Werk

Titel: Coordinate free definitions of cross ratio.

Autor: Simonis, J.; van de Craats, J. de

Jahr: 1985

PURL: https://resolver.sub.uni-goettingen.de/purl?378850199_0040|log19

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Coordinate free definitions of cross ratio

1. Introduction

Let P_1, P_2, P_3, P_4 be four distinct points on the projective line $\mathbf{P}(V)$, i.e., four distinct one-dimensional linear subspaces of the two-dimensional vector space V over a field K. Their cross ratio $[P_1, P_2; P_3, P_4]$ usually is defined by means of coordinates (see, e.g., Berger [1], p. 153–155). It might be of interest of show that an elegant coordinate free alternative is possible. Our main tools are parallel projections in V on lines through the origin. We shall define cross ratio as the trace of the composition of two such endomorphisms. As a matter of fact, it is well-known that the notion of trace is a coordinate free concept.
The derivation of a few simple results on traces in section 3 paves the way for a complete treatment of cross ratio in section 4. Next we show that our definition is equivalent to the usual ones with coordinates, and finally we present two alternative possibilities for invariant definitions of cross ratio.

1.1. We shall employ the following elementary properties of the trace $\operatorname{tr}(\varphi)$, where φ is an endomorphism of V:
(i) $\operatorname{tr}(\varphi + \psi) = \operatorname{tr}(\varphi) + \operatorname{tr}(\psi)$,
(ii) $\operatorname{tr}(\varphi \cdot \psi) = \operatorname{tr}(\psi \cdot \varphi)$,
(iii) if φ has two eigenvalues λ_1, λ_2, then $\operatorname{tr}(\varphi) = \lambda_1 + \lambda_2$ (recall that V is two-dimensional).

2. The projections π^i_j

2.1 *Definition*

For $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, let π^i_j: $V \to V$ be the projection on P_j in the direction P_i.

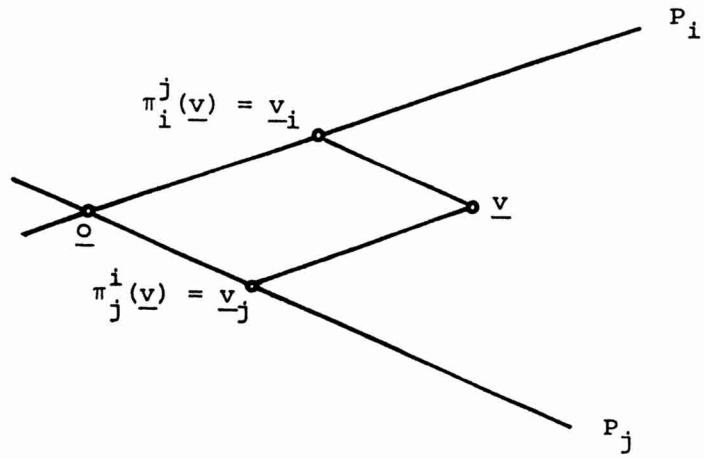

Figure 1

(Recall that any $\underline{v} \in V$ can be written in a unique way as $\underline{v} = \underline{v}_i + \underline{v}_j$ with $\underline{v}_i \in P_i, \underline{v}_j \in P_j$. Then, by definition, $\pi_j^i(\underline{v}) = \underline{v}_j$.)

2.2. We list some obvious properties of the π_j^i:

(i) $(\pi_j^i)^2 = \pi_j^i$,

(ii) $\pi_j^i + \pi_i^j = id$ (the identity mapping),

(iii) $\operatorname{tr}(\pi_j^i) = 1$ (cf. 1.1 (iii)).

3. The endomorphisms $\pi_j^i \pi_l^k$

3.1. The eigenspaces of the endomorphisms $\pi_4^1 \pi_3^2$ obviously are P_2 and P_4; the corresponding eigenvalues being 0 and λ for some $\lambda \neq 0$. Thus $\lambda = \operatorname{tr}(\pi_4^1 \pi_3^2)$. Figure 2 illustrates the effect of $\pi_4^1 \pi_3^2$ on an eigenvector $\underline{p} \in P_4$.

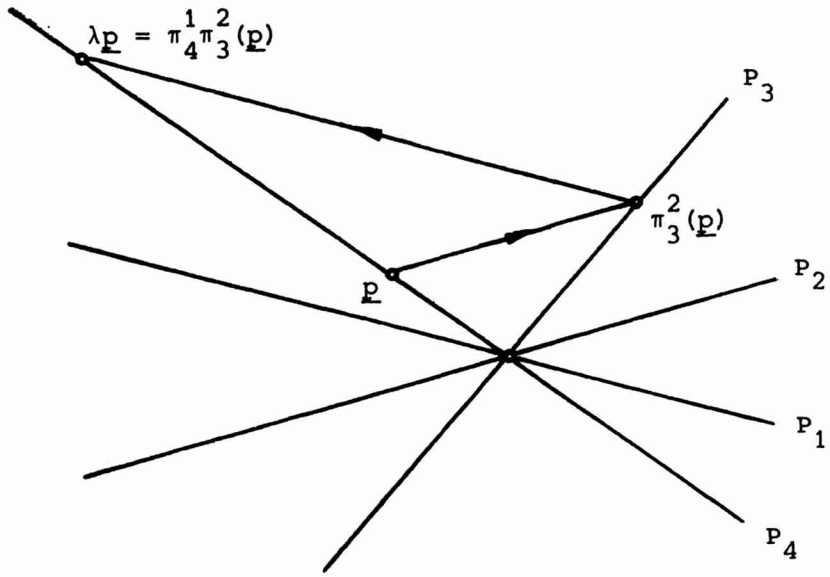

Figure 2

3.2 *Lemma*

$$\operatorname{tr}(\pi_4^1 \pi_3^2) = \operatorname{tr}(\pi_3^2 \pi_4^1), \tag{i}$$

$$\operatorname{tr}(\pi_4^1 \pi_3^2) \cdot \operatorname{tr}(\pi_4^2 \pi_3^1) = 1, \tag{ii}$$

$$\operatorname{tr}(\pi_4^1 \pi_3^2) + \operatorname{tr}(\pi_4^1 \pi_2^3) = 1. \tag{iii}$$

Proof:

(i) This instance of 1.1 (ii) can also be verified directly:

$\pi_3^2 \pi_4^1(\pi_3^2(\underline{p})) = \pi_3^2(\pi_4^1 \pi_3^2(\underline{p})) = \pi_3^2(\lambda \underline{p}) = \lambda \pi_3^2(\underline{p})$

($\underline{p} \in P_4$, see fig. 2), so the nonzero eigenvalue of $\pi_3^2 \pi_4^1$ is λ.

(ii) $\pi_4^2 \pi_3^1$ maps $\pi_4^1 \pi_3^2(\underline{p})$ onto $\underline{p}$ (reverse the arrows in fig. 2!). So the nonzero eigenvalue of $\pi_4^2 \pi_3^1$ is $\dfrac{1}{\lambda}$.

(iii) $\pi_4^1\pi_3^2 + \pi_4^1\pi_2^3 = \pi_4^1(\pi_3^2 + \pi_2^3) = \pi_4^1$.
Now apply 1.1 (i) and 2.2 (iii).

3.3 *Corollary*

Since the permutations (12) (34), (12) and (23) generate the group of permutations of $\{1,2,3,4\}$, we can express $\operatorname{tr}(\pi_j^i\pi_l^k)$ as a function of $\operatorname{tr}(\pi_4^1\pi_3^2)$ for any four distinct indices i,j,k,l. In particular, we have

$$\operatorname{tr}(\pi_4^1\pi_3^2) = \operatorname{tr}(\pi_3^2\pi_4^1) = \operatorname{tr}(\pi_1^4\pi_2^3) = \operatorname{tr}(\pi_2^3\pi_1^4).$$

4. Cross ratio

4.1 *Definition*

The *cross ratio* $[P_1, P_2; P_3, P_4]$ of the ordered quadruple (P_1, P_2, P_3, P_4) is the element

$$[P_1, P_2; P_3, P_4] = \operatorname{tr}(\pi_4^1\pi_3^2) \in K.$$

4.2 *Proposition*

$$[P_1, P_2; P_3, P_4] = [P_2, P_1; P_4, P_3] = [P_4, P_3; P_2, P_1] = [P_3, P_4; P_1, P_2], \tag{i}$$
$$[P_1, P_2; P_3, P_4] \cdot [P_2, P_1; P_3, P_4] = 1, \tag{ii}$$
$$[P_1, P_2; P_3, P_4] + [P_1, P_3; P_2, P_4] = 1. \tag{iii}$$

Proof: 3.2, 3.3.

4.3 *Proposition*

If $\psi: V \to V'$ is an isomorphism of V onto another K-vector space V', and if $P_i' = \psi(P_i)$ $(i = 1,2,3,4)$, then

$$[P_1, P_2; P_3, P_4] = [P_1', P_2'; P_3', P_4'].$$

Proof: $\bar{\pi}_j^i = \psi\pi_j^i\psi^{-1}$ is the projection on P_j' along P_i', so

$$\begin{aligned}[P_1', P_2'; P_3', P_4'] &= \operatorname{tr}(\bar{\pi}_4^1\bar{\pi}_3^2) = \operatorname{tr}(\psi\pi_4^1\pi_3^2\psi^{-1}) = \operatorname{tr}(\psi^{-1}\psi\pi_4^1\pi_3^2)\\ &= \operatorname{tr}(\pi_4^1\pi_3^2) = [P_1, P_2; P_3, P_4].\end{aligned}$$

4.4 *Remark*

From $P_1 \neq P_2$ and $P_3 \neq P_4$ it follows that $\lambda = [P_1, P_2; P_3, P_4] \neq 1$ (cf. fig. 2), and from $P_1 \neq P_3$, $P_2 \neq P_4$ it follows that $\lambda = [P_1, P_2; P_3, P_4] \neq 0$.
Figure 2 strongly suggests how to *define* $[P_1, P_2; P_3, P_4]$ if two of the P_i coincide. Indeed, one usually defines

$$[P_1, P_1; P_3, P_4] = [P_1, P_2; P_3, P_3] = 1,$$
$$[P_1, P_2; P_1, P_4] = [P_1, P_2; P_3, P_2] = 0$$

and also

$$[P_1, P_2; P_3, P_1] = [P_1, P_2; P_2, P_4] = \infty .$$

4.5 *Proposition*

If $[P_1, P_2; P_3, P_4] = [Q_1, Q_2; Q_3, Q_4]$ and if $P_i = Q_i$ for three values of i, then $P_i = Q_i$ for all four values of i.

Proof: On account of proposition 4.2 it is sufficient to prove that from $[P_1, P_2; P_3, P_4] = [P_1, Q_2; P_3, P_4]$ it follows that $P_2 = Q_2$. Renaming, for convenience, $Q_2 = P_5$, we have $\operatorname{tr}(\pi_4^1 \pi_3^2) = \operatorname{tr}(\pi_4^1 \pi_3^5)$.

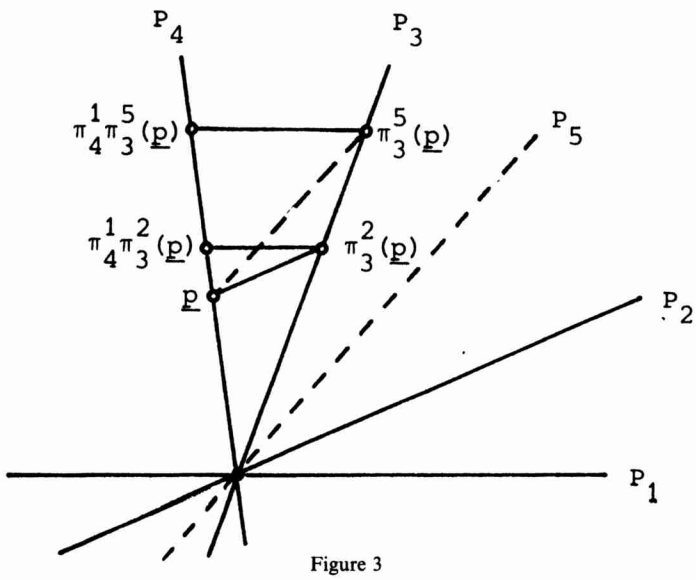

Figure 3

This implies that $\pi_4^1 \pi_3^2(\underline{p}) = \pi_4^1 \pi_3^5(\underline{p})$ for all eigenvectors $\underline{p} \in P_4$, whence $P_2 = P_5 (= Q_2)$.

4.6 *Corollary*

If $P_1, P_2, P_3, P_4 \in \mathbf{P}(V)$, $Q_1, Q_2, Q_3, Q_4 \in \mathbf{P}(V')$, where V' is another two-dimensional K-vector space, and if

$$[P_1, P_2; P_3, P_4] = [Q_1, Q_2; Q_3, Q_4]$$

then there exists an isomorphism $\psi : V \to V'$ such that $\psi(P_i) = Q_i$ $(i = 1, 2, 3, 4)$.

Proof: There exists an isomorphism $\psi : V \to V'$ such that

$$\psi(P_i) = Q_i \quad (i = 1, 2, 3).$$

By proposition 4.3 we have

$$([Q_1, Q_2; Q_3, Q_4] =)[P_1, P_2; P_3, P_4]$$
$$= [\psi(P_1), \psi(P_2); \psi(P_3), \psi(P_4)] = [Q_1, Q_2; Q_3, \psi(P_4)]$$

whence $\psi(P_4) = Q_4$.

5. Calculation of cross ratio in coordinates

5.1 *Proposition*

If, with respect to some coordinate system in V, the one-dimensional subspaces P_i are spanned by vectors with coordinates (x_i, y_i), then

$$[P_1, P_2; P_3, P_4] = \frac{\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \begin{vmatrix} x_2 & x_4 \\ y_2 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_4 \\ y_1 & y_4 \end{vmatrix} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}}.$$

Proof: Writing $[ij] = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$ we have

$$\pi_j^i = \frac{1}{[ij]} \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_j & -x_j \\ -y_i & x_i \end{pmatrix} = \frac{1}{[ij]} \begin{pmatrix} -x_j y_i & x_j x_i \\ -y_j y_i & y_j x_i \end{pmatrix}$$

so

$$\operatorname{tr}(\pi_4^1 \pi_3^2) = \frac{1}{[14][23]} (x_4 y_1 x_3 y_2 - x_4 x_1 y_3 y_2 - y_4 y_1 x_3 x_2 + y_4 x_1 y_3 x_2) = \frac{[13][24]}{[14][23]}.$$

5.2. If, in particular, one chooses coordinates such that $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (0, 1)$, $(x_3, y_3) = (1, 1)$, $(x_4, y_4) = (\lambda, 1)$ for some $\lambda \in K$ (this is always possible), then we have

$$[P_1, P_2; P_3, P_4] = \lambda.$$

This also shows that for any $\lambda \in K$, $\lambda \neq 0, 1$, there are four distinct P_1, P_2, P_3, P_4 such that $[P_1, P_2; P_3, P_4] = \lambda$.

6. Alternative

The following formulae provide alternative invariant definitions of the cross ratio $\lambda = [P_1, P_2; P_3, P_4]$:

$$\pi_4^1 \pi_3^2 + \pi_2^3 \pi_1^4 = \lambda \cdot id, \tag{i}$$
$$\det(id - \pi_4^1 \pi_2^3) = \lambda. \tag{ii}$$