} = x_1, \ \ldots . \ w_n = x_{n-m},$$

fit,

$$10. \quad \varDelta = \Sigma \pm \left(\frac{\partial x_{n-m+1}}{\partial \alpha}\right)\left(\frac{\partial x_{n-m+2}}{\partial \alpha_1}\right) \ldots . \left(\frac{\partial x_n}{\partial \alpha_{m-1}}\right)$$

$$= \left\{\Sigma \pm \frac{\partial w}{\partial x_{n-m+1}} \cdot \frac{\partial w_1}{\partial x_{n-m+2}} \ldots . \frac{\partial w_{m-1}}{\partial x_n}\right\}^{-1}.$$

Porro e (4.) obtinetur,

$$W_m = X, \quad W_{m+1} = X_1, \ \ldots . \ W_n = X_{n-m}.$$

Hinc eruitur

Propositio II.

„Aequationum differentialium

$$dx : dx_1 \dots : dx_n = X : X_1 \dots : X_n,$$

quarum M est Multiplicator, inventis m Integralibus,

$$w = \alpha, \quad w_1 = \alpha_1, \quad \dots \quad w_{m-1} = \alpha_{m-1},$$

si exhibentur x_{n-m+1}, x_{n-m+2}, ... x_n per x, x_1, ... x_{n-m} atque Constantes Arbitrarias α, α_1, ... α_{m-1}, aequationum differentialium reductarum

$$dx : dx_1 \dots : dx_{n-m} = X : X_1 \dots : X_{n-m},$$

evadit Multiplicator,

$$M \Sigma \pm \left(\frac{\partial x_{n-m+1}}{\partial \alpha}\right) \left(\frac{\partial x_{n-m+2}}{\partial \alpha_1}\right) \dots \left(\frac{\partial x_n}{\partial \alpha_{m-1}}\right)$$

$$= M \left\{ \Sigma \pm \frac{\partial w}{\partial x_{n-m+1}} \cdot \frac{\partial w_1}{\partial x_{n-m+2}} \dots \frac{\partial w_{m-1}}{\partial x_n} \right\}^{-1}.\text{"}$$

Si eaedem aequationes differentiales propositae per diversa Integralium systemata reducuntur, Multiplicatores diversorum aequationum differentialium reductarum systematum ex eorum uno deduci possunt. Qua in re semper supponitur, unumquodque Integrale quod reductioni inservit sua affici Constante Arbitraria, ideoque aequationes differentiales reductas omnes ingredi Constantes Arbitrarias, quibus Integralia quorum ope reductio effecta est afficiuntur.

Sint enim rursus Integralia reductioni adhibenda,

$$w = \alpha, \quad w_1 = \alpha_1, \quad \dots \quad w_{m-1} = \alpha_{m-1},$$

atque aequationes differentiales reductae, inter variabiles w_m, w_{m+1}, ... w_n exhibitae,

$$11. \quad dw_m : dw_{m+1} \dots : dw_n = W_m : W_{m+1} \dots : W_n.$$

Eaedem aequationes differentiales propositae (1.) ope Integralium,

$$u = \beta, \quad u_1 = \beta_1, \quad \dots \quad u_{k-1} = \beta_{k-1},$$

reducantur ad has, inter variabiles u_k, u_{k+1}, ... u_n exhibitas,

$$12. \quad du_k : du_{k+1} \dots : du_n = U_k : U_{k+1} \dots : U_n.$$

Sit M Multiplicator aequationum differentialium propositarum, sint respective N et K Multiplicatores aequationum differentialium reductarum (11.) et (12.): erit secundum Prop. I.

$$13. \quad N = M \left\{ \Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \dots \frac{\partial w_n}{\partial x_n} \right\}^{-1},$$

$$K = M \left\{ \Sigma \pm \frac{\partial u}{\partial x} \cdot \frac{\partial u_1}{\partial x_1} \dots \frac{\partial u_n}{\partial x_n} \right\}^{-1},$$

unde

$$14.\qquad K = N\frac{\Sigma \pm \frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}}{\Sigma \pm \frac{\partial u}{\partial x}\cdot\frac{\partial u_1}{\partial x_1}\cdots\frac{\partial u_n}{\partial x_n}}.$$

Quae formula supponit, in aequationibus differentialibus reductis (11.) et (12.) ita definiri quantitates differentialibus proportionales ut fiat,

$$\frac{\partial w_m}{W_m} = \frac{\partial u_k}{U_k}.$$

Si ipsae $w, w_1, \ldots w_n$ per $u, u_1, \ldots u_n$ exprimuntur, formulam (14.) notae propositionis beneficio (D. F. §. 10. (5.)) concinnius sic exhibere licet,

$$15.\qquad K = N\Sigma \pm \frac{\partial w}{\partial u}\cdot\frac{\partial w_1}{\partial u_1}\cdots\frac{\partial w_n}{\partial u_n}.$$

Quae formula generalis duos amplectitur casus, quo aequationes differentiales propositae per eadem Integralia reducuntur, sed reductae inter diversas variabiles exhibentur, et quo per diversa Integralia reductae inter easdem variabiles exhibentur.

Etenim ponendo $k = m$ atque

$$u = w,\quad u_1 = w_1,\quad \ldots\quad u_{m-1} = w_{m-1},$$

sequitur e (15.), si eaedem aequationes differentiales propositae per eadem Integralia,

$$w = \alpha,\quad w_1 = \alpha_1,\quad \ldots\quad w_{m-1} = \alpha_{m-1},$$

reducantur ad $n-m$ aequationes differentiales inter $n-m+1$ variabiles w_m, $w_{m+1}, \ldots w_n$ vel ad alias inter variabiles $u_m, u_{m+1}, \ldots u_n$, fieri

$$16.\qquad K = N\Sigma \pm \frac{\partial w_m}{\partial u_m}\cdot\frac{\partial w_{m+1}}{\partial u_{m+1}}\cdots\frac{\partial w_n}{\partial u_n},$$

ubi $w_m, w_{m+1}, \ldots w_n$ expressae supponuntur per variabiles $u_m, u_{m+1}, \ldots u_n$ atque Constantes Arbitrarias $\alpha, \alpha_1, \ldots \alpha_{m-1}$.

Si vero rursus $k = m$ atque

$$u_m = w_m,\quad u_{m+1} = w_{m+1},\quad \ldots\quad u_n = w_n,$$

vel si aequationes differentiales propositae per hoc m Integralium systema

$$w = \alpha,\quad w_1 = \alpha_1,\quad \ldots\quad w_{m-1} = \alpha_{m-1},$$

aut per hoc,

$$u = \beta,\quad u_1 = \beta_1,\quad \ldots\quad u_{m-1} = \beta_{m-1},$$

reducuntur ad $n-m$ aequationes differentiales diversas inter easdem $n-m+1$ variabiles $w_m, w_{m+1}, \ldots w_n$: abit formula (15.) in hanc,

$$17.\qquad K = N\Sigma \pm \frac{\partial w}{\partial \beta}\cdot\frac{\partial w_1}{\partial \beta_1}\cdots\frac{\partial w_{m-1}}{\partial \beta_{m-1}},$$

siquidem in formando Determinante Functionali supponitur expressas esse w, w_1, ... w_{m-1} per variabiles w_m, w_{m+1}, ... w_n atque Constantes Arbitrarias β, β_1, ... β_m.

Principium ultimi Multiplicatoris sive quomodo cognito Multiplicatore systematis aequationum differentialium vulgarium ultima integratio ad Quadraturas revocatur.

§. 11.

Propositionum I. et. II. §. pr. prae ceteris memorabilis est casus $m = n - 1$, quo omnibus praeter unum inventis Integralibus una integranda restat aequatio differentialis primi ordinis inter duas variabiles. Eo casu Multiplicator aequationis differentialis reductae redit in Multiplicatorem *Eulerianum*, qui eam per se integrabilem reddit sive ad Quadraturas revocat. Unde ponendo $n = m - 1$ e Propp. I. et II. §. pr. memorabiles prodeunt Propositiones, quae novum constituunt principium, e quo Calculus Integralis haud parum incrementi capit. Quod *principium ultimi Multiplicatoris* appellare convenit.

Propositio I.

„*Propositis aequationibus differentialibus*

$$dx : dx_1 \dots : dx_n = X : X_1 \dots : X_n$$

habeatur Multiplicator M sive solutio quaecunque aequationis differentialis partialis,

$$\frac{\partial . MX}{\partial x} + \frac{\partial . MX_1}{\partial x_1} \dots + \frac{\partial . MX_n}{\partial x_n} = 0;$$

porro inventa sint Integralia praeter unum omnia,

$$w = \alpha, \quad w_1 = \alpha_1, \quad \dots \quad w_{n-2} = \alpha_{n-2},$$

designantibus α etc. Constantes Arbitrarias, quibus ipsae functiones w, w_1 etc. non afficiantur; sumtis ex arbitrio duabus ipsarum x, x_1, ... x_n functionibus w_{n-1}, w_n, fiat,

$$X \frac{\partial w_{n-1}}{\partial x} + X_1 \frac{\partial w_{n-1}}{\partial x_1} \dots + X_n \frac{\partial w_{n-1}}{\partial x_n} = W_{n-1},$$

$$X \frac{\partial w_n}{\partial x} + X_1 \frac{\partial w_n}{\partial x_1} \dots + X_n \frac{\partial w_n}{\partial x_n} = W_n,$$

erit ultimum Integrale

$$\int \frac{M\{W_n dw_{n-1} - W_{n-1} dw_n\}}{\Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \dots \frac{\partial w_n}{\partial x_n}} = \text{Const.}"$$

Propositio II.

„*Inventis aequationum differentialium,*

$$dx : dx_1 \dots : dx_n = X : X_1 \dots : X_n,$$

Integralibus praeter unum omnibus,

$$w = \alpha, \quad w_1 = \alpha_1, \quad \ldots. \quad w_{n-2} = \alpha_{n-2},$$

ac designante M solutionem quamcunque aequationis differentialis partialis,

$$0 = \frac{\partial . MX}{\partial x} + \frac{\partial . MX_1}{\partial x_1} \ldots. + \frac{\partial . MX_n}{\partial x_n},$$

exprimantur

$$x_2, \quad x_3, \quad \ldots. \quad x_n, \quad X, \quad X_1, \quad M$$

per x et x_1 atque Constantes Arbitrarias

$$\alpha, \quad \alpha_1, \quad \ldots. \quad \alpha_{n-2}:$$

erit ultima aequatio integralis,

$$\int \frac{M\{X_1 dx - X dx_1\}}{\Sigma \pm \frac{\partial w}{\partial x_2} . \frac{\partial w_1}{\partial x_3} \ldots. \frac{\partial w_{n-2}}{\partial x_n}} = \text{Const."}$$

In duabus Propositionibus antecedentibus quantitas sub integrationis signo posita evadit differentiale completum, ubi expressiones in bina differentialia ducta per easdem duas variabiles exhibentur inter quas aequatio differentialis reducta locum habet. Similiter in sequentibus etsi pressis verbis non adnotetur, quoties formula integralis Constanti Arbitrariae aequiparatur, innuitur sub signo integrationis haberi differentiale completum.

In Propp. antecedentibus loco divisionis per Determinantia Functionalia,

$$\Sigma \pm \frac{\partial w}{\partial x} . \frac{\partial w_1}{\partial x_1} \ldots. \frac{\partial w_n}{\partial x_n},$$

$$\Sigma \pm \frac{\partial w}{\partial x_2} . \frac{\partial w_1}{\partial x_3} \ldots. \frac{\partial w_{n-2}}{\partial x_n},$$

etiam multiplicatio institui potuisset per Determinantia Functionalia sensu inverso formata (Det. Funct. §. 9.). Quod ubi fit, erit in altera Propositione ultima aequatio integralis,

$$1. \quad \int M\Delta (W_n \, dw_{n-1} - W_{n-1} \, dw_n) = \text{Const.},$$

posito

$$2. \quad \Delta = \Sigma \pm \frac{\partial x_2}{\partial \alpha} . \frac{\partial x_3}{\partial \alpha_1} \ldots. \frac{\partial x_n}{\partial \alpha_{n-2}} . \frac{\partial x}{\partial w_{n-1}} . \frac{\partial x_1}{\partial w_n}$$
$$= \left\{ \Sigma \pm \frac{\partial w}{\partial x} . \frac{\partial w_1}{\partial x_1} \ldots. \frac{\partial w_n}{\partial x_n} \right\}^{-1},$$

vel in altera

$$3. \quad \int M\Delta (X_1 dx - X dx_1) = \text{Const.},$$

posito

$$4. \quad \Delta = \Sigma \pm \frac{\partial x_2}{\partial \alpha} . \frac{\partial x_3}{\partial \alpha_1} \ldots. \frac{\partial x_n}{\partial \alpha_{n-2}} = \left\{ \Sigma \pm \frac{\partial w}{\partial x_2} . \frac{\partial w_1}{\partial x_3} \ldots. \frac{\partial w_{n-2}}{\partial x_n} \right\}^{-1}.$$

In formandis Determinantibus functionalibus (2.) et (4.) supponitur, aut ipsa $n-2$ Integralia dari novasque quoque variabiles w_{n-1}, w_n per $x, x_1, \dots x_n$ expressas esse, aut per integrationes transactas variabiles omnes expressas esse per binas w_{n-1}, w_n vel x, x_1 atque per Constantes Arbitrarias quae singulis integrationibus accedunt. Generalius si reductio ad aequationem differentialem primi ordinis inter duas variabiles efficitur ope $n-1$ aequationum integralium quarumcunque,

$$\Pi = 0, \quad \Pi_1 = 0, \quad \dots \quad \Pi_{n-2} = 0,$$

quae afficiuntur totidem Constantibus Arbitrariis

$$\alpha, \quad \alpha_1, \quad \dots \quad \alpha_{n-2},$$

poni poterit in formula (2.)

$$5. \quad \Delta = \frac{\Sigma \pm \frac{\partial \Pi}{\partial \alpha} \cdot \frac{\partial \Pi_1}{\partial \alpha_1} \cdots \frac{\partial \Pi_{n-2}}{\partial \alpha_{n-2}}}{\Sigma \pm \frac{\partial \Pi}{\partial x_2} \cdot \frac{\partial \Pi_1}{\partial x_3} \cdots \frac{\partial \Pi_{n-2}}{\partial x_n} \cdot \frac{\partial w_{n-1}}{\partial x} \cdot \frac{\partial w_n}{\partial x_1}},$$

vel in formula (4.),

$$6. \quad \Delta = \frac{\Sigma \pm \frac{\partial \Pi}{\partial \alpha} \cdot \frac{\partial \Pi_1}{\partial \alpha_1} \cdots \frac{\partial \Pi_{n-2}}{\partial \alpha_{n-2}}}{\Sigma \pm \frac{\partial \Pi}{\partial x_2} \cdot \frac{\partial \Pi_1}{\partial x_3} \cdots \frac{\partial \Pi_{n-2}}{\partial x_n}}$$

(Cf. ***Det. Funct.*** §. 10.). Formula antecedens prae ceteris cum fructu adhibetur. Aequationibus enim integralibus inventis saepissime per varias eliminationes eiusmodi formas induere licet, pro quibus Determinantia functionalia, quae numeratorem et denominatorem fractionis antecedentis constituunt, sine molestia inveniantur. Commode etiam adhiberi potest ad Determinantia functionalia formanda propositio, valorem Determinantium functionalium,

$$\Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \cdots \frac{\partial w_n}{\partial x_n}, \quad \Sigma \pm \frac{\partial w}{\partial x_2} \cdot \frac{\partial w_1}{\partial x_3} \cdots \frac{\partial w_{n-2}}{\partial x_n},$$

non mutari, si ante differentiationes partiales transigendas functio quaeque w_i ope aequationum,

$$7. \quad w = \alpha, \quad w_1 = \alpha_1, \quad \dots \quad w_{i-1} = \alpha_{i-1},$$

mutationes quascunque subeat. Inservire possunt aequationes (7.) ad eliminandas e quaque functione w_i variabiles

$$x_n, \quad x_{n-1}, \quad \dots \quad x_{n-i+1}.$$

Quo facto si abit w_i in Π_i, erunt

$$\Pi - \alpha = 0, \quad \Pi_1 - \alpha_1 = 0, \quad \dots \quad \Pi_{n-2} - \alpha_{n-2} = 0,$$

aequationes integrales, quales per integrationem et eliminationem successivam inveniuntur. Porro fit

$$8.\quad X\pm\frac{\partial w}{\partial x_2}\cdot\frac{\partial w_1}{\partial x_3}\cdots\frac{\partial w_{n-2}}{\partial x_n} = \frac{\partial\Pi}{\partial x_n}\cdot\frac{\partial\Pi_1}{\partial x_{n-1}}\cdots\frac{\partial\Pi_{n-2}}{\partial x_2}.$$

Cf. §. 3. Si vero adhibentur variabilium expressiones quales ex eliminatione successiva prodeunt, videlicet ipsius x_n expressio per x, x_1, ... x_{n-1}, α; ipsius x_{n-1} expressio per x, x_1, ... x_{n-2}, α, α_1 etc., abit Determinans

$$X\pm\frac{\partial x_2}{\partial\alpha}\cdot\frac{\partial x_3}{\partial\alpha_1}\cdots\frac{\partial x_n}{\partial\alpha_{n-2}}$$

in productum

$$\left(\frac{\partial x_n}{\partial\alpha}\right)\left(\frac{\partial x_{n-1}}{\partial\alpha_1}\right)\cdots\left(\frac{\partial x_2}{\partial\alpha_{n-2}}\right),$$

ubi uncis innuo esse x_{n-i} ipsarum x, x_1, ... x_{n-i-1}, α, α_1, ... α_i functionem. Quibus substitutis in (4.), fit

$$9.\quad \Delta = \left(\frac{\partial x_n}{\partial\alpha}\right)\left(\frac{\partial x_{n-1}}{\partial\alpha_1}\right)\cdots\left(\frac{\partial x_2}{\partial\alpha_{n-2}}\right) = \frac{1}{\frac{\partial\Pi}{\partial x_n}\cdot\frac{\partial\Pi_1}{\partial x_{n-1}}\cdots\frac{\partial\Pi_{n-2}}{\partial x_2}}.$$

Hinc sequentes emergunt Propositiones.

Propositio III.

„Aequationum differentialium vulgarium,

$$dx : dx_1 \ldots : dx_n = X : X_1 \ldots : X_n,$$

quarum M est Multiplicator, inventis per integrationem et eliminationem successivam aequationibus integralibus praeter unam omnibus,

$$\Pi = \alpha,\quad \Pi_1 = \alpha_1,\quad \ldots\quad \Pi_{n-2} = \alpha_{n-2},$$

ubi Π_i est functio variabilium x, x_1, ... x_{n-i} atque Constantium Arbitrariarum α, α_1, ... α_{i-1}: fit ultima aequatio integralis,

$$\int\frac{M\{X_1\,dx - X\,dx_1\}}{\frac{\partial\Pi}{\partial x_n}\cdot\frac{\partial\Pi_1}{\partial x_{n-1}}\cdots\frac{\partial\Pi_{n-2}}{\partial x_2}} = \text{Const.“}$$

Propositio IV.

„Aequationum differentialium vulgarium

$$dx : dx_1 \ldots : dx_n = X : X_1 \ldots : X_n,$$

quarum M est Multiplicator, inventis per integrationem et eliminationem successivam expressionibus ipsius x_n per x, x_1, ... x_{n-1} atque Constantem Arbitrariam α; ipsius x_{n-1} per x, x_1, ... x_{n-2} atque Constantes Arbitrarias α, α_1 etc., denique ipsius x_2 per x, x_1 atque Constantes Arbitrarias α, α_1, ... α_{n-2}, dabitur aequatio inter x et x_1 per formulam,

$$\int\left(\frac{\partial x_n}{\partial\alpha}\right)\left(\frac{\partial x_{n-1}}{\partial\alpha_1}\right)\cdots\left(\frac{\partial x_2}{\partial\alpha_{n-2}}\right)M(X_1\,dx - X\,dx_1) = \text{Const.“}$$

In utraque Propositione functiones sub signo integrationis ope aequationum integralium inventarum per x et x_1 exprimendae sunt.

Quod e Multiplicatore aequationum differentialium propositarum eruitur Multiplicator aequationis differentialis, in quam post inventa praeter unum omnia Integralia problema redit, id eo maioris momenti est, quia huius ultimae aequationis differentialis primi ordinis inter duas variabiles valde latere potest Multiplicator, dum systematis aequationum differentialium propositarum sponte se offert. Veluti quod in gravissimis quaestionibus evenit, si ipsarum X, X_1 etc. expressiones ita sunt comparatae, ut identice habeatur,

$$\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\cdots\cdot+\frac{\partial X_n}{\partial x_n}=0,$$

aequationum differentialium propositarum Multiplicator *unitati* aequalis evadit; aequationis autem postremo integrandae Multiplicator secundum antecedentia aequatur Determinanti Functionali, cui valor complicatus competere potest. Casu illo particulari in quatuor Propositionibus antecedentibus ponere licet $M=1$; quod ubi ex gr. in Prop. IV: facimus, emergit haec:

Propositio V.

„Proponantur aequationes differentiales simultaneae,

$$dx:dx\ldots\ldots:dx_n = X:X_1\ldots\ldots:X_n,$$

designantibus X, X_1, etc. variabilium x, x_1 etc. functiones pro quibus identice habeatur,

$$\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\cdots\cdot+\frac{\partial X_n}{\partial x_n}=0;$$

inventis aequationum propositarum $n-1$ Integralibus, $n-1$ Constantes Arbitrarias α, α_1, ... α_{n-2} involventibus, exprimantur X et X_1 atque variabiles x_2, x_3, ... x_n per x, x_1 atque istas Constantes Arbitrarias α, α_1, ... α_{n-2}: erit ultimum Integrale,

$$\int\left(\Sigma\pm\frac{\partial x_2}{\partial\alpha}\cdot\frac{\partial x_3}{\partial\alpha_1}\cdots\cdot\frac{\partial x_n}{\partial\alpha_{n-2}}\right)\left\{X_1dx-Xdx_1\right\}=\text{Const.},$$

ubi expressio sub integrationis signo differentiale completum existit."

Propositionis antecedentis afferam exempla pro $n=2$ et $n=3$.

I. „Proponantur aequationes differentiales

$$dx:dy:dz = X:Y:Z,$$

designantibus X, Y, Z variabilium x, y, z functiones, pro quibus identice fiat,

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0;$$

invento uno Integrali involvente Constantem Arbitrariam α, exprimantur X, Y, z per x, y, α, erit alterum Integrale,

$$\int \frac{\partial z}{\partial \alpha} \{Y dx - X dy\} = \text{Const.}"$$

II. „Proponantur aequationes differentiales

$$dt : dx : dy : dz = T : X : Y : Z,$$

designantibus T, X, Y, Z variabilium t, x, y, z functiones, pro quibus identice fiat,

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0,$$

inventis duobus Integralibus involventibus Constantes Arbitrarias α et β, exprimantur T, X, y, z per t, x, α, β; erit tertium Integrale,

$$\int \left(\frac{\partial y}{\partial \alpha} \cdot \frac{\partial z}{\partial \beta} - \frac{\partial y}{\partial \beta} \cdot \frac{\partial z}{\partial \alpha} \right) (X dt - T dx) = \text{Const.}"$$

Quae exempla non sine molesto calculo verificantur.

Quibus casibus Multiplicator aequationum differentialium per aequationes integrales *particulares* reductarum ex aequationum differentialium propositarum Multiplicatore eruitur. Principium ultimi Multiplicatoris sine Determinantium adiumento comprobatum.

§. 12.

Si aequationes integrales, aequationibus differentialibus reducendis adhibitae, sunt particulares, in genere non licet Multiplicatorem aequationum differentialium reductarum e Multiplicatore propositarum deducere. In Prop. II. §. 10., quae docet quomodo aequationum differentialium propositarum et reductarum Multiplicatores a se invicem pendeant, possunt quidem Constantibus Arbitrariis quibus Integralia afficiuntur valores particulares tribui: supponitur autem ipsa cognita esse aequationum differentialium propositarum Integralia generalia. Quae tamen suppositio necessaria non est. Etenim si aequationes integrales reductioni adhibendae alia post aliam investigantur, sufficit unamquamque aequationem integralem inventam ita comparatam esse, ut differentiata per aequationes differentiales propositas identica reddatur, simul omnibus *ipsam praecedentibus* aequationibus integralibus accitis. Neque vero propositum succederet si ex aequationibus integralibus reductioni adhibitis duae pluresve ita comparatae essent, ut quaeque earum differentiata per aequationes differentiales propositas identica reddi non possit nisi simul omnes reliquae aequationes integrales, nullo ordine observato, in auxilium vocentur.

Antecedentia cum e formulis traditis patent tum ope propositionis elementaris directe demonstrantur, quoties aequationes integrales alia post aliam inventae ad variabiles successive eliminandas adhibentur. Sit enim aequationum differentialium propositarum primum Integrale inventum,

$$F = \alpha;$$

cujus ope e quantiiatibus X, X_1, ... X_{n-1} eliminetur x_n. Ponendo $m=1$ in Prop. II §. 10. sequitur, *Multiplicatorem aequationum differentialium reductarum,*

$$1. \quad dx : dx_1 \ldots . : dx_{n-1} = X : X_1 \ldots . : X_{n-1},$$

aequari Multiplicatori aequationum differentialium propositarum diviso per $\frac{\partial F}{\partial x_n}$ *sive quantitati*

$$\frac{M}{\frac{\partial F}{\partial x_n}},$$

in qua variabilis x_n *per aequationem* $F=\alpha$ *éliminanda est.* Constans α in hac propositione fundamentali arbitraria est ideoque valor ei quicunque tribui potest particularis.

Tributo in functionibus X, X_1, ... X_{n-1} Constanti α quam implicant valore particulari, sit aequationum (1.) Integrale,

$$F_1 = \alpha_1.$$

Quod non erit Integrale aequationum differentialium propositarum. Quippe aequatio $dF_1=0$ per aequationes differentiales propositas identica non redditur nisi simul Constans α ubique functioni F aequatur. Quae Constantis α eliminatio ubi fit in functione F_1, aequatio $F_1=\alpha_1$ evadit Integrale aequationum differentialium propositarum. Sed ea Constantis α eliminatio fieri non potest si ei in aequationibus differentialibus reductis (1.) tribuitur valor particularis, neque igitur eo casu ex aequationum differentialium reductarum Integrali Integrale propositarum restituere licet.

Eliminata x_{n-1} ope aequationis $F_1=\alpha_1$, obtinentur e (1.) aequationes differentiales denuo reductae,

$$2. \quad dx : dx_1 \ldots . dx_{n-2} = X : X_1 \ldots . : X_{n-2}.$$

Quarum Multiplicator secundum eandem regulam derivatur e Multiplicatore aequationum (1.), atque hic e Multiplicatore aequationum differentialium propositarum erutus est, videlicet dividendo per $\frac{\partial F_1}{\partial x_{n-1}}$, unde prodit aequationum (2.) Multiplicator,

$$\frac{M}{\frac{\partial F_1}{\partial x_n}\cdot\frac{\partial F_1}{\partial x_{n-1}}},$$

quae quantitas variabilibus x_n et x_{n-1} per aequationes $F=\alpha$, $F_1=\alpha_1$ eliminatis solarum $x, x_1, \ldots. x_{n-2}$ functio evadit. Unde aequationum differentialium (2.) erutus est Multiplicator, quamquam reductio facta est per duas aequationes $F=\alpha$, $F_1=\alpha_1$, quarum tantum altera est aequationum differentialium propositarum Integrale, altera non est neque ad tale revocari potest, si Constanti α tributus est valor particularis.

Rursus tributo Constanti α_1 valore particulari quocunque, aequationum (2.) quaeratur Integrale, quo invento aequationes differentiales (2.) ulterius reduci possunt, reductarumque per eandem regulam constabit Multiplicator. Sic pergendo successive eruantur m aequationes integrales,

$$3.\quad F=\alpha,\quad F_1=\alpha_1,\quad \ldots.\quad F_{m-1}=\alpha_{m-1},$$

in quibus $\alpha, \alpha_1, \ldots. \alpha_{m-1}$ sint Constantes particulares quaecunque; quarum aequationum integralium ope revocatis $X, X_1, \ldots. X_{n-m}$ ad solarum x, $x_1, \ldots. x_{n-m}$ functiones, aequationum differentialium ad quas successiva eliminatione pervenitur,

$$4.\quad dx:dx_1\ldots.:dx_{n-m} = X:X_1\ldots.:X_{n-m},$$

eruitur Multiplicator,

$$5.\quad \frac{M}{\frac{\partial F}{\partial x_n}\cdot\frac{\partial F_1}{\partial x_{n-1}}\cdots\frac{\partial F_{m-1}}{\partial x_{n-m+1}}},$$

quae quantitas et ipsa per aequationes (3.) ad solarum $x, x_1, \ldots. x_{n-m}$ functionem revocanda est. Aequationes (3.) reductionibus successivis inservientes hic ita comparatae sunt ut quaeque $F_i=\alpha_i$ sit Integrale aequationum differentialium,

$$dx:dx_1\ldots:dx_{n-i} = X:X_1\ldots.:X_{n-i},$$

variabilibus $x_n, x_{n-1}, \ldots. x_{n-i+1}$ e $X, X_1, \ldots. X_{n-i}$ eliminatis ope aequationum ipsam $F_i=\alpha_i$ praecedentium,

$$F=\alpha,\quad F_1=\alpha_1,\quad \ldots.\quad F_{i-1}=\alpha_{i-1}.$$

Si $m=n-1$, formula (5.) suppeditat Multiplicatorem aequationis differentialis primi ordinis inter duas variabiles x et x_1,

$$6.\quad X_1dx - Xdx_1 = 0,$$

quae post inventas aequationes integrales,

$$7.\quad F=\alpha,\quad F_1=\alpha_1,\quad \ldots.\quad F_{n-2}=\alpha_{n-2},$$

unica integranda restat. Multiplicatore sic invento,

$$\frac{M}{\frac{\partial F}{\partial x_n}\cdot\frac{\partial F_1}{\partial x_{n-1}}\cdots\frac{\partial F_{n-2}}{\partial x_2}},$$

laeva pars aequationis (6.) evadit differentiale completum, unde eius integratio ad Quadraturas revocatur sive fit ultima aequatio integralis,

$$8.\quad \int \frac{M(X_1 dx - X dx_1)}{\frac{\partial F}{\partial x_n}\cdot\frac{\partial F_1}{\partial x_{n-1}}\cdots\frac{\partial F_{n-2}}{\partial x_2}} = \text{Const.}$$

Qua in formula adiumento aequationum integralium inventarum (7.) quantitates, sub integrationis signo in differentialia dx et dx_1 ductae, per solas x et x_1 exprimendae sunt.

Cum antecedentibus Constantes α, α_1, ... α_{n-2} sint particulares *quaecunque,* earum valorem etiam generalem seu indefinitam servare licet, quo facto formula (8.) redit in Prop. III. §. pr. Vice versa Prop. III. §. pr., in qua designant α, α_1, ... α_{n-2} Constantes Arbitrarias, eum quoque amplectitur casum quo post quamque novam integrationem Constanti Arbitrariae qua afficitur valor tribuitur particularis. Quod intelligitur observando, aequationibus differentialibus Constantes Arbitrarias involventibus, idem earum Integrale obtineri posse, sive ante sive post integrationem Constantibus Arbitrariis illis valores particulares tribuas.

Necessarium non est, ut quaeque nova aequatio integralis inveniatur ut Integrale ipsarum aequationum differentialium ad quas propositae reducuntur eliminato per aequationes integrales ante inventas aequali variabilium numero; generalius ea esse poterit Integrale aequationum differentialium propositarum, per aequationes integrales ante ipsam inventas quocunque modo transformatarum. Aequationum enim differentialium propositarum per Integrale $F=\alpha$ transformatarum sit Integrale $F_1=\alpha_1$; aequationum differentialium propositarum per binas aequationes $F=\alpha$, $F_1=\alpha_1$ transformatarum sit Integrale $F_2=\alpha_2$, per tres aequationes $F=\alpha$, $F_1=\alpha_1$, $F_2=\alpha_2$ transformatarum sit Integrale $F_3=\alpha_3$, et ita porro, ubi Constantes α, α_1 etc. poterunt arbitrariae esse sive particulares quaecunque. Quibus positis, ex aequatione integrali $F=\alpha$ et aequationibus differentialibus propositis sequi debet, $dF_1=0$; unde per aequationem $F=\alpha$ eliminata x_n e functionibus X, X_1, ... X_{n-1}, F_1, fieri debet $F_1=\alpha_1$ Integrale aequationum differentialium,

$$dx : dx_1 \ldots : dx_{n-1} = X : X_1 \ldots : X_{n-1}.$$

Ex aequationibus integralibus $F=\alpha$, $F_1=\alpha_1$ et aequationibus differentialibus propositis sequi debet $dF_2=0$; unde per aequationes $F=\alpha$, $F_1=\alpha_1$ eliminatis x_n et x_{n-1} e functionibus $X, X_1, \dots X_{n-2}$, fieri debet $F_2=\alpha_2$ Integrale aequationum differentialium,

$$dx:dx_1\,\dots:dx_{n-2} = X:X_1\dots:X_{n-2},$$

et ita porro. Generaliter si primum functiones F_1, F_2 etc. ratione illa generaliori qua eas definivi obtinebantur, ac deinde e quaque F_i eliminantur x_n, $x_{n-1}, \dots x_{n-i+1}$ per aequationes $F=\alpha$, $F_1=\alpha_1, \dots F_{i-1}=\alpha_{i-1}$, eaedem functiones F, F_1, F_2 etc. prodeunt quas in formulis (5. et. 8.) consideravi. Ea autem reductione adhibita abit Determinans functionale

$$\Sigma\pm\frac{\partial F}{\partial x_n}\cdot\frac{\partial F_1}{\partial x_{n-1}}\cdots\frac{\partial F_{m-1}}{\partial x_{n-m+1}}$$

in simplex productum

$$\frac{\partial F}{\partial x_n}\cdot\frac{\partial F_1}{\partial x_{n-1}}\cdots\frac{\partial F_{m-1}}{\partial x_{n-m+1}},$$

quod formulae (5.) denominatorem afficit (§. 3.). Unde si functionibus F, F_1, F_2 etc. generaliorem significatum servare placet, formula (5.) evadere debet,

$$9.\qquad \frac{M}{\Sigma\pm\frac{\partial F}{\partial x_n}\cdot\frac{\partial F_1}{\partial x_{n-1}}\cdots\frac{\partial F_{m-1}}{\partial x_{n-m+1}}},$$

ideoque etiam formula (8.)

$$10.\qquad \int\frac{M\{X_1\,dx-X\,dx_1\}}{\Sigma\pm\frac{\partial F}{\partial x_n}\cdot\frac{\partial F_1}{\partial x_{n-1}}\cdots\frac{\partial F_{n-2}}{\partial x_2}} = \text{Const.}$$

Definitio functionum F, F_1 etc. amplectitur casum quo omnes aequationes $F_i=\alpha_i$ sunt ipsarum aequationum differentialium Integralia generalia. Unde e simplice propositione elementari tradita derivatur principium ultimi Multiplicatoris, si reductio ad aequationem differentialem primi ordinis inter duas variabiles per Integralia generalia fit, simulque monstrantur casus maxime generales quibus invenire liceat ultimum Multiplicatorem, etsi aequationes integrales reductioni adhibitae sint particulares.

Addam demonstrationem propositionis fundamentalis qua antecedentibus vidimus principium ultimi Multiplicatoris via maxime elementari adeoque absque ullo Determinantium adiumento superstrui.

Propositio.

„*Sit F solutio quaecunque aequationis*

$$X\frac{\partial F}{\partial x}+X_1\frac{\partial F}{\partial x_1}\dots+X_n\frac{\partial F}{\partial x_n}=0,$$

exclusa Constante; sit porro M *solutio quaecunque aequationis*

$$\frac{\partial . MX}{\partial x} + \frac{\partial . MX_1}{\partial x_1} \dots + \frac{\partial . MX_n}{\partial x_n} = 0,$$

Constante non exclusa: posito

$$N = \frac{M}{\frac{\partial F}{\partial x_n}},$$

ipsisque N, X, X_1, ... X_{n-1} *per* x, x_1, ... x_{n-1}, F *expressis, fit* N *solutio aequationis*

$$\frac{\partial . NX}{\partial x} + \frac{\partial . NX_1}{\partial x_1} \dots + \frac{\partial . NX_{n-1}}{\partial x_{n-1}} = 0."$$

Demonstratio.

Ponatur

$$\frac{\partial F}{\partial x_n} = u;$$

differentiando variabilis x_n respectu aequationem identicam,

$$X\frac{\partial F}{\partial x} + X_1\frac{\partial F}{\partial x_1} \dots + X_n\frac{\partial F}{\partial x_n} = 0,$$

prodit,

$$X\frac{\partial u}{\partial x} + X_1\frac{\partial u}{\partial x_1} \dots + X_n\frac{\partial u}{\partial x_n}$$
$$+ \frac{\partial X}{\partial x_n}.\frac{\partial F}{\partial x} + \frac{\partial X_1}{\partial x_n}.\frac{\partial F}{\partial x_1} \dots + \frac{\partial X_n}{\partial x_n}.\frac{\partial F}{\partial x_n} = 0.$$

Innuendo uncis quibus differentialia partialia includantur exhiberi X, X_1 etc. per x, x_1, ... x_{n-1}, F, fit

$$\frac{\partial X_i}{\partial x_n} = \left(\frac{\partial X_i}{\partial F}\right)\frac{\partial F}{\partial x_n} = \left(\frac{\partial X_i}{\partial F}\right)u.$$

Quam formulam in aequatione praecedente substituendo atque per u dividendo prodit,

$$X\frac{\partial \log u}{\partial x} + X_1\frac{\partial \log u}{\partial x_1} \dots + X_n\frac{\partial \log u}{\partial x_n}$$
$$+ \left(\frac{\partial X}{\partial F}\right)\frac{\partial F}{\partial x} + \left(\frac{\partial X_1}{\partial F}\right)\frac{\partial F}{\partial x_1} \dots + \left(\frac{\partial X_n}{\partial F}\right)\frac{\partial F}{\partial x_n} = 0.$$

Haec formula detrahatur de sequente, quae ex ea qua M definitur fluit,

$$X\frac{\partial \log M}{\partial x} + X_1\frac{\partial \log M}{\partial x_1} \dots + X_n\frac{\partial \log M}{\partial x_n}$$
$$+ \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} \dots + \frac{\partial X_n}{\partial x_n} = 0,$$

simulque observetur haberi pro indicis i valoribus 1, 2, ... $n-1$,

$$\frac{\partial X_i}{\partial x_i} = \left(\frac{\partial X_i}{\partial x_i}\right) + \left(\frac{\partial X_i}{\partial F}\right)\frac{\partial F}{\partial x_i},$$

prodit ponendo $\frac{M}{u} = N$,

$$X\frac{\partial \log N}{\partial x} + X_1\frac{\partial \log N}{\partial x_1} \dots + X_n\frac{\partial \log N}{\partial x_n}$$
$$+ \left(\frac{\partial X}{\partial x}\right) + \left(\frac{\partial X_1}{\partial x_1}\right) \dots + \left(\frac{\partial X_{n-1}}{\partial x_{n-1}}\right) = 0.$$

Fit autem

$$X\frac{\partial \log N}{\partial x} + X_1\frac{\partial \log N}{\partial x_1} \dots + X_n\frac{\partial \log N}{\partial x_n}$$
$$= X\left(\frac{\partial \log N}{\partial x}\right) + X_1\left(\frac{\partial \log N}{\partial x_1}\right) \dots + X_{n-1}\left(\frac{\partial \log N}{\partial x_{n-1}}\right)$$
$$+ \frac{\partial \log N}{\partial F}\left\{X\frac{\partial F}{\partial x} + X_1\frac{\partial F}{\partial x_1} \dots + X_n\frac{\partial F}{\partial x_n}\right\}$$
$$= X\left(\frac{\partial \log N}{\partial x}\right) + X_1\left(\frac{\partial \log N}{\partial x_1}\right) \dots + X_{n-1}\left(\frac{\partial \log N}{\partial x_{n-1}}\right),$$

aggregato in $\left(\frac{\partial \log N}{\partial F}\right)$ ducto identice evanescente. Unde aequatio antecedens sic quoque exhiberi potest:

$$X\left(\frac{\partial \log N}{\partial x}\right) + X_1\left(\frac{\partial \log N}{\partial x_1}\right) \dots + X_{n-1}\left(\frac{\partial \log N}{\partial x_{n-1}}\right)$$
$$+ \left(\frac{\partial X}{\partial x}\right) + \left(\frac{\partial X_1}{\partial x_1}\right) \dots + \frac{\partial X_{n-1}}{\partial x_{n-1}} = 0,$$

quae per N multiplicata suppeditat,

$$\left(\frac{\partial . NX}{\partial x}\right) + \left(\frac{\partial . NX_2}{\partial x_1}\right) \dots + \left(\frac{\partial . NX_{n-1}}{\partial x_{n-1}}\right) = 0,$$

quae est formula demonstranda.

Vidimus supra, propositione antecedente iteratis vicibus adhibita erui aequationum differentialium reductarum Multiplicatorem e Multiplicatore propositarum. Sed ad hunc finem non necesse est ut hic ipse cognoscatur sed sufficit eius cognoscere valorem quem per aequationes integrales reductioni adhibitas induere potest. Si problema ad aequationem differentialem primi ordinis inter x et x_1 revocatum est, definitur M aequationibus,

$$11. \quad \frac{d \log M}{dx} = -\frac{1}{X}\left\{\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} \dots + \frac{\partial X_n}{\partial x_n}\right\} X_1 dx = X dx_1,$$

in quibus *post differentiationes partiales factas* eliminandae sunt x_2, x_3, ... x_n. Si aequationes integrales, quarum ope reductiones et eliminationes propositae operantur, particulares sunt, evenire potest ut e formulis (11.) eruatur valor ipsius M in principio ultimi Multiplicatoris requisitus, neque tamen inveniri queat ipsius M valor generalis sive ipsarum aequationum differentialium propositarum Multiplicator. Directe aequationis differentialis,

$$X_1 dx - X dx_1 = 0,$$

definitur Multiplicator P per formulam,

$$12. \quad \frac{d \log P}{dx} = -\frac{1}{X}\left\{\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1}\right\},$$

in cuius dextra parte X et X_1 ***ante differentiationes partiales transigendas*** per solas x et x_1 exprimendae sunt. Potest autem evenire ut via non pateat qua ipsum P e (12.) eruatur, dum ipsius M determinatio per formulam (11.) in promtu est. Quae adeo, nullis cognitis aequationibus integralibus, in amplis gravissimisque problematis succedit, unde pro quibuscunque aequationibus integralibus reductioni adhibitis sive completis sive dicta ratione inventis particularibus ultimus Multiplicator constat.

De usu Multiplicatoris in integrandis systematis quibusdam aequationum differentialium specialibus.

§. 13.

Systema aequationum differentialium propositarum ita comparatum esse potest ut ultima Integratio sponte in Quadraturam redeat. Quod evenit si unius variabilis differentiale tantum, non ipsa in aequationibus differentialibus invenitur. Ponamus ipsam x esse variabilem a qua simul omnes functiones vacuae sint $X, X_1, \dots X_n$: redire constat integrationem n aequationum differentialium inter $n+1$ variabiles,

$$1. \quad dx : dx_1 : dx_2 \dots : dx_n = X : X_1 : X_2 \dots : X_n,$$

in integrationem $n-1$ aequationum differentialium inter n variabiles unamque Quadraturam. Integratis enim aequationibus,

$$2. \quad dx_1 : dx_2 \dots : dx_n = X_1 : X_2 \dots : X_n,$$

quae sunt $n-1$ aequationes differentiales inter n variabiles $x_1, x_2, \dots x_n$, exhiberi poterunt variabiles $x_1, x_2, \dots x_n$ per earum unam veluti x_1: unde, expressa $\frac{X}{X_1}$ per x_1, dabit simplex Quadratura ipsius x valorem,

$$3. \quad x = \int \frac{X dx_1}{X_1} + \text{Const.}$$

Iam cognito aequationum differentialium (1.) Multiplicatore quaeritur, quemnam ex eo fructum ad integrationem perficiendam percipere liceat, cum ultima integratio sua sponte in Quadraturam redeat. Quod ut cognoscatur, inter duo casus distinguendum erit, prout datus aequationum differentialium (1.) Multiplicator a variabili x afficiatur sive non afficiatur.

Aequationum differentialium (2.) systema vocabo *proprium,* quo distinguatur a systemate *proposito* aequationum differentialium (1.), cuius integratio componitur ex integratione systematis proprii et Quadratura. Si datus systematis propositi Multiplicator M et ipse a variabili x vacuus est, idem erit systematis proprii Multiplicator. Tum enim evanescente termino $\frac{\partial . MX}{\partial x}$, satisfaciet aequationum differentialium (1.) Multiplicator aequationi,

$$\frac{\partial . MX_1}{\partial x_1} + \frac{\partial . MX_2}{\partial x_2} \dots + \frac{\partial . MX_n}{\partial x_n} = 0,$$

eadem autem aequatione definitur aequationum differentialium (2.) Multiplicator. Quoties igitur datus systematis propositi (1.) Multiplicator et ipse variabili x vacat, systematis proprii ultima integratio ad Quadraturas revocari potest, sive quod idem est, ***systematis aequationum differentialium propositarum duae ultimae integrationes per Quadraturas absolvuntur.***

Vice versa si datur systematis proprii (2.) Multiplicator N, qui erit solarum variabilium x_1, x_2, ... x_n functio, idem erit systematis propositi (1.) Multiplicator. Evanescente enim termino $\frac{\partial . NX}{\partial x}$, functio N, quae huic aequationi satisfacere debet,

$$0 = \frac{\partial . NX_1}{\partial x_1} + \frac{\partial . NX_2}{\partial x_2} \dots + \frac{\partial . NX_n}{\partial x_n},$$

etiam huic satisfaciet qua systematis propositi Multiplicator definitur,

$$0 = \frac{\partial . NX}{\partial x} + \frac{\partial . NX_1}{\partial x_1} \dots + \frac{\partial . NX_n}{\partial x_n}.$$

Inventis autem omnibus systematis proprii Integralibus,

$$4. \quad f_1 = \alpha_1, \quad f_2 = \alpha_2, \quad \dots \quad f_{n-1} = \alpha_{n-1},$$

ubi Constantes Arbitrariae α_1 etc. dextram aequationum partem occupant, erit aequationum (2.) Multiplicator,

$$5. \quad N = \frac{1}{X_n} \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_{n-1}}{\partial x_{n-1}}.$$

Qui igitur systematis quoque propositi Multiplicator erit. Unde si systematis propositi datur Multiplicator M, variabilem x implicans, simulque systema proprium complete integratum est, duo innotescunt systematis propositi Multiplicatores M et N. Quibus cognitis, secundum §. 4. systematis propositi constabit Integrale,

$$6. \quad \frac{N}{M} = \frac{1}{MX_n} \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_{n-1}}{\partial x_{n-1}} = \text{Const.}$$

Quo Integrali dabitur x per $x_1, x_2, \dots x_n$, sive ope Integralium (4.) expressis $x_2, x_3, \dots x_n$ per x_1, dabitur x per x_1. Unde ***si innotescit systematis propositi Multiplicator variabili x affectus, post systematis proprii integrationem completam, non amplius opus erit Quadratura, quam formula*** (3.) ***poscebat ad inveniendum ipsius x valorem per x_1 expressum.***

Fieri potest ut solo cognito systematis propositi Multiplicatore variabili x affecto, absque ulla integratione eruantur systematis proprii unum plurave Integralia. Expressa enim per (4.) quantitate $\frac{X}{X_1}$ per $x_1, \alpha_1, \alpha_2, \dots \alpha_{n-1}$, in functione

$$\int \frac{X \partial x_1}{X_1},$$

post factam integrationem, Constantium α_1, α_2 etc. loco restituamus functiones f_1, f_2 etc., quo facto prodeat variabilium $x_1, x_2, \dots x_n$ functio

$$\xi = \int \frac{X\, d x_1}{X_1}:$$

erit e (3.), designante α_n novam Constantem Arbitrariam,

$$x - \xi = \alpha_n$$

systematis propositi Integrale. Sit rursus variabilium $x_1, x_2, \dots x_n$ functio N systematis proprii ideoque etiam systematis propositi Multiplicator, erit secundum §. 4. expressio generalis Multiplicatoris systematis propositi,

$$M = \Pi(x-\xi, f_1, f_2, \dots f_{n-1}).N.$$

Cognito igitur valor ipsius M, variabili x affecto, erit $\frac{\partial \log M}{\partial x}$ ipsarum $x-\xi$, $f_1, f_2, \dots f_{n-1}$ functio,

$$\frac{\partial \log M}{\partial x} = \Phi(x-\xi, f_1, f_2, \dots f_{n-1}).$$

Unde ponendo

$$7. \quad \frac{\partial \log M}{\partial x} = u,$$

atque ex hac aequatione quaerendo ipsius x valorem per $u, x_1, x_2, \dots x_n$ expressum, prodit

$$x = \xi + \psi(u, f_1, f_2, \dots f_{n-1}),$$

designante ψ certam ipsarum $u, f_1, f_2, \dots f_{n-1}$ functionem. Quaerendo igitur e (7.) ipsius x valorem per $u, x_1, x_2, \dots x_n$ expressum, atque in ea expressione ipsius u loco ponendo varios valores constantes arbitrarios, differentiae quantitatum provenientium erunt solarum $f_1, f_2, \dots f_n$ functiones, ideoque Constantibus Arbitrariis aéquiparatae suppeditabunt systematis proprii

Integralia. Methodus hic tradita semper succedit si non tantum M sed etiam $\frac{\partial \log M}{\partial x}$ ipsam x involvit atque ψ non solius u vel Φ non solius $x-\xi$ functio est. Quoties autem $\Phi = \frac{\partial \log M}{\partial x}$ solius $x-\xi$ functio est, erit $\frac{\partial \Phi}{\partial x}$ ipsius Φ functio. Unde *e systematis propositi Multiplicatore cognito M semper deducere licet absque integratione systematis proprii unum plurave Integralia, quoties $\frac{\partial^2 \log M}{\partial x^2}$ non ipsius $\frac{\partial \log M}{\partial x}$ functio est.* Similiter demonstratur, *cognito systematis propositi Integrali, variabili x affecto, $v=\alpha$, designante α Constantem Arbitrariam, ex eo semper derivari posse unum plurave systematis proprii Integralia, nisi $\frac{\partial v}{\partial x}$ ipsius v functio sit.* Nam cum esse debeat v quantitatum $x-\xi$, f_1, f_2, ... f_{n-1} functio, ex aequatione $v=\alpha$ sequitur huiusmodi

$$x = \xi+\psi(\alpha, f_1, f_2, \dots f_{n-1});$$

unde eruendo e $v=\alpha$ ipsius x valore in eoque ponendo ipsius α loco varios valores constantes arbitrarios, differentiae expressionum provenientium Constantibus Arbitrariis aequiparatae suppeditabunt systematis proprii Integralia.

Ut habeatur exemplum quo systematis propositi Multiplicator variabili x affectus innotescit ideoque post systematis proprii integrationem completam ipsa x per $x_1, x_2, \dots x_n$ absque Quadratura exprimitur, ponamus $X=1$ simulque fieri

$$\frac{\partial X_1}{\partial x_1}+\frac{\partial X_2}{\partial x_2}\dots+\frac{\partial X_n}{\partial x_n} = c,$$

designante c quantitatem constantem; quod inter alia evenit, si X_1, X_2 etc. variabilium x_1, x_2 etc. functiones sunt lineares. Dabitur systematis propositi Multiplicator per formulam

$$\frac{d \log M}{dx}+c = 0, \quad \text{unde} \quad M = e^{-cx}.$$

Hinc sequitur e (6.) sumendo logarithmos,

$$x = -\frac{1}{c}\log\left(\frac{1}{X_n}\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\dots\frac{\partial f_{n-1}}{\partial x_{n-1}}\right)+\text{Const.}$$

Cognitione igitur Multiplicatoris in hoc exemplo non reductionem aequationis differentialis ad Quadraturas sed Quadraturam lucramur.

Antecedentibus demonstratum est, *si aequationum differentialium* (1.), *in quibus X, X_1 etc. solarum x_1, x_2, ... x_n functiones sunt, detur Multiplicator et ipse variabili x vacans, duas postremas integrationes per Quadraturam absolvi; si Multiplicator variabili x afficiatur, ultimam aequationem integralem ipsam sine Quadratura obtineri.* Quae propositio sic amplificatur.

Ponamus functiones X_{m+1}, X_{m+2}, ... X_n vacuas esse a variabilibus x, x_1, ... x_m, simulque X, X_1, ... X_m nisi ab iisdem variabilibus vacuae sunt, certe satisfacere conditioni,

$$7.\quad \frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}\dots\dots+\frac{\partial X_m}{\partial x_m}=0.$$

Eo casu aequationes differentiales propositae (1.) sic tractabuntur, ut primum aequationum differentialium inter solas x_{m+1}, x_{m+2}, ... x_n locum habentium,

$$8.\quad dx_{m+1}:dx_{m+2}\dots\dots:dx_n = X_{m+1}:X_{m+2}\dots\dots:X_n,$$

quaerantur Integralia,

$$9.\quad f_1=\alpha_1,\quad f_2=\alpha_2,\quad \dots\dots\quad f_{n-m-1}=\alpha_{n-m-1},$$

eorumque ope exprimantur variabiles x_{m+1}, x_{m+2}, ... x_n per earum unam x_{m+1}; quibus factis superest ut integrentur aequationes differentiales inter ipsas x, x_1, ... x_{m+1} locum habentes,

$$10.\quad dx:dx_1\dots\dots:dx_{m+1} = X:X_1\dots\dots:X_{m+1}.$$

Per conditionem (7.) constat, aequationum differentialium propositarum (1.) Multiplicatorem, a variabilibus x, x_1, ... x_m vacuum, eundem esse aequationum differentialium (8.) Multiplicatorem, et vice versa harum Multiplicatorem ipsarum quoque aequationum differentialium (1.) Multiplicatorem esse. Designante enim M quantitatem a variabilibus x, x_1, ... x_m vacuam, sequitur e (7.),

$$\frac{\partial.MX}{\partial x}+\frac{\partial.MX_1}{\partial x_1}\dots\dots+\frac{\partial.MX_m}{\partial x_m}=0,$$

unde pro eiusmodi ipsius M valore conditio ut M aequationum (1.) sit Multiplicator,

$$\frac{\partial.MX}{\partial x}+\frac{\partial.MX_1}{\partial x_1}\dots\dots+\frac{\partial.MX_n}{\partial x_n}=0,$$

convenit cum conditione ut M aequationum (8.) Multiplicator sit,

$$\frac{\partial.MX_{m+1}}{\partial x_{m+1}}+\frac{\partial.MX_{m+2}}{\partial x_{m+2}}\dots\dots+\frac{\partial.MX_n}{\partial x_n}=0.$$

Aequationum differentialium* (10.) *semper assignare licet Multiplicatorem. Nam cum ipsarum x_{m+2}, x_{m+3}, ... x_n expressiones per x_{m+1} e (9.) petitae ab ipsis x, x_1, ... x_m vacuae sint, conditio (7.) valebit etiam post harum expressionum substitutionem. Qua substitutione cum X_{m+1} in solius x_{m+1} functionem abeat, valebit etiam aequatio (7.), si loco ipsarum X_i ponitur $\frac{X_i}{X_{m+1}}$. Unde sequitur, ***aequationum differentialium* (10.) *Multiplicatorem esse*** $\frac{1}{X_{m+1}}$. Qua de re ***aequationum differentialium* (10.) *ultima integratio semper solis Quadraturis absolvitur.***

Si datur Multiplicator aequationum differentialium propositarum (1.), variabilibus $x, x_1, \ldots . x_m$ non affectus, idem erit aequationum (8.) Multiplicator, ideoque eo casu cum aequationum (8.) tum aequationum (10.) ultima integratio Quadraturis absolvitur. Iam vero sit aequationum differentialium propositarum (1.) datus Multiplicator M variabilibus $x, x_1, \ldots . x_m$ affectus. Inventis aequationum differentialium (8.) Integralibus (9.), earum fit Multiplicator

$$N = \frac{1}{X_{m+1}} \Sigma \pm \frac{\partial f_1}{\partial x_{m+2}} \cdot \frac{\partial f_2}{\partial x_{m+3}} \cdots \frac{\partial f_{n-m-1}}{\partial x_n}$$

idemque ex antecedentibus fit Multiplicator aequationum differentialium propositarum (1.). Quarum igitur cognitis duobus Multiplicatoribus M et N, datur absque Quadratura Integrale

$$\frac{N}{M} = \frac{1}{MX_{m+1}} \Sigma \pm \frac{\partial f_1}{\partial x_{m+2}} \cdot \frac{\partial f_2}{\partial x_{m+3}} \cdots \frac{\partial f_{n-m-1}}{\partial x_n} = \text{Const.}$$

Quod substituendo ipsarum $x_{m+2}, x_{m+3}, \ldots . x_n$ valores per x_{m+1} exhibitos in aequationum (10.) Integrale abit. Harum aequationum praeterea vidimus ultimam integrationem Quadraturis absolvi. Unde propositis aequationibus differentialibus,

$$dx : dx_1 \ldots . : dx_n = X : X_1 \ldots . : X_n,$$

in quibus functiones $X_{m+1}, X_{m+2}, \ldots . X_n$ variabilibus $x, x_1, \ldots . x_m$ vacant simulque fit

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} \cdots + \frac{\partial X_m}{\partial x_m} = 0,$$

si datur Multiplicator et ipse variabilibus $x, x_1, \ldots . x_m$ vacans, duae integrationes per Quadraturas absolvuntur; si vero datus Multiplicator variabilibus $x, x_1, \ldots . x_n$ afficitur, una aliqua aequatio integralis absque omni Quadratura constabit atque altera integratio Quadraturis efficietur.

Antecedentia exemplo esse possunt, ad aequationes differentiales integrandas e Multiplicatoris cognitione semper fructum aliquem percipi, etsi ultima integratio absque eius auxilio Quadraturis absolvi possit. Neque nessarium est ut in antecedentibus aequationes (4.) sint Integralia ipsarum aequationum differentialium (2.), vel aequationes (9.) sint Integralia ipsarum aequationum differentialium (8.). Nam secundum ea quae §. 12. tradidi, Constanti Arbitrariae post quamque novam integrationem accedenti valorem tribuere licet particularem quemcunque. Sufficit ut quaelibet aequatio $f_i = \text{Const.}$ sit Integrale aequationum differentialium quocunque modo transformatarum per aequationes integrales ante eam inventas,

$$f_1 = \alpha_1, \quad f_2 = \alpha_2, \quad \ldots . \quad f_{i-1} = \alpha_{i-1},$$

in quibus ad dextram habentur quantitates constantes quaecunque particulares.