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Boundary value problems for the stationary Vlasov-Maxwell system

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Abstract. The Vlasov-Maxwell equations provide a kinetic description of the flow of particles in a self-consistent electromagnetic field. The aim of this paper is to prove the existence of stationary solutions for boundary value problems with arbitrary large data. The main idea consists in using explicit upper solutions for the Vlasov equation that allow to bound the particles concentration and flux. A key point is that the electric field is repulsive. The mathematical analysis is first given for the relativistic Vlasov-Maxwell system. Next, the results are extended to classic mechanics, systems with several species of particles and Boltzmann-Vlasov-Poisson problems.

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1. Introduction

A lot of studies in physics and applied physics are based on Vlasov-Maxwell equations. These equations model the transport processes of charged particles in a selfconsistent electromagnetic field. A few domains of application are: particle accelerators, electron guns, semiconductors and so on ... Recently the works of R.J. Diperna and P.L. Lions [7, 8] have allowed significant progress in the mathematical studies of kinetic equations. The have proved existence of solutions of the Vlasov-Maxwell system for the Cauchy problem in a free space. However it remains in this field a lot of difficult problems.

To my knowledge, there are few mathematical works on boundary value problem for stationary kinetic equations. P. Degond has given in [2] particular solutions for the relativistic Vlasov-Maxwell system. C. Greengard and P.A. Raviart [11] have studied a one dimensional problem for the Vlasov-Poisson equations. They obtain the existence of stationary solutions and give some conditions on the data that guarantee uniqueness. The aim of this paper is to provide stationary solutions of boundary value problems for Vlasov-Maxwell systems with arbitrary large data in any kind of geometris with minor restrictions on the regularity of the boundaries. Some results and the basic ideas have been presented in [13]. This work is a first step in the mathematical study of boundary value problems for plasma physics. Thus in a forthcoming paper [4] we use some of the techniques developped in the present paper to provide an asymptotic analysis of the Vlasov-Poisson system. It gives a mathematical proof of the Child-Langmuir law for cylinders and spheres. This result has already been obtained in [5] in the case of the plane diode.

This work begins with the study of linear stationary Vlasov equations. The Vlasov problem reads

$$(1)\qquad \begin{cases} v(p).\nabla_x f(x,p) + F(x,p).\nabla_p f(x,p) = 0; \quad x \in \Omega; \quad p \in \mathbb{R}^3 \\ f(x,p) = f_0(x,p); \ (x,p) \in \Sigma^-. \end{cases}$$

The function $f(x, p)$ denotes the particle distribution depending on the position x and the momentum p. Ω is a smooth bounded domain of $\mathbb{R}^3$ and Σ^- ist the subset of the boundary $\partial\Omega \times \mathbb{R}^3$ where the velocities are pointing inwards

$$(2)\qquad \Sigma^{\pm} = \{(x,p) \in \partial\Omega \times \mathbb{R}^3;\ v(p).\nu(x) \gtrless 0\}.$$

Above, we have denoted by $\nu(x)$ the unit outward normal to $\partial\Omega$ at x. The velocity $v(p)$ is the gradient of the given energy function $\varepsilon(p)$ of the particles

$$(3)\qquad v(p) = \nabla_p \varepsilon(p).$$

The force field $F(x, p)$ reads

$$(4)\qquad F(x,p) = q(-\nabla_x \phi(x) + v(p) \wedge B(x)).$$

The electrostatic potential ϕ and the magnetic field B are first assumed to be known. The constant q is the elementary charge of the particles.

Since the value zero lies in the spectrum of Vlasov operators, there is no uniqueness for the boundary value problem (1). We prove existence by considering the following perturbed problem

$$(5)\qquad \begin{cases} \alpha f_\alpha(x,p) + v(p).\nabla_x f_\alpha(x,p) + F(x,p).\nabla_p f_\alpha(x,p) = 0; \quad x \in \Omega; \quad p \in \mathbb{R}^3 \\ f_\alpha(x,p) = f_0(x,p); \ (x,p) \in \Sigma^-. \end{cases}$$

Results due to C. Bardos [1] on first order hyperbolic equations guarantee that the problem (5) is well posed for $\alpha > 0$ and for a C^1 force field F. We show that the sequence f_α converges towards the solution of the Vlasov problem (1) that is minimal for the order relation $f \leq g$ among the set of solutions of (1). Next we give a weak formulation of (1). This formulation allows to get rid of restrictions on the regularity of the force field F. This second section is ended by introducing upper solutions that will be usefull in the following. They read

$$(6)\qquad g(x,p) = G(\varepsilon(p) + q\phi(x)).$$

We point out that the function $\varepsilon(p) + q\phi(x)$ is nothing else than the total energy of the particles which is invariant along their trajectories.

In section 3 the non linear Vlasov-Maxwell system is investigated. From now on the electromagnetic field is selfconsistent. It satisfies the stationary Maxwell equations

(7) $$-\Delta\phi(x) = \frac{q}{\varepsilon_0}\varrho(x); \quad x \in \Omega.$$

(8) $$\nabla \wedge B(x) = \mu_0 q j(x); \quad \nabla . B(x) = 0; \quad x \in \Omega.$$

The constant ε_0 and μ_0 are the vacuum permittivity and permeability. The concentration ϱ and the flux j depend on the particle distribution f through the relations

(9) $$\varrho(x) = \int_{\mathbb{R}^3} f(x,p)\,dp; \quad j(x) = \int_{\mathbb{R}^3} v(p) f(x,p)\,dp; \quad x \in \Omega.$$

In this section we first recall some classical results on the magnetostatic problem (8). We point out that the necessary conditions for the problem to be well posed are

$$\nabla . j = 0; \quad \partial\Omega \text{ is connected.}$$

Then boundary conditions for the stationary Maxwell equations are

(10) $$\phi(x) = \phi_0(x); \quad B(x).\nu(x) = b(x); \quad x \in \partial\Omega.$$

The existence of a solution (f, ϕ, B) for the Vlasov-Maxwell problem (1), (7), (8), (10) with arbitrary large data will be obtained by the application of the Schauder fixed point theorem. Unfortunately the non uniqueness of the solutions for the Vlasov problem does not allow to directly apply the fixed point procedure. Therefore we have to introduce a regularized problem.

Section 4 is devoted to the proof of the existence of solutions for this regularized problem. The a-priori estimates are obtained with the upper solutions (6) of the Vlasov problem. Indeed the maximum principle applied to (7) implies that the electrostatic energy $q\phi$ is uniformly bounded from below. Therefore, if the function G is non increasing, we get with some constant C_0:

(11) $$0 \le f(x,p) \le G(\varepsilon(p) + q\phi(x)) \le G(\varepsilon(p) + C_0).$$

It provides apriori estimates on the density ϱ and the flux j that allow to apply the Schauder theorem. We point out that changing the sign of the right hand side of (7) which would correspond to an attractive (gravitational) potential would not allow to obtain such an estimate any more. In this case, the techniques used in this paper would only provide solutions for small data. We do not focus on this problem because the physical meaning of boundary value problems for gravitational potential is unclear.

In section 5, we investigate other models for which the above analysis is successful. First the Vlasov-Poisson system with several species of particles is investigated. The way to obtain a-priori estimates is a little more complicated than in the preceeding sections. However, once they have been derived, the techniques are the same. Next we study the Vlasov-Poisson system with reflection conditions for the distribution f and with mixed conditions (Dirichlet and Neumann boundary conditions) for the potential. Slight differences appear in the analysis of the linear Vlasov problem and in the procedure of regularization of the electrostatic potential. Finally the analysis of a kinetic model for semiconductors is performed. A linear Boltzmann operator is involved in the problem. After giving a proof of existence of solutions for the Boltzmann-Vlasov equation, we demonstrate that Maxwellian distributions are upper solutions. This allows to perform the proofs of sections 3 and 4 for the Boltzmann-Vlasov-Poisson system.

We end this paper in section 6 with some remarks. We first point out that the compactness results of velocity averages of [7] which are essential in the analysis of the Cauchy problem for the time dependent Vlasov-Maxwell system have not been used. The reason is that the stationary Maxwell equations are no more hyperbolic but elliptic ones. The regularizing effects are those of the Poisson equation. They allow to control the non linearity $F.\nabla_p f = \nabla_p.(Ff)$. However the regularity results on the integrals of the distribution f with respect to momentums of [7, 10] would be useful in the investigation of problems that involve non linear Boltzmann operators. Next we give some examples of non uniqueness of solutions. To construct these solutions we profit of the difference of charges between two species of particles or between one species and a ionized background (impurities in semiconductors or heavy ions in plasmas). When only one charge is involved, C. Greengard and P.A. Raviart has given in [11] an example of multiple solutions in the one dimensional case and for an entering data which is a Dirac distribution. They have also proved a uniqueness result for entering data which are non increasing with respect to v. However this problem of uniqueness is still open for more complicated geometries.

2. The Vlasov equation

In this section we assume that the force field is given. Then the Vlasov equation (5) is a linear hyperbolic equation. Therefore let us recall classical results on first order hyperbolic equations.

Let Q be a smooth domain of $\mathbb{R}^N$. We denote by $C_b^p(Q)$ the space of functions whose derivatives of order less or equal to p are continuous and bounded and by $C_0^p(Q)$ the space of functions which are moreover compactly supported. Let A be a vector field

$$A \in (C_b^p(Q))^N.$$

We denote $\nu(x)$ the unit outward normal to the boundary Σ of Q at x, Σ^- is the part of

Σ where A is pointing inwards and $d\sigma(x)$ is the superficial measure of Σ. Let us introduce the unbounded operator Λ_p on $L^p(Q)$

$$\Lambda_p\colon D(\Lambda_p) = \{u \in L^p(Q);\ A.\nabla u \in L^p(Q);\ u_{|\Sigma^-} = 0\} \to L^p(Q)$$
$$u \to A.\nabla u.$$

We denote by $\nabla.A$ the divergence of the vector field A. Then we get

Theorem 1. (C. Bardos [1]) *Let* $1 \leq p < \infty$. *The operators* Λ_p *are* $\frac{\omega}{p}$ *maximal accretif with* $\omega = \|\nabla.A\|_{L^\infty(Q)}$. *It follows that for any function* S *in* $L^p(Q)$, *any entering data* u_0 *such that*

$$\int_{\Sigma^-} |A(x).\nu(x)|\,|u_0(x)|^p\,d\sigma(x) < \infty$$

and for any $\lambda > \frac{\omega}{p}$ *the problem*

$$\begin{cases} \lambda u(x) + A(x).\nabla u(x) = S(x); & x \in Q;\\ h_{|\Sigma^-} = u_0 \end{cases}$$

has a unique solution in $L^p(Q)$ *which satisfies the (weak) maximum principle.*

We now apply this result for the Vlasov equation. Let Ω be a smooth open set of $\mathbb{R}^3$. We denote $d\gamma$ its superficial measure. We assume

$$v \in (C_b^1(\mathbb{R}^3))^3 \quad \text{or} \quad v(p) = \frac{p}{m}; \tag{12}$$

$$\phi \in C_b^2(\Omega); \quad B \in (C_b^1(\Omega))^3. \tag{13}$$

In (12) m is the mass of the particles. We point out that (12) is satisfied for a relativistic velocity field. Indeed if c denotes the light speed we get

$$v(p) = \frac{cp}{\sqrt{|p|^2 + m^2c^2}}; \quad v \in (C_b^1(\mathbb{R}^3))^3.$$

Remark. The force field F given by (4) is divergence free with respect to p. It follows that the vector field $(v(p), F(x,p))$ is divergence free with respect to (x,p). Thus we get

$$\omega = \|\nabla_{(x,p)}.(v, F)\|_{L^\infty(\Omega\times\mathbb{R}^3)} = 0.$$

Proposition 1. *Under the hypotheses (12) and (13) the problem*

$$\begin{cases} \alpha f_\alpha(x,p) + v(p).\nabla_x f_\alpha(x,p) + F(x,p).\nabla_p f_\alpha(x,p) = S(x,p); & x \in \Omega;\ p \in \mathbb{R}^3\\ f_\alpha(x,p) = f_0(x,p); \quad (x,p) \in \Sigma^- \end{cases} \tag{14}$$

where F is given by (4) has a unique solution in $L^p(\Omega \times \mathbb{R}^3)$ for any $\alpha > 0$, any source term S in $L^p(\Omega \times \mathbb{R}^3)$ and any entering data f_0 that satisfies

$$\int_{\Sigma^-} |v(p).\nu(x)|\,|f_0(x,p)|^p d\gamma(x)\,\mathrm{d}p < \infty. \tag{15}$$

The solution f satisfies the maximum principle.

Proof. It is a straightforward application of theorem 1 except in the case where $v(p) = p/m$. Although the result is widely known let us sketch a proof. First let us consider compactly supported data. By bounding velocities for sufficiently large p we get solutions. By application of the maximum principle we show that they are compactly supported. Then we a posteriori verify that this solutions satisfy (14). It remains only to approximate any data by compactly supported functions. □

We want to get rid of the absorption term αf_α. This is not straightforward since zero belongs to the spectrum of the operators Λ_p. Indeed we get

Example 1. Non uniqueness for the Vlasov problem. We put $\Omega = \{x; |x| < 1\}$, $\phi(x) = -(1 - |x|^2)$, $q = 1$. We assume that there is an energy $\varepsilon(p)$ which satisfies (3) and $\varepsilon(p) \geq \varepsilon(0)$. For any magnetic field B and any function G in $C_0^1(\mathbb{R})$ which satisfies

$$G(t) = 0 \quad \text{for} \quad t \geq 0; \qquad G(t) \geq 0 \quad \text{for} \quad t \leq 0$$

the functions $f(x,.p) = G(\varepsilon(p) - \varepsilon(0) + \phi(x))$ are all solutions of

$$\begin{cases} v(p).\nabla_x f(x,p) + F(x,p).\nabla_p f(x,p) = 0; & x \in \Omega;\ p \in \mathbb{R}^3 \\ f(x,p) = 0; \quad (x,p) \in \Sigma^-. \end{cases}$$

Nevertheless we get

Theorem 2. *We always assume (12) and (13). Then for any entering data f_0 that belongs to $L^\infty(\Sigma^-)$, the sequence f_α of solutions of*

$$\begin{cases} \alpha f_\alpha(x,p) + v(p).\nabla_x f_\alpha(x,p) + F(x,p).\nabla_p f_\alpha(x,p) = 0; & x \in \Omega;\ p \in \mathbb{R}^3 \\ f_\alpha(x,p) = f_0(x,p); \quad (x,p) \in \Sigma^-. \end{cases} \tag{16}$$

converges a.e. towards the minimal solution f in $L^\infty(\Omega \times \mathbb{R}^3)$ of the stationary Vlasov problem

$$\begin{cases} v(p).\nabla_x f(x,p) + F(x,p).\nabla_p f(x,p) = 0; & x \in \Omega;\ p \in \mathbb{R}^3 \\ f(x,p) = f_0(x,p); \quad (x,p) \in \Sigma^- \end{cases} \tag{17}$$

A solution f of (17) is minimal if for any solutions g of (17) the inequality $|g| \leq |f|$ implies that $g = f$. First we establish a lemma which will be useful in the sequel.

Lemma 1. *Let f_n be a sequence such that*

$$\begin{cases} v(p).\nabla_x f_n(x,p) + F_n(x,p).\nabla_p f_n(x,p) = S_n(x,p); & x \in \Omega;\ p \in \mathbb{R}^3 \\ f_n(x,p) = f_0(x,p); \quad (x,p) \in \Sigma^-. \end{cases}$$

We assume that

$$F_n \to F \text{ in } L^1_{\text{loc}}(\bar{\Omega} \times \mathbb{R}^3);$$
$$S_n \to S \text{ in } L^p_{\text{loc}}(\bar{\Omega} \times \mathbb{R}^3) \text{ weak (or weak * if } p = \infty);$$
$$f_n \to f \text{ in } L^\infty(\Omega \times \mathbb{R}^3) \text{ weak } *.$$

Then the function f is a solution of

$$(18) \qquad \begin{cases} v(p).\nabla_x f(x,p) + F(x,p).\nabla_p f(x,p) = S(x,p); & x \in \Omega;\ p \in \mathbb{R}^3 \\ f(x,p) = f_0(x,p); \quad (x,p) \in \Sigma^-. \end{cases}$$

Proof of lemma 1. Let us introduce a weak formulation of the Vlasov problem (18). We define

$$V = \{\theta \in C^1_0(\mathbb{R}^6);\ \theta_{|\Sigma^+} = 0\}.$$

A function f is a weak solution of (18) if and only if

$$(19) \qquad \begin{cases} \text{for any function } \theta \text{ in } V \\ -\int_{\Omega \times \mathbb{R}^3} f(v.\nabla_x \theta + F.\nabla_p \theta)\,dx\,dp = \int_{\Omega \times \mathbb{R}^3} S\theta\,dx\,dp + \int_{\Sigma^-} v.\nu f_0 \theta\,d\gamma(x)\,dp. \end{cases}$$

In the above formulation we have used that F is divergence free with respect to p. Thus the term $F.\nabla_p f$ is equal to $\nabla_p.(fF)$. This allows to pass to the limit in the nonlinear term $F_n.\nabla_p f_n$. Using this formulation the lemma is now obvious. □

Proof of theorem 2. By treating separately the sequence $f_\alpha^+ = \frac{1}{2}(|f_\alpha| + f_\alpha)$ and $f_\alpha^- = \frac{1}{2}(|f_\alpha| - f_\alpha)$ we may assume that f_0 is non negative. Then the maximum principle implies

$$0 \le f_\alpha \le \|f_0\|_{L^\infty(\Sigma^-)}.$$

Let $g = f_\alpha - f_\beta$, this function satisfies

$$\begin{cases} \alpha g + v.\nabla_x g + F.\nabla_p g = (\beta - \alpha) f_\beta; & x \in \Omega;\ p \in \mathbb{R}^3 \\ g_{|\Sigma^-} = 0. \end{cases}$$

Therefore g has the sign of $\beta - \alpha$ because f_β is non negative. It follows that the sequence f_α is non increasing. Thus we get

$$f_\alpha \to f = \sup_{\alpha > 0} f_\alpha \text{ a.e.} \qquad 0 \le f \le \|f_0\|_{L^\infty(\Sigma^-)}.$$

By lemma 1, f is a solution of (17). Now let g be an other solution of (17). A regularization procedure shows that the function g^+ also satisfies (17) since f_0 is non negative. We put $h = g^+ - f_\alpha$, we get

$$\begin{cases} \alpha h + v.\nabla_x h + F.\nabla_p h = \alpha g^+; & x \in \Omega;\ p \in \mathbb{R}^3 \\ h_{|\Sigma^-} = 0. \end{cases}$$

Thus the function h is non negative and we obtain

$$0 \le f = \sup_{\alpha > 0} f_\alpha \le g^+$$

which concludes. □

Remark. The notion of minimal solutions is not very useful in the sequel. Indeed we are unable to prove a stability result of minimal solutions under perturbations of the force field.

Upper solutions. The maximum principle for the Vlasov equation allows to obtain L^∞ estimates. If f_α is a solution of (14) we get

$$\|f_\alpha\|_{L^\infty(\Omega \times \mathbb{R}^3)} \le \|f_0\|_{L^\infty(\Sigma^-)} + \frac{1}{\alpha} \|S\|_{L^\infty(\Omega \times \mathbb{R}^3)}.$$

But it is not sufficient to bound the concentration and the flux related to the distribution f. Therefore we have to introduce upper solutions that are rapidly decreasing for high momentums. The derivation of explicit solutions of the Vlasov equation is closely related to the existence of invariants of the trajectories (see [2]). We only use the energy invariant.

Proposition 2. *Under the hypothesis (12) and (13), if moreover we assume that the velocity field satisfies (3) for some energy* ε, *the function* $g(x, p) = G(\varepsilon(p) + q\phi(x))$ *satisfies*

$$v.\nabla_x g + F.\nabla_p g = 0; \quad x \in \Omega;\ p \in \mathbb{R}^3$$

for any non negative function G *in* $C^1(\mathbb{R})$. *It follows that for any entering data* f_0 *that verifies*

$$0 \le f_0 \le g; \quad (x, p) \in \Sigma^-$$

the solution f_α *of the Vlasov problem (16) with an absorption term is estimated by*

$$0 \le f_\alpha \le g; \quad x \in \Omega;\ p \in \mathbb{R}^3. \tag{20}$$

Proof. A computation gives

$$v.\nabla_x g + F.\nabla_p g = (v.q\nabla_x \phi + q(-\nabla_x \phi + v \wedge B).\nabla_p \varepsilon)\, G'(\varepsilon + q\phi).$$

Using (3) we get

$$v.\nabla_x g + F.\nabla_p g = (v \wedge B).v\, G'(\varepsilon + q\phi) = 0.$$

The conclusion of the proposition follows from the maximum principle. □

Remark. The result is false for the Vlasov problem (17) since we have no more uniqueness of the solutions. But it follows from theorem 2 that the minimal solutions also satisfies (20). However in the following sections we only use upper solutions for the perturbed Vlasov problem (16).

The assumption of regularity on the force field F (13) is very restrictive. As it has been remarked by Diperna and Lions in [8], only very weak assumptions on the convetive field are needed to obtain solutions for hyperbolic equations of first order. But, then, there is no more result of uniqueness. This last question is closely related to the existence of characteristics (see [8]). Although this result will not be needed in the sequel, we end this paragraph with the corresponding result for the stationary Vlasov problem.

Theorem 3. *We assume that the velocity field satisfies (13) and that the force field verifies*

$$F \in L^1_{\mathrm{loc}}(\bar{\Omega} \times \mathbb{R}^3).$$

Then for any function f_0 in $L^\infty(\Sigma^-)$, there is at least one solution f which belongs to $L^\infty(\Omega \times \mathbb{R}^3)$ of the boundary value problem for the Vlasov equation (17).

Proof. Let F_n be a regularizing sequence of the force field F. The minimal solutions f_n given by theorem 2 are uniformly bounded. We conclude using lemma 1. □

3. The Vlasov-Maxwell system

In this section we restrict our attention to the relativistic case. Thus we have

$$(21) \qquad \varepsilon(p) = c\sqrt{|p|^2 + m^2 c^2}; \quad v(p) = \nabla_p \varepsilon(p) = \frac{cp}{\sqrt{|p|^2 + m^2 c^2}}$$

The Vlasov-Maxwell system reads

$$(22) \qquad v(p).\nabla_x f(x,p) + F(x,p).\nabla_p f(x,p) = 0; \qquad x \in \Omega;\ p \in \mathbb{R}^3,$$

$$(23) \qquad -\Delta_x \phi(x) = \frac{q}{\varepsilon_0}\varrho(x); \qquad x \in \Omega$$

$$(24) \qquad \nabla_x \wedge B(x) = \mu_0 q j(x); \quad \nabla_x . B(x) = 0; \qquad x \in \Omega$$

$$(25) \qquad \varrho(x) = \int_{\mathbb{R}^3} f(x,p)\,dp; \quad j(x) = \int_{\mathbb{R}^3} v(p) f(x,p)\,dp; \quad x \in \Omega$$

$$(26) \qquad F(x,p) = q(-\nabla_x \phi(x) + v(p) \wedge B(x)) \qquad x \in \Omega;\ p \in \mathbb{R}^3.$$

This system is supplemented with the boundary conditions

$$(27) \qquad f(x,p) = f_0(x,p); \qquad (x,p) \in \Sigma^-$$

(28) $\phi(x) = \phi_0(x); \qquad x \in \partial\Omega$

(29) $B(x).\nu(x) = b(x); \qquad x \in \partial\Omega.$

Before analyzing the nonlinear problem, we recall some results on the magnetostatic problem (24), (29).

Lemma 2. *Let Ω be a smooth open set of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is locally compact. Let $\partial\Omega_i$, $i = 1, \ldots, M$, be the components of $\partial\Omega$. Then a function h in $(L^2(\Omega))^3$ reads*

$$h = \nabla \wedge u; \quad u \in (L^2(\Omega))^3$$

if and only if

$$\nabla.h = 0; \quad \int_{\partial\Omega_i} h(x).\nu(x)\,d\gamma(x) = 0; \quad 1 \le i \le M.$$

If these conditions are satisfied, there is a unique solution in $(L^2(\Omega))^3$ of

$$\nabla \wedge u = h; \quad \nabla.u = 0; \quad u.\nu_{|\partial\Omega} = 0.$$

This solution belongs to $(H^1(\Omega))^3$ and we get

$$\|u\|_{H^1(\Omega)} \le C(\Omega)\,\|h\|_{L^2(\Omega)}.$$

For this result we refer to [6]. The regularity of the solution u is demonstrated in [9, chap. 7, th. 6.1]. The estimate is then a consequence of the closed graph theorem.

We point out that the condition of divergence free is necessarily satisfied by the flux j of a solution of the Vlasov equation. It is a property of mass conservation that is obtained by integrating (22) with respect to p. On the contrary the conditions

$$\int_{\partial\Omega_i} j(x).\nu(x)\,d\gamma(x) = 0; \quad 1 \le i \le M$$

are not generally verified if $M \ge 2$. On the other hand, if $\partial\Omega$ is connected, i.e. $M = 1$, this condition is an obvious consequence of the assumption $\nabla.j = 0$. Therefore we assume

(H1) Ω is a smooth bounded set of $\mathbb{R}^3$. Its boundary $\partial\Omega$ is compact and connected.

Furthermore we impose that the boundary data satisfy

(H2) For some $C_0 > 0$ and some $\gamma > \frac{3}{2}$ $\; 0 \le f_0 \le C_0(1 + |p|^2)^{-\gamma}$

(H3) $\phi_0 \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$

(H4) $b \in H^{-1/2}(\partial\Omega); \quad \int_{\partial\Omega} b\,d\gamma(x) = \langle b, 1\rangle_{H^{-1/2}, H^{1/2}} = 0.$

The main result of this section is

Theorem 4. *Under the hypothesis (H1), ..., (H4), the stationary Vlasov-Maxwell system (21), ..., (29) has at least one solution* (f, ϕ, B) *which verifies*

$$0 \leq f \leq C(1+|p|^2)^{-\gamma}; \quad \text{for some } C > 0,$$

$$\phi \in H^1(\Omega); \quad B \in H(\text{div}, \text{curl}, \Omega).$$

In the theorem above, $H(\text{div}, \text{curl}, \Omega)$ denotes the space of functions u such that u, $\nabla \wedge u$, $\nabla . u$ belong to $L^2(\Omega)$. First of all we begin with an easy result about the boundary data.

Lemma 3. *Let us suppose that* ϕ_0 *and b satisfy (H3) and (H4). Then there are two functions* Φ_0 *and* B_0 *such that*

$$\Phi_0 \in H^1(\Omega) \cap L^\infty(\Omega); \quad \Delta_x \Phi_0 = 0; \qquad \Phi_{0|\partial\Omega} = \Phi_0$$

$$B_0 \in H(\text{div}, \text{curl}, \Omega); \quad \nabla_x \wedge B_0 = 0; \quad \nabla_x . B_0 = 0; \quad B_0 . \nu_{|\partial\Omega} = b.$$

Proof. The existence of Φ_0 is obvious. If b satisfies (H4), let ψ be a solution of the Neumann problem

$$\psi \in H^1(\Omega); \quad \Delta_x \psi = 0; \quad \frac{\partial \psi}{\partial \nu}_{|\partial\Omega} = b.$$

We immediately verify that $B_0 = \nabla_x \psi$ has the requested properties. □

Sketch of proof of theorem 4. The main idea that has been presented in [13], is to use a fixed point procedure on the electromagnetic field. Indeed we remark that in view of (23) and (28) the potential satisfies for any non negative concentration ϱ

$$(30) \qquad q\phi \geq q\Phi_0.$$

It follows from proposition 2 that we are able to find a non increasing function G for which we get

$$(31) \qquad 0 \leq f(x,p) \leq C(f_0, \Phi_0)\, G(\varepsilon(p) + q\Phi(x)) \leq C(f_0, \Phi_0)\, G(\varepsilon(p) + q\Phi_0(x)).$$

We point out that if we consider repulsive forces, the inequality (30) is reversed. We have no more a-priori estimate on the distribution f and the mathematical analysis performed in this paper fails.

Now the stimate (31) allows to obtain uniform bounds on the flux j and the concentration ϱ. Then we obtain compactness properties for F in L^2. So let us define the following map

$(\phi, B) \to f_{\phi,B}$ solution of the Vlasov problem $\to \varrho_{\phi,B}$, $j_{\phi,B}$ concentration anf flux of $f_{\phi,B} \to (\phi_1, B_1)$ solution of Maxwell equations with sources $\varrho_{\phi,B}$ and $j_{\phi,B}$.

We hope to apply the Schauder fixed point theorem to this map. Unfortunately the above map is not well defined since we have no uniqueness for the Vlasov problem. Therefore we have to introduce a perturbed problem. To recover a uniqueness property we have to put an absorption term in the Vlasov equation and to regularized the force field.

Regularization of the force field. For any $\alpha > 0$, we define a regularized force field F_α in the following way.

$$F_\alpha = F_\alpha(\phi, B) = q(-\nabla_x \psi_\alpha(\phi) + v(p) \wedge H_\alpha(B)).$$

The modified magnetic field H_α is given by

$$H_\alpha(B) = \bar{B} * \zeta_\alpha$$

where ζ_α is a regularizing sequence

$$\zeta_\alpha(x) = \frac{1}{\alpha^3} \zeta\left(\frac{x}{\alpha}\right); \quad \int_{\mathbb{R}^3} \zeta(x)\, dx = 1; \quad \zeta \in C_0^\infty(\mathbb{R}^3)$$

and where $\bar{B}$ is the extension of B by zero outside Ω.

The regularization of the potential is a little more complicated. Indeed we want to preserve the a-priori estimate on the distribution (31). Therefore we have to impose on the modified potential

$$\psi_\alpha \in C_b^2(\Omega); \quad \psi_{\alpha|\partial\Omega} = \phi_0; \quad q\psi_\alpha \geq q\Phi_0$$

for any ϕ such that

$$\phi \in H^1(\Omega); \quad \phi_{|\partial\Omega} = \phi_0; \quad q\phi \geq q\Phi_0.$$

A possible choice is to let

$$\psi_\alpha = \Phi_0 + (I - \alpha\Delta)^{-2}(\phi - \Phi_0) \tag{32}$$

where the operator Δ is considered as an unbounded operator on $L^2(\Omega)$ whose domain is $H^2 \cap H_0^1(\Omega)$. Let us remark that $(I - \alpha\Delta)^{-2}(\phi - \Phi_0)$ belongs to $H^4(\Omega)$ and then to $C_b^2(\Omega)$. Thus, in order that ψ_α belongs to $C_b^2(\Omega)$, we have to assume

(H5) $\quad \Phi_0 \in C_b^2(\Omega).$

Then the properties of the above regularization are summarized in the following

Lemma 4. *The map $F_\alpha = F_\alpha(\phi, B)$ is continuous from $H^1(\Omega) \times L^2(\Omega)$ into $C_b^1(\Omega \times \mathbb{R}^3)$. For any potential ϕ such that*

$$\phi \in H^1(\Omega); \quad \phi_{|\partial\Omega} = \phi_0; \quad q\phi \geq q\Phi_0.$$

the modified potential $\psi_\alpha = \psi_\alpha(\phi)$ satisfies

$$\psi_\alpha \in C_b^2(\Omega); \quad \psi_{\alpha|\partial\Omega} = \phi_0; \quad q\psi_\alpha \geq q\Phi_0.$$

Furthermore for any sequence (α_n, ϕ_n, B_n) *such that*

$$(\phi_n) \text{ is uniformly bounded in } H^2(\Omega); \quad \phi_{n|\partial\Omega} = \phi_0; \quad \phi_n \to \phi \text{ in } H^1(\Omega)$$

$$B_n \to B \text{ in } L^2(\Omega); \quad \alpha_n \to 0$$

the regularized force field converges and we have

$$F_{\alpha_n}(\phi_n, B_n) \to F = q(-\nabla_x \phi + v(p) \wedge B) \text{ in } L^2_{\text{loc}}(\bar{\Omega} \times \mathbb{R}^3).$$

Proof. The continuity of F_α is obvious. The second point is a consequence of the definition (32) and of the maximum principle. Let (α_n, ϕ_n, B_n) a sequence defined as in the lemma. First we get

$$H_{\alpha_n}(B_n) = (\bar{B}_n - \bar{B}) * \zeta_{\alpha_n} + \bar{B} * \zeta_{\alpha_n} \to \bar{B} \text{ in } L^2(\mathbb{R}^3)$$

Thus we obtain

$$v(p) \wedge H_{\alpha_n} \wedge (B_n) \to v(p) \wedge B \text{ in } L^2_{\text{loc}}(\bar{\Omega} \times \mathbb{R}^3).$$

Next we use that for any u in $H^2 \cap H^{1,0}(\Omega)$ we get

$$(33) \qquad \|(I - \alpha\Delta)^{-2}(u)\|_{H^2(\Omega)} \leq C\|u\|_{H^2(\Omega)}; \qquad C \text{ independent of } \alpha.$$

Indeed if we put $v_\alpha = (I - \alpha\Delta)^{-1}(u)$ we have

$$\|\Delta v_\alpha\|_{L^2(\Omega)} \leq \|\Delta u\|_{L^2(\Omega)}.$$

But the norm $\|\Delta v\|_{L^2(\Omega)}$ is equivalent to the norm $\|v\|_{L^2(\Omega)}$ in the space $H^2 \cap H^1_0(\Omega)$. Therefore for some constant C independent of α we obtain

$$\|v_\alpha\|_{H^2(\Omega)} \leq C\|u\|_{H^2(\Omega)}.$$

Repeating this argument once more leads to (33). From (33) we deduce that $\psi_{\alpha_n}(\phi_n)$ given by (32) is uniformy bounded in $H^2(\Omega)$ since $\phi_n - \Phi_0$ is uniformly bounded in $H^2 \cap H^{1,0}(\Omega)$. Furthermore it easy to verify that

$$\psi_{\alpha_n}(\phi_n) \to \phi \text{ in } L^2(\Omega).$$

It follows that we have

$$\psi_{\alpha_n}(\phi_n) \to \phi \text{ in } H^1(\Omega)$$

which allows to conclude. □

The modified Vlasov-Maxwell system. We introduce the following regularized Vlasov-Maxwell problem

$$(34) \qquad \begin{cases} \alpha f + v.\nabla_x f + F_\alpha(\phi, B).\nabla_p f = 0; & x \in \Omega;\ p \in \mathbb{R}^3 \\ f_{|\Sigma^-} = f_0. \end{cases}$$

$$(35) \qquad \begin{cases} -\Delta_x \phi = \dfrac{q}{\varepsilon_0}\varrho; \quad \varrho(x) = \int_{\mathbb{R}^3} f(x,p)\,dp; \quad x \in \Omega \\ \phi_{|\partial\Omega} = \phi_0. \end{cases}$$

$$(36)\qquad \begin{cases} \nabla_x \wedge B = \mu_0 q \left(j - \alpha \dfrac{\varepsilon_0}{q} \nabla_x \phi \right); \quad \nabla_x . B = 0; \quad x \in \Omega \\ j(x) = \int\limits_{\mathbb{R}^3} v(p) f(x,p) dp; \quad x \in \Omega \\ B . \nu_{|\partial\Omega} = b . \end{cases}$$

The magnetostatic problem has been modified because the flux of a solution of (34) is no more divergence free. Instead we obtain

$$\alpha \varrho + \nabla_x . j = 0 .$$

In view of (35) this equation gives

$$-\alpha \frac{\varepsilon_0}{q} \Delta_x \phi + \nabla_x . j = \nabla_x . \left(j - \alpha \frac{\varepsilon_0}{q} \nabla_x \phi \right) = 0 .$$

showing that $j - \alpha \dfrac{\varepsilon_0}{q} \nabla_x \phi$ is divergence free.

Proposition 3. *Let* $\alpha > 0$. *Under the hypothesis (H1), ..., (H5), the modified Vlasov-Maxwell system (34), ..., (36) has at least one solution* $(f_\alpha, \phi_\alpha, B_\alpha)$ *which satisfies uniformly with respect to* α

$$(37)\qquad 0 \le f_\alpha \le C(C_0, \| \Phi_0 \|_{L^\infty(\Omega)}) (1 + |p|^2)^{-\gamma}$$

ϕ_α *is uniformly bounded in* $H^2(\Omega)$; $B_\alpha - B_0$ *is uniformly bounded in* $H^1(\Omega)$.

The demonstration of this proposition is given in the following section. We are now ready to prove theorem 4.

Proof of theorem 4. First le us assume that Φ_0 satisfies (H5). The above proposition gives a solution $(f_\alpha, \phi_\alpha, B_\alpha)$ of the modified problem for any $\alpha > 0$. In view of the uniform estimates there is a subsequence $\alpha_n \to 0$ and a triple (f, ϕ, B) such that $(f_n, \phi_n, B_n) = (f_{\alpha_n}, \phi_{\alpha_n}, B_{\alpha_n})$ satisfies

$f_n \to f$ in $L^\infty(\Omega \times \mathbb{R}^3)$ weak star;

(ϕ_n) is uniformly bounded in $H^2(\Omega)$; $\phi_{n|\partial\Omega} = \phi_0$; $\phi_n \to \phi$ in $H^1(\Omega)$;

$B_n \to B$ in $L^2(\Omega)$.

We deduce from lemma 4 that

$$F_{\alpha_n}(\phi_n, B_n) \to F = q(-\nabla_x \phi + v(p) \wedge B) \text{ in } L^2_{\text{loc}}(\bar{\Omega} \times \mathbb{R}^3).$$

Then lemma 1 allows to conclude that f is a solution of (22), (27). Moreover in view of (37) and of the condition (H2), $\gamma > 3/2$, we get

$$\varrho_n(x) = \int_{\mathbb{R}^3} f_n(x,p)\,dp \quad \to \varrho(x) = \int_{\mathbb{R}^3} f(x,p)\,dp \text{ in } L^\infty(\Omega) \text{ weak star,}$$

$$j_n(x) = \int_{\mathbb{R}^3} v(p) f_n(x,p)\,dp \to j(x) = \int_{\mathbb{R}^3} v(p) f(x,p)\,dp \text{ in } L^\infty(\Omega) \text{ weak star.}$$

Then it is straightforward to pass to the limit in (35) (36) to obtain (25), (26), (28) and (29). Thus (f, ϕ, B) is the desired solution. It remains to get rid of the restriction (H5). For that we introduce a sequence $\Phi_{0,n}$ such that

$$\Phi_{0,n} \in C_b^2(\Omega); \quad \Phi_{0,n} \to \Phi_0 \text{ in } H^1(\Omega); \quad \|\Phi_{0,n}\|_{L^\infty(\Omega)} \le C_1.$$

Let (f_n, ϕ_n, B_n) the corresponding solutions. We always have

$$0 \le f_n \le C(C_0, C_1)(1 + |p|^2)^{-\gamma}.$$

Therefore ϱ_n and j_n are uniformly bounded in $L^\infty(\Omega)$ and for a subsequence

$$f_n \to f \text{ in } L^\infty(\Omega \times \mathbb{R}^3) \text{ weak star}$$

$$\varrho_n \to \varrho \text{ in } L^\infty(\Omega) \text{ weak star,} \quad j_n \to j \text{ in } L^\infty(\Omega) \text{ weak star}$$

From lemma 2 and 3 we deduce

$$\phi_n - \Phi_{0,n} \text{ is uniformly bounded in } H^2(\Omega);$$
$$B_n - B_0 \quad \text{is uniformly bounded in } H^1(\Omega).$$

Hence we obtain for a subsequence

$$\phi_n \to \phi \text{ in } H^1(\Omega); \quad B_n \to B \text{ in } L^2(\Omega).$$

The limits ϕ and B are solutions of the Maxwell problem. Moreover we get

$$F_n = q(-\nabla_x \phi_n + v(p) \wedge B_n) \to F = q(-\nabla_x \phi + v(p) \wedge B) \text{ in } L^2_{\text{loc}}(\bar{\Omega} \times \mathbb{R}^3).$$

Thus we apply lemma 1 to pass to the limit in the Vlasov problem. □

4. Existence for the modified Vlasov-Maxwell system

The aim of this paragraph is to demonstrate proposition 3. The ideas have been presented in the sketch of the proof of theorem 4 of the previous section.

Definition of a map on the electromagnetic field. First let us define

$$\Xi = \{(\phi, B) \in H^1(\Omega) \times L^2(\Omega); \quad \phi_{|\partial\Omega} = \phi_0; \quad q\phi \ge q\Phi_0\}.$$

It is straightforward to verify that Ξ is a nonempty convex closed set of $H^1(\Omega) \times L^2(\Omega)$. For a pair (ϕ, B) in Ξ we denote by $f_{\phi \cdot B}$ the *unique* solution of the modified Vlasov problem (34) (see proposition 1). Then we put

$$\varrho_{\phi,B}(x) = \int_{\mathbb{R}^3} f_{\phi,B}(x,p)\,dp; \quad j_{\phi,B}(x) = \int_{\mathbb{R}^3} v(p) f_{\phi,B}(x,p)\,dp.$$

Let (ϕ_1, B_1) be the solution of (35) and (36) with the corresponding concentration and flux. The map Γ is defined by $\Gamma(\phi, B) = (\phi_1, B_1)$.

Proposition 4. *The map Γ is a continuous and compact map from Ξ into itself for the topology of $H^1(\Omega) \times L^2(\Omega)$.*

A consequence of the above proposition and of the Schauder fixed point theorem is the existence of a solution for the modified Vlasov-Maxwell system. First we establish

Lemma 5. *For any (ϕ, B) in Ξ, the solution $f_{\phi,B}$ of (34) satisfies*

$$0 \leq f_{\phi,B} \leq C(C_0, \|\Phi_0\|_{L^\infty(\Omega)})\,(1+|p|^2)^{-\gamma}.$$

Proof. We use upper solutions of the Vlasov equation given by proposition 2. We point out that in view of (21)

$$c|p| \leq \varepsilon(p) \leq mc^2 + c|p|.$$

We denote $C_1 = \|q\Phi_0\|_{L^\infty(\Omega)}$, then it follows from lemma 4

$$\varepsilon(p) + q\psi_\alpha(x) = \varepsilon(p) + q\phi_0(x) \leq c|p| + mc^2 + C_1; \quad \text{on } \Sigma \tag{38}$$

$$\varepsilon(p) + q\psi_\alpha(x) \geq \varepsilon(p) + q\Phi_0(x) \geq c|p| - C_1; \quad \text{on } \Omega \times \mathbb{R}^3. \tag{39}$$

We choose a function G as follows

$$G(t) = C_0\left(1 + \frac{1}{c^2}(t - mc^2 - C_1)^2\right)^{-\gamma}; \quad t > mc^2 + C_1$$

$$G(t) = C_0; \quad t \leq mc^2 + C_1.$$

The function G is C^1 and non increasing. We put $g(x,p) = G(\varepsilon(p) + q\psi_\alpha(x))$. Since G is non increasing we deduce from (38)

$$g(x,p) \geq G(c|p| + mc^2 + C_1) = C_0(1+|p|^2)^{-\gamma}; \quad \text{on } \Sigma.$$

With (H2), it gives

$$0 \leq f_0 \leq g; \quad \text{on } \Sigma^-.$$

Hence proposition 2 leads to

$$0 \leq f_{\phi,B} \leq g; \quad \text{on } \Omega \times \mathbb{R}^3.$$

Using (39), we obtain

$$0 \leq f_{\phi,B} \leq g \leq G(c|p| - C_1) \leq C(C_0, C_1)\,(1+|p|^2)^{-\gamma}; \quad \text{on } \Omega \times \mathbb{R}^3$$

which concludes. □

Proof of proposition 4. The proof is divided in two steps. First we show that Γ is well defined and compact. The second step is devoted to the proof of continuity.

Compactness. It follows from lemma 4 that for any (ϕ, B) in Ξ, the concentration $\varrho_{\phi,B}$ and the flux $j_{\phi,B}$ satisfy

$$0 \leq \varrho_{\phi,B} \leq C_2; \quad |f_{\phi,B}| \leq C_2$$

for some constant C_2 that depends only on C_0 and C_1. Then the solution of

$$-\Delta_x \eta = \frac{q}{\varepsilon_0} \varrho_{\phi,B}; \quad \eta \in H_0^1(\Omega)$$

is uniformly bounded in $H^2(\Omega)$ and satisfies $q\eta \geq 0$. Therefore the function $\phi_1 = \Phi_0 + \eta$ lies in a bounded set of $H^2(\Omega)$ that is a compact set of $H^1(\Omega)$, by the Rellich theorem. It satisfies

$$\phi_{1|\partial\Omega} = \phi_0; \quad q\phi_1 \geq q\Phi_0 .$$

The function $j_{\phi,B} - \alpha \frac{\varepsilon_0}{q} \nabla_x \phi_1$ is in a bounded set of $L^2(\Omega)$ and verifies

$$\nabla_x . \left(j_{\phi,B} - \alpha \frac{\varepsilon_0}{q} \nabla_x \phi_1 \right) = \nabla_x . j_{\phi,B} + \alpha \varrho_{\phi,B} = 0 .$$

The last equality is obtained by integrating (34) with respect to p. We apply lemma 2 to show that the solution D of

$$\nabla_x \wedge D = \mu_0 q \left(j_{\phi,B} - \alpha \frac{\varepsilon_0}{q} \nabla_x \phi_1 \right); \quad \nabla_x . D = 0; \quad D . \nu_{|\partial\Omega} = 0$$

belongs to a bounded set of $H^1(\Omega)$. Therefore $B_1 = B_0 + D$ belongs to a compact set of $L^2(\Omega)$. Thus we have proved that (ϕ_1, B_1) lies in a compact subset of Ξ. □

Continuity. Let (ϕ_n, B_n) be a sequence in Ξ such that

$$\phi_n \to \phi \text{ in } H^1(\Omega); \quad B_n \to B \text{ in } L^2(\Omega) .$$

Lemma 4 implies that the force field $F_\alpha(\phi_n, B_n)$ converges in $C_b^1(\Omega \times \mathbb{R}^3)$ towards $F_\alpha(\phi, B)$. Then we deduce from lemma 1 that for any subsequence $f_n = f_{\phi_n, B_n}$ which converges in $L^\infty(\Omega \times \mathbb{R}^3)$ weak star the limit is $f_{\phi,B}$. Hence a consequence of lemma 5 is

$$f_n \to f_{\phi,B} \text{ in } L^\infty(\Omega \times \mathbb{R}^3) \text{ weak star;}$$

$$\varrho_n \to \varrho_{\phi,B} \text{ in } L^2(\Omega) \text{ weak;} \quad j_n \to j_{\phi,B} \text{ in } L^2(\Omega) \text{ weak} .$$

Since the sequence $\Gamma(\phi_n, B_n) = (\phi_{n,1}, B_{n,1})$ belongs to a copmpact set of $H^1(\Omega) \times L^2(\Omega)$, the last convergences show that

$$(\phi_{n,1}, B_{n,1}) \to (\phi_n, B_n) \text{ in } H^1(\Omega) \times L^2(\Omega) . \quad \square$$

Proof of proposition 3. Proposition 4 and lemma 5 establish the existence of a solution $(f_\alpha, \phi_\alpha, B_\alpha)$ which satisfy (37). Furthermore we deduce from the proof of the compactness of Γ that ϕ_α belongs to a bounded set of $H^2(\Omega)$ and that $B_\alpha - B_0$ belongs to a bounded set of $H^1(\Omega)$. □

5. Other kinetic models

First let us point out that without any change the preceeding proofs provide solutions to the Vlasov-Maxwell system for classical mechanics and to Vlasov-Poisson equations for classical or relativistic mechanics. The condition (H2) about the decay of the entering data becomes $\gamma > 2$ in the first case and is always $\gamma > 3/2$ in the others. It allows to estimate the concentration and the flux of the solutions of the Vlasov equation. Now we analyze models for which slight changes in the proofs of section 2, 3 and 4 also give existence results.

Models with several species of particles. The Vlasov-Poisson equations for several species of particles in classical mechanics read

$$\text{(40)} \quad \begin{cases} v.\nabla_x f_s - \dfrac{q_s}{m_s} \nabla_x \phi . \nabla_v f_s = 0; \quad x \in \Omega;\ v \in \mathbb{R}^3;\ s = 1, \dots, S \\ f_{s|\Sigma^-} = f_{0,s};\ s = 1, \dots, S \end{cases}$$

$$\text{(41)} \quad \begin{cases} -\Delta_x \phi = \dfrac{1}{\varepsilon_0}\left(N + \sum_{s=1}^{S} q_s \varrho_s\right); \quad \varrho_s(x) = \int_{\mathbb{R}^3} f_s(x, v)\, dv; \quad x \in \Omega \\ \phi_{|\partial\Omega} = \phi_0 . \end{cases}$$

Above, q_s and m_s are the elementary charge and the mass of particles of the species s. The function $N = N(x)$ is a fixed background charge concentration. It accounts for steady heavy ions in a plasma or for the doping profile in a semiconductor.

We always assume that Ω is a smooth domain of $\mathbb{R}^3$ and

$$\text{(42)} \quad 0 \le f_{0,s} \le C_{0,s}(1 + |v|^2)^{-\gamma}; \quad s = 1, \dots, S; \quad \gamma > 3/2$$

$$\text{(43)} \quad \phi_0 \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$$

$$\text{(44)} \quad N \in L^\infty(\Omega) .$$

Theorem 5. *If the conditions (42), (43) and (44) are satisfied, the Vlasov-Poisson system with S species of particles (40), (41) has at least one solution* $(\phi, (f_s)_{s=1,\dots,S})$ *that verify*

$$0 \le f_s \le C_s(1 + |v|^2)^{-\gamma}; \quad s = 1, \dots, S,$$
$$\phi \in H^1(\Omega) \cap L^\infty(\Omega) .$$

The following analysis of the Vlasov-Poisson equations is easily extended to Vlasov-Maxwell systems and to relativistic mechanics. The proof of the corresponding theorems are left to the reader.

Proof. The only difference with the previous sections is the derivation of the uniform estimate on the potential. Then we only detail this point. Let us define

$$G(t) = (1 + t^2)^{-\gamma/2} \text{ for } t > 0; \quad G(t) = 1 \text{ for } t \le 0$$

$$g_s(x, v) = C_s G\left(\frac{m_s v^2}{2} + q_s \phi(x)\right).$$

The constants C_s that depend only on ϕ_0 and $C_{0,s}$, are chosen such that

$$g_s \ge C_{0,s}(1 + |v|^2)^{-\gamma} \ge f_{0,s}; \quad \text{on } \Sigma^-.$$

Then the functions f_s are estimated by

$$0 \le f_s \le g_s; \quad \text{on } \Omega \times \mathbb{R}^3. \tag{45}$$

We assume that the charges q_s are positive for $s = 1, \dots, P$, and negative for $s = P + 1, \dots, S$. We introduce

$$n_s(\phi) = q_s C_s \int_{\mathbb{R}^3} G\left(\frac{m_s v^2}{2} + q_s \phi\right) dv.$$

Since G is non increasing, the charge concentration n_s is a non decreasing function with respect to ϕ. It is positive for $s = 1, \dots, P$, and negative for $s = P + 1, \dots, S$. Therefore we deduce from (41) and (45) that

$$\begin{cases} \dfrac{1}{\varepsilon_0}\left(N + \sum\limits_{P+1}^{S} n_s(\phi)\right) \le -\Delta_x \phi \le \dfrac{1}{\varepsilon_0}\left(N + \sum\limits_{1}^{P} n_s(\phi)\right) \\ \phi_{|\partial\Omega} = \phi_0 . \end{cases} \tag{46}$$

Let us introduce the two solutions θ_1 and θ_2 of the problems

$$\begin{cases} -\Delta_x \theta_1 = \dfrac{1}{\varepsilon_0}\left(N + \sum\limits_{P+1}^{S} n_s(\theta_1)\right) \\ \theta_{1|\partial\Omega} = \phi_0 . \end{cases} \qquad \begin{cases} -\Delta_x \theta_2 = \dfrac{1}{\varepsilon_0}\left(N + \sum\limits_{1}^{P} n_s(\theta_2)\right) \\ \theta_{2|\partial\Omega} = \phi_0 \end{cases}$$

Since the function n_s are non increasing these problems have unique solutions. It follows from (43) and (44) that they belong to $H^1(\Omega) \cap L^\infty(\Omega)$.

Lemma 6. *Let ϕ be a function of $H^1(\Omega)$ which satisfies (46) then $\theta_1 \le \phi \le \theta_2$.*

Let us suppose this lemma. It provides an a-priori estimate on the potential ϕ in $H^1(\Omega) \cap L^\infty(\Omega)$. It leads with (45) to an a-priori estimate on the distributions f_s. Now the proof of theorem 5 can be carried out as in section 3 and 4 except that the convex Ξ of section 4 has to be replaced by

$$\Xi = \{\phi \in H^1(\Omega);\ \phi_{|\partial\Omega} = \phi_0;\ \theta_1 \le \phi \le \theta_2\}. \quad \square$$

Proof of lemma 6. We use the Stampacchia method. Let H be a C^1 function such that

$$H(t) = \text{ for } t \le 0; \quad H(t) > 0 \text{ for } t > 0; \quad 0 \le H'(t) \le M < \infty.$$

Since the function $\theta_1 - \phi$ belongs to $H_0^1(\Omega)$, the function $H(\theta_1 - \phi)$ lies in $H_0^1(\Omega)$. We get

$$\nabla H(\theta_1 - \phi) = H'(\theta_1 - \phi)\nabla(\theta_1 - \phi).$$

We multiply the inequation

$$-\Delta(\theta_1 - \phi) + \frac{1}{\varepsilon_0}\sum_{P+1}^{S} n_s(\phi) - n_s(\theta_1) \leq 0$$

by $H(\theta_1 - \phi)$ snd we integrate over Ω. We obtain

$$\int_\Omega H'(\theta_1 - \phi)\,|\nabla(\theta_1 - \phi)|^2 dx$$
$$+ \frac{1}{\varepsilon_0}\int_\Omega \sum_{P+1}^{S} (n_s(\phi) - n_s(\theta_1))\,H(\theta_1 - \phi)\,dx \leq 0.$$

The first integral is the integral of a non negative function. For the second one we have

$$(n_s(\phi) - n_s(\theta_1))\,H(\theta_1 - \phi) = 0 \text{ for } \theta_1 \leq \phi;$$
$$(n_s(\phi) - n_s(\theta_1))\,H(\theta_1 - \phi) \geq 0 \text{ for } \theta_1 \geq \phi$$

because n_s is non increasing. It follows that both integrals vanish. Hence we get

$$H'(\theta_1 - \phi)\,|\nabla(\theta_1 - \phi)|^2 = 0 \quad \text{a.e.}$$

Thus

$$\nabla H(\theta_1 - \phi) = 0 \quad \text{a.e.}$$

We finally deduce that $\theta_1 \leq \phi$ a.e. In the same way we obtain $\phi \leq \theta_2$. □

Reflection and Neumann boundary conditions. It is of physical interest to impose a reflection condition on a part of the boundary for the Vlasov problem and a Neumann condition for the Poisson equation. The problem reads

$$(47) \qquad \begin{cases} v.\nabla_x f - \dfrac{q}{m}\nabla_x\phi.\nabla_v f = 0; & x \in \Omega;\ v \in \mathbb{R}^3 \\ f_{|\Sigma_1^-} = f_0; \quad R(f)_{|\Sigma_2} = f_{|\Sigma_2} \end{cases}$$

$$(48) \qquad \begin{cases} -\Delta_x \phi = \dfrac{q}{\varepsilon_0}\varrho; \quad \varrho(x) = \int_{\mathbb{R}^3} f(x,v)\,dv; & x \in \Omega \\ \phi_{|\partial\Omega_1} = \phi_0.; \quad \dfrac{\partial\phi}{\partial v}_{|\partial\Omega_2} = g. \end{cases}$$

where we define

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2; \quad \Sigma_1^- = \{(x,v) \in \Sigma^-;\ x \in \partial\Omega_1\};$$
$$\Sigma_2 = \{(x,v) \in \Sigma;\ x \in \partial\Omega_2\}.$$

The reflection operator R is defined on Σ by

$$R(f)(x, v) = f(x, R_x v); \quad R_x v = v - 2v.\nu(x)\,\nu(x).$$

The differences with section 2, 3 and 4 appear in the analysis of the linear Vlasov problem and in the regularization procedure of the potential. Let us first give a precise result. We introduce the potential Φ_0 solution of

$$\begin{cases} -\Delta_x \Phi_0 = 0; & x \in \Omega \\ \Phi_{0|\partial\Omega_1} = \phi_0.; & \dfrac{\partial \Phi_0}{\partial \nu}_{|\partial\Omega_2} = g. \end{cases}$$

We assume

(49) $\quad \Phi_0$ belongs to $H^1(\Omega) \cap L^\infty(\Omega)$,

(50) $\quad 0 \le f_0 \le C_0(1 + |v|^2)^{-\gamma}; \quad \gamma > 3/2.$

Theorem 6. *Under the assumptions (49), (50) the problem (47), (48) has at least one solution (f, ϕ) which satisfies*

$$0 \le f \le C(1 + |v|^2)^{-\gamma}; \quad \phi \textit{ belongs to } H^1(\Omega); \quad q\phi \ge q\Phi_0.$$

The results on the linear Vlasov problem that are needed for the proof of the above theorem are summarized below.

Proposition 5. *Let ϕ a potential in $C_b^2(\Omega)$ and f_0 an entering data which satisfies*

$$\int_{\Sigma_1^-} |v.\nu(x)|\,|f_0(x, v)|^p \, d\gamma(x)\, dv < \infty.$$

Then for any $\alpha > 0$ the problem

$$\begin{cases} \alpha f + v.\nabla_x f - \dfrac{q}{m} \nabla_x \phi . \nabla_v f = 0; & x \in \Omega;\ v \in \mathbb{R}^3 \\ f_{|\Sigma_1^-} = f_0; \quad R(f)_{|\Sigma_2} = f_{|\Sigma_2} \end{cases}$$

has a unique solution in $L^p(\Omega \times \mathbb{R}^3)$. Moreover if we get

$$f_0 \le g \textit{ on } \Sigma_1^-$$

where $g(x, v) = G\left(\dfrac{mv^2}{2} + q\phi(x)\right)$ and G is C^1, the solution f satisfies

$$f \le g \textit{ on } \Omega \times \mathbb{R}^3.$$

Proof of proposition 5. The results of this proposition are well known. But a reference where a proof is given seems difficult to find. So let us sketch a proof.

Uniqueness. If f is a smooth solution we multiply the Vlasov equation by $f\,|f|^{p-2}$ and we integrate over $\Omega \times \mathbb{R}^3$. Thus we obtain the estimate

$$\int_{\Omega\times\mathbb{R}^3} |f|^p dx\,dp + \int_{\Sigma_1^-} |v.\nu|\,|f|^p d\gamma(x)\,dv \le \int_{\Sigma_1^-} |v.\nu|\,|f_0|^p d\gamma(x)\,dv.$$

As in [1], this estimate is proved for any solution by a density argument. It implies uniqueness.

Existence and maximum principle. Let f_0 be a bounded and compactly supported entering data. There is a function

$$g(x,v) = G\left(\frac{mv^2}{2} + q\phi(x)\right); \quad G \in C_0^1(\mathbb{R}),$$

such that

$$|f_0| \le g \quad \text{on} \quad \Sigma_1^-.$$

Let us consider the sequence

$$f^{(0)} = 0,$$

$$\begin{cases} \alpha f^{(n+1)} + v.\nabla_x f^{(n+1)} - \dfrac{q}{m}\nabla_x\phi.\nabla_v f^{(n+1)} = 0; & x\in\Omega;\ v\in\mathbb{R}^3 \\ f^{(n+1)}{}_{|\Sigma_1^-} = f_0; \quad f^{(n+1)}{}_{|\Sigma_2^-} = R(f^{(n)})_{|\Sigma_2^-}. \end{cases}$$

From proposition 1 we deduce by induction that $f^{(n)}$ is uniformly estimated by

$$|f^{(n)}| \le g \quad \text{on} \quad \Omega\times\mathbb{R}^3.$$

Then it is just an exercise to verify that a weak limit of $h^{(N)} = \frac{1}{N}\sum_{n=1}^{N} f^{(n)}$ provides a solution to our problem. Moreover it follows from this construction that this solution satisfies the maximum principle. This result is extended to any entering data by density arguments. □

Now the only difference with sections 3 and 4 is the way to regularize the potential. The difficulty is that singularities may occur at the interface between Neumann and Dirichlet conditions. First we assume instead of (H5) that there exists an open subset ω of $\partial\Omega_1$ such that $\mathrm{int}(\partial\Omega_1) \supset \omega$ where we denote by $\mathrm{int}(\partial\Omega_1)$ the interior of $\partial\Omega_1$, and $\omega\times\mathbb{R}^3 \supset \mathrm{supp}(f_0)$. We assume in addition that $\Phi_0 \in C_b^2(\Omega)$. Then there is a function θ in $C_0^\infty(\mathbb{R}^3)$ such that

$$\theta_{|\omega} = 1; \quad \mathrm{int}(\partial\Omega_1) \supset \mathrm{supp}(\theta_{|\partial\Omega}).$$

We define the regularized potential $\psi_\alpha(\phi)$ by

$$\psi_\alpha(\phi) = (I-\alpha\Delta)^{-2}(\theta(\phi-\Phi_0)) + \theta\Phi_0 + (1-\theta)(\phi * \zeta_\alpha)$$

where ζ_α is a regularizing sequence. We verify easily that we have an analogous of lemma 4 for this modified potential. □

Boltzmann-Vlasov-Poisson system. We focus our attention on a Boltzmann-Vlasov-Poisson system which occurs in semiconductors physics. We refer to [3, 12] for some mathematical results on the time dependent system and for references about the physical background. The problem reads

$$(51)\qquad \begin{cases} v.\nabla_x f - \dfrac{q}{m} \nabla_x \phi . \nabla_v f - Q(f) = 0; & x \in \Omega;\ v \in \mathbb{R}^3 \\ f_{|\Sigma^-} = f_0 \end{cases}$$

$$(52)\qquad \begin{cases} -\Delta_x \phi = \dfrac{1}{\varepsilon_0}(N - q\varrho); \quad \varrho(x) = \int_{\mathbb{R}^3} f(x, v)\, dv; & x \in \Omega \\ \phi_{|\partial\Omega} = \phi_0 . \end{cases}$$

The charge density N is a known doping profile. The Boltzmann operator Q is a linear integral operator that models the collisions of the particles (electrons or holes) with the semiconductor crystal. For a function $g = g(x, v)$, it is given by

$$(53)\qquad Q(g)(x, v) = \int_{\mathbb{R}^3} s(x, v, w)\,(M(v)\, g(x, w) - M(w)\, g(x, v))\, dw .$$

The function M is a centered Maxwellian with a fixed temperature

$$M(v) = \left(2\pi \frac{k_B T}{m}\right)^{-3/2} \exp\left(-\frac{mv^2}{2k_B T}\right).$$

The positive real numbers k_B and T are the Boltzmann constant and the temperature. The integral kernel s is a non negative measure that satisfies

$$s \in L^\infty(\Omega \times \mathbb{R}^3_v;\ M_b(\mathbb{R}^3)); \quad s(x, v, w) = s(x, w, v),$$

where $M_b(\mathbb{R}^3)$ is the space of bounded measures and the last equality means that for any continuous functions f and g we get

$$\int_{\Omega \times \mathbb{R}^3} \int_{\mathbb{R}^3} s(x, v, w) f(x, w)\, dw\, g(x, v)\, dv = \int_{\Omega \times \mathbb{R}^3} \int_{\mathbb{R}^3} s(x, v, w)\, g(x, w)\, dw\, f(x, v)\, dv .$$

The collision frequency σ is the function of $L^\infty(\Omega \times \mathbb{R}^3)$ defined by

$$\sigma(x, v) = \int_{\mathbb{R}^3} s(x, v, w)\, M(w)\, dw .$$

Thus an other expression of the operator Q is

$$\begin{aligned} Q(g)(x, v) &= M(v) \int_{\mathbb{R}^3} s(x, v, w)\, g(x, w)\, dw - \sigma(x, v)\, g(x, v) \\ &= Q^+(g)(x, v) - \sigma g(x, v) . \end{aligned}$$

Some properties of the collision operator are summarized below.

Proposition 6. *The operators Q and Q^+ are continuous operators on $L^1(\Omega \times \mathbb{R}^3)$. The Maxwellians ϱM, $\varrho \in \mathbb{R}$, belong to the nullspace of Q. For any integrable function f we get*

$$\int_{\Omega \times \mathbb{R}^3} Q(f)(x,v)\,dx\,dv = 0.$$

Proof. For a continuous function f, using the symmetry of the kernel we obtain

$$\int_{\Omega \times \mathbb{R}^3} |Q^+(f)(x,v)|\,dx\,dv \le \int_{\Omega \times \mathbb{R}^3} |Q^+(|f|)(x,v)|\,dx\,dv$$
$$= \int_{\Omega \times \mathbb{R}^3} \sigma(x,v)\,|f(x,v)|\,dx\,dv,$$

$$\|Q^+(f)\|_{L^1(\Omega \times \mathbb{R}^3)} \le \|\sigma\|_{L^\infty(\Omega \times \mathbb{R}^3)} \|f\|_{L^1(\Omega \times \mathbb{R}^3)}.$$

It proves that Q^+ and Q can be extended into bounded operators on $L^1(\Omega \times \mathbb{R}^3)$. The last two properties are obvious in view of (53) and of the symmetry of s. □

About the linear Boltzmann-Vlasov problem we obtain

Proposition 7. *Let ϕ a potential in $C_b^2(\Omega)$ and f_0 an entering data which satisfies*

$$\int_{\Sigma^-} |v.\nu(x)|\,|f_0(x,v)|\,d\gamma(x)\,dv < \infty.$$

then for any $\alpha > 0$ the problem

$$\text{(54)} \qquad \begin{cases} \alpha f + v.\nabla_x f - \dfrac{q}{m} \nabla_x \phi . \nabla_v f - Q(f) = 0; & x \in \Omega;\ v \in \mathbb{R}^3 \\ f_{|\Sigma^-} = f_0 \end{cases}$$

has a unique solution in $L^1(\Omega \times \mathbb{R}^3)$. Moreover if for some constant C_0 we get

$$\text{(55)} \qquad f_0 \le C_0 \exp\left(-\frac{1}{k_B T}\left(\frac{mv^2}{2} + q\phi(x)\right)\right); \quad \text{on } \Sigma^-$$

the solution f satisfies

$$\text{(56)} \qquad f \le C_0 \exp\left(-\frac{1}{k_B T}\left(\frac{mv^2}{2} + q\phi(x)\right)\right); \quad \text{on } \Omega \times \mathbb{R}^3.$$

Proof. Let us consider the following map Λ on $L^1(\Omega \times \mathbb{R}^3)$

$$\begin{cases} \Lambda(f) = h \\ (\alpha + \sigma)h + v.\nabla_x h - \dfrac{q}{m} \nabla_x \phi . \nabla_v h = Q^+(f); & x \in \Omega;\ v \in \mathbb{R}^3 \\ h_{|\Sigma^-} = f_0 . \end{cases}$$

We denote

$$|||f||| = \int_{\Omega \times \mathbb{R}^3} (\alpha + \sigma(x, v)) \, |f(x, v)| \, dx \, dv.$$

Since α is positive and since σ is a nonnegative function of $L^\infty(\Omega \times \mathbb{R}^3)$ the norm $|||\,.\,|||$ is equivalent to the norm of $L^1(\Omega \times \mathbb{R}^3)$. Let $\delta = \Lambda(f) - \Lambda(g)$. We have

$$\begin{cases} (\alpha + \sigma)\delta + v.\nabla_x \delta - \dfrac{q}{m} \nabla_x \phi . \nabla_v \delta = Q^+(f - g); \quad x \in \Omega; \ v \in \mathbb{R}^3 \\ \delta_{|\Sigma^-} = 0. \end{cases}$$

If δ is a smooth function, we multiply the equation above by sign(δ) ans we integrate it over $\Omega \times \mathbb{R}^3$. We obtain

$$|||\delta||| \leq \int_{\Omega \times \mathbb{R}^3} |Q^+(f - g)| \, dx \, dv \leq \int_{\Omega \times \mathbb{R}^3} \sigma |(f - g)| \, dx \, dv.$$

This estimate is easily generalized to any function δ by density arguments (see [1]). Then we obtain

$$|||\delta||| = |||\Lambda(f) - \Lambda(g)||| \leq k |||f - g||| \quad \text{with } k = \frac{\sigma_\infty}{\alpha + \sigma_\infty} < 1;$$
$$\sigma_\infty = \|\sigma\|_{L^\infty(\Omega \times \mathbb{R}^3)}.$$

Thus the map Λ is a contraction. Since a solution of the Vlasov problem is a fixed point for Λ, we obtain existence and uniqueness. Let f_0 be an entering data that satisfies (55). Let us consider the following sequence

$$f^{(0)} = 0; \quad f^{(n+1)} = \Lambda(f^{(n)}).$$

Since Λ is a contraction this sequence converges towards the solution f of the Vlasov problem. We point out that the function

$$g = C_0 \exp\left(-\frac{1}{k_B T} \left(\frac{mv^2}{2} + q\phi(x) \right) \right)$$

satisfies $g = \Lambda(g)$ because $Q(g) = Q^+(g) - \sigma g = 0$ (proposition 6). Let us suppose that $f^{(n)} \leq g$, then we get

$$\begin{cases} (\alpha + \sigma)\delta + v.\nabla_x \delta - \dfrac{q}{m} \nabla_x \phi . \nabla_v \delta = Q^+(f^{(n)} - g) \leq 0; \quad x \in \Omega; \ v \in \mathbb{R}^3 \\ \delta_{|\Sigma^-} = f_0 - g \leq 0 \end{cases}$$

with $\delta = f^{(n+1)} - g$. It follows $\delta \leq 0$, $f^{(n+1)} \leq g$. It remains to pass to the limit to obtain the desired estimate (56). □

Thanks to the above proposition, the techniques of section 3 and 4 leads to

Theorem 7. *Let us assume*

$$\phi_0 \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega); \quad f_0 \leq C_0 \exp\left(-\frac{mv^2}{2k_B T} \right); \quad \text{on } \Sigma^-.$$

then there is at least one solution (f, ϕ) of the Boltzmann-Vlasov-Poisson equation that satisfies

$$\phi \in H^1(\Omega) \cap L^\infty(\Omega); \quad f \leq C_1 \exp\left(-\frac{mv^2}{2k_B T}\right); \quad on\ \Omega \times \mathbb{R}^3.$$

6. Counter-examples and remarks

Let us point out that we have not used the mean regularity results of [7, 10]. The regularizing effects of the stationary Maxwell equations are sufficient to control the non-linearity that appears in the Vlasov equation (lemma 1). The situation would be different if the model included a non-linear Boltzmann operator. Then mean compactness results would be necessary. Such a problem arises in the kinetic theory of semiconductors. The Cauchy problem in a free space has been studied in [12]. In a forthcoming paper we will also give an analysis of boundary problems for the stationary equations. Although in the previous sections, the results of [7] are not necessary in the proof of existence, they provide some indications on the regularity of the solutions. Indeed for a solution that has a compact support with respect to velocities (it is the case if the entering data is compactly supported) the corresponding concentration belongs to $H^{1/4}_{\mathrm{loc}}(\Omega)$ [7, theorem 3]. It follows that the potential belongs to $H^{2+1/4}_{\mathrm{loc}}(\Omega)$.

We now focus our attention on the question of uniqueness of solutions. We give two counter-examples. They are based on the idea to trap particles with a potential created by a background charge density or by an other species. In the following examples, the solutions of the Vlasov equation depend only on $|v|^2$. It follows that a magnetic field has no effect on these distributions. Thus, the following multiple solutions of the Vlasov-Poisson equations give also multiple solutions for the Vlasov-Maxwell systems.

Example 2. Particles trapped by a background charge density. In dimensionless variables, the Vlasov-Poisson problem that we deal with reads

$$\begin{cases} v.\nabla_x f - \nabla_x \phi . \nabla_v f = 0 \\ f_{|\Sigma^-} = 0 \end{cases}$$

$$\begin{cases} -\Delta_x \phi = \varrho - N; \quad \varrho = \int_{\mathbb{R}^3} f dv \\ \phi_{|\partial\Omega} = 0. \end{cases}$$

Let F be a smooth function such that

(57) $\quad F(t) = 0$ for $t \geq 0$; $\quad F(t) > 0$ for $t < 0$.

We let n_0 be an arbitrary positive real number, $n_0 > 0$. We define ψ by

$$-\Delta_x \psi = -n_0; \quad \psi_{|\partial\Omega} = 0.$$

We let the background charge density N be equal to

$$N(x) = \int_{\mathbb{R}^3} F\left(\psi(x) + \frac{v^2}{2}\right) dv + n_0$$

and finally we define Φ_0 by

$$-\Delta_x \Phi_0 = -N; \quad \Phi_{0|\partial\Omega} = 0.$$

The two solution of the Vlasov-Poisson problem are $(f = 0, \phi = \Phi_0)$ and $\left(f = F\left(\psi(x) + \frac{v^2}{2}\right), \phi = \psi\right)$. The function $F\left(\psi(x) + \frac{v^2}{2}\right)$ is not vanishing since the function ψ is negative on Ω.

Example 3. Particles trapped by an other species. A Vlasov-Poisson system for two species reads

$$\begin{cases} v.\nabla_x f_1 - \nabla_x \phi.\nabla_v f_1 = 0 \\ f_{1|\Sigma^-} = 0 \end{cases} \qquad \begin{cases} v.\nabla_x f_2 + \nabla_x \phi.\nabla_v f_2 = 0 \\ f_{2|\Sigma^-} = (2\pi)^{-3/2} \exp\left(-\frac{v^2}{2}\right) \end{cases}$$

$$\begin{cases} -\Delta_x \phi = \varrho_1 - \varrho_2; \quad \varrho_i = \int_{\mathbb{R}^3} f_i dv \\ \phi_{|\partial\Omega} = 0. \end{cases}$$

Let Φ_0 be the solution of

$$-\Delta_x \Phi_0 = -\exp(\Phi_0); \quad \Phi_{0|\partial\Omega} = 0.$$

One solution of the problem is given by

$$\left(f_1 = 0, f_2 = (2\pi)^{-3/2} \exp\left(\Phi_0(x) - \frac{v^2}{2}\right), \phi = \Phi_0\right).$$

Let F be a smooth non increasing function that satisfies (57). We put

$$\varrho_F(\phi) = \int_{\mathbb{R}^3} F\left(\frac{v^2}{2} + \phi\right) dv.$$

The function ϱ_F is non increasing with respect to ϕ. Therefore there is a unique solution ϕ_F of the non-linear elliptic problem

$$-\Delta_x \phi = -\exp(\phi) + \varrho_F(\phi); \quad \phi_{|\partial\Omega} = 0.$$

The potential ϕ_F is not equal to Φ_0. Otherwise we get $\varrho_F(\Phi_0) = 0$ which is impossible since Φ_0 is negative on Ω. Indeed in view of (57) we get $\varrho_F(\phi) > 0$ for $\phi > 0$. Thus a family of solutions is given by

$$\left(f_1 = F\left(\frac{v^2}{2} + \phi_F\right), f_2 = (2\pi)^{-3/2} \exp\left(\phi_F - \frac{v^2}{2}\right), \phi = \phi_F\right).$$

For Vlasov-Poisson systems with only positive (or negative) charges we are not able to give such counter-examples. In [11], multiple solutions are described when the entering data is a Dirac distribution. We think that even for smooth entering data multiple solutions may occur. Indeed let us consider the following situation

$$\begin{cases} v.\nabla_x f - \nabla_x \phi . \nabla_v f = 0 \\ f_{|\Sigma^-} = F(v^2) \end{cases} \qquad \begin{cases} -\Delta_x \phi = \varrho;\ \varrho = \int_{\mathbb{R}^3} f dv \\ \phi_{|\partial\Omega} = 0 . \end{cases}$$

Then if Φ is a solution of

$$-\Delta_x \Phi = n(\Phi); \quad \Phi_{|\partial\Omega} = 0; \quad n(\phi) = \int_{\mathbb{R}^3} F\left(\frac{v^2}{2} + \phi\right) dv .$$

$\left(f = F\left(\frac{v^2}{2} + \Phi\right), \phi = \Phi\right)$ is a solution of the Vlasov-Poisson system. But if F is chosen in order that the function n is not non-decreasing, the above non-linear elliptic problem may have several solutions.

As a consequence of the non-uniqueness of solutions, these solutions do not necessary have the same symmetries as the data. However, working with symmetric functions, the techniques presented in this paper provide symmetric solutions. Such a result is derived in [4].

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