Die beiden Körper L und S sind in dem Sinne extremal, als sie die in I) bzw. II) links stehenden Linearformen der drei fundamentalen Masszahlen M, F und V zu einem Maximum bzw. zu einem Minimum machen.

Da sich ein beliebiger konvexer Rotationskörper durch einen Äquatorschnitt in zwei Halbkörper zerlegen lässt, ist es im Hinblick auf das noch nicht restlos geklärte Problem des vollständigen Ungleichungssystems für M, F und V (Hauptproblem) eine sich aufdrängende taktische Massnahme, passende Extremaleigenschaften der Halbkörper zu studieren. Unsere Ungleichungen liefern starke Indizien dafür, dass sich jeder Extremalkörper des Hauptproblems aus höchstens drei Segmenten zusammensetzen lässt, wobei als zulässige Segmente lediglich Zylinder Z und abgestumpfte Kappenkörper der Kugelhalblinse L in Betracht fallen. Hierbei ist zu beachten, dass der Segmenttyp L zahlreiche wichtige Sonderfälle umfasst, nämlich etwa Kugellinsenkappenkörper, Kugelschicht, Kegel, Kegelstumpf, Linse usw., wobei immer nur die entsprechenden Halbkörperformen gemeint sind. – Die vielen Einzelergebnisse der Untersuchungen von H. BIERI (Bern) [vgl. insb. «Beitrag zu einem Extremalproblem über konvexe Rotationskörper», Experientia *14*, 1958, 113–116 und die dort zitierte Literatur] haben stets die Vorrangstellung der oben genannten elementaren Körpertypen erkennen lassen, und einzelne davon sind bei Teillösungen zum Hauptproblem bereits gesichert.

Die hier nur kurz mitgeteilten Resultate I) und II) wurden im Rahmen einer kleinen Spezialvorlesung über Elementarmathematik vom höheren Standpunkt aus im SS 1963 erörtert. Diese werden in einer sich allgemeiner auf drei Minkowskische Quermassintegrale n-dimensionaler konvexer Rotationskörper beziehenden Note an anderer Stelle eingehender begründet werden.

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Higher Moments of Plane Convex Sets

It is well known that if every chord of a plane convex set bisects its area, then the set is radially symmetric. In Theorem 2 this is generalized to the equality of nth moments where n is even (the case $n = 0$ being the case quoted). However if n is odd, each chord through a given point can cut the set into two sets of equal nth moments around the chord even though the set is not radially symmetric; this is Theorem 1. Theorem 3 states that the center of gravity of a mirror-symmetric set corresponding to equality of higher moments has at least four normals through it.

We will assume that the bounding curves of the convex sets under consideration have a continuously turning tangent line. The proof of Theorem 3 goes through unchanged if we relax this condition at the two points of the boundary on the line of symmetry.

Let K be a plane convex set and L a line in the plane and not intersecting the interior of K. By the nth moment of K around L we shall mean $\int_K f(P)\, dA$ where $f(P)$ is the nth power of the distance from P to L. (We take $f(P)$ to be nonnegative; hence the nth moment of K around L is positive.)

Theorem 1. Let n be an odd positive integer. There exists a plane convex set K, not radially symmetric, and a point P^* in K such that every chord of K that passes through P^* cuts K into two sets whose nth moments around the chord are equal.

Proof. Let f be a function of period 2π such that the Fourier Series of $[f(\theta)]^{n+2}$, $(a_0/2) + \sum_{j=i}^{\infty} a_j \cos j\,\theta + b_j \sin j\,\theta$, has $a_j = 0 = b_j$ for $j = 1, 2, \ldots, n$. Furthermore, choose the remaining coefficients in such a way that the set K whose border has the equation $r = f(\theta)$ is convex and not symmetric with respect to the pole, which is designated P^*.

Consider a chord of K through P^* and making an angle α with the polar axis, which we also take to be an x axis.

We wish to prove that

$$\int_K (x \cos\alpha + y \sin\alpha)^n \, dA = 0 \tag{1}$$

To establish (1) it suffices to show that

$$\int_K x^r y^s \, dA = 0 \tag{2}$$

for each pair of nonnegative integers r and s such that $r + s = n$. Now

$$\int_K x^r y^s \, dA = \int_0^{2\pi} \int_0^{f(\theta)} r^{n+1} \cos^r\theta \sin^s\theta \, dr \, d\theta = [1/(n+2)] \int_0^{2\pi} [f(\theta)]^{n+2} \cos^r\theta \sin^s\theta \, d\theta .$$

Since $\cos^r\theta \sin^s\theta$ is a linear combination of $\cos i\theta$ and $\sin j\theta$ where $0 < i, j \leqslant n$, it follows that (2) holds. The theorem is proved.

Theorem 2. Let K be a plane convex set, n a nonnegative even integer, and P^* a point in K such that every chord through P^* cuts K into two sets of equal nth moments around the chord. Then K is radially symmetric around P^*.

Proof. Let the border of K have the equation $r = f(\theta)$ relative to a fixed polar coordinate system whose pole is at P^*. Then for each a in $[0, \pi]$ we have

$$\int_a^{a+\pi} \int_0^{f(\theta)} [r \sin(\theta - a)]^n \, r \, dr \, d\theta = \int_{a+\pi}^{a+2\pi} \int_0^{f(\theta)} [r \sin(\theta - a)]^n \, r \, dr \, d\theta$$

hence

$$\int_a^{a+\pi} [f(\theta)]^{n+2} \sin^n(\theta - a) \, d\theta = \int_{a+\pi}^{a+2\pi} [f(\theta)]^{n+2} \sin^n(\theta - a) \, d\theta . \tag{3}$$

Now

$$\int_{a+\pi}^{a+2\pi} [f(\theta)]^{n+2} \sin^n(\theta - a) \, d\theta = \int_a^{a+\pi} [f(\theta + \pi)]^{n+2} \sin^n(\theta - a) \, d\theta .$$

Thus if we let $g(\theta) = [f(\theta)]^{n+2} - [f(\theta + \pi)]^{n+2}$, (3) is equivalent to

$$\int_a^{a+\pi} g(\theta) \sin^n(\theta - a) \, d\theta = 0 \tag{4}$$

for each $a \in [0, \pi]$. Note that $g(\theta + \pi) = -g(\theta)$. From (4) we will show that $g(\theta) = 0$ for all θ; this will complete the proof.

Differentiation of (4) with respect to a shows that

$$\int_a^{a+\pi} g(\theta) \sin^{n-1}(\theta - a) \cos(\theta - a) \, d\theta = 0 \tag{5}$$

and a similar differentiation of (5) shows that

$$\int_a^{a+\pi} g(\theta) \, [(n-1) \sin^{n-2}(\theta - a) \, (-\cos^2(\theta - a)) + \sin^n(\theta - a)] \, d\theta = 0 . \tag{6}$$

From (4), (6), and the identity $\cos^2(\theta - a) = 1 - \sin^2(\theta - a)$, it follows that

$$\int_a^{a+\pi} g(\theta) \sin^{n-2}(\theta - a) \, d\theta = 0 .$$

Repeated application of these two steps shows that

$$\int_a^{a+\pi} g(\theta)\, d\theta = 0$$

for all a. Thus $g(a + \pi) = g(a)$ for all a. Combining this with the identity $g(a + \pi) = -g(a)$ yields $g(a) = 0$ for all a and the proof is done.

In the next theorem we shall refer to a 'normal' to a curve. By this we will mean a line that is perpendicular to a support line of the curve and passes through a point of contact. 'Normal' to a surface, used in the corollary following, is defined similarly.

Theorem 3. Let K be a plane convex set symmetric with respect to the y axis and furnished with the density $|x|^\alpha$ where α is a nonnegative real number. Then through the center of gravity of K pass at least four normals to the border of K.

Proof. Two such normals are parallel to the y axis. In view of the symmetry of K it suffices to prove the existence of one more normal through the center of gravity.

Let the x axis be placed in such a way that the center of gravity is at the origin. We have then

$$\int_K |x|^\alpha y\, dA = 0 \tag{7}$$

Exploiting the symmetry of K and using polar coordinates, we translate (7) into

$$\int_{-\pi/2}^{\pi/2} \int_0^{f(\theta)} (r \cos\theta)^\alpha r \sin\theta\, r\, dr\, d\theta = 0\,, \tag{8}$$

where $r = f(\theta)$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, describes that part of K not to the left of the y axis. Thus

$$\int_{-\pi/2}^{\pi/2} [f(\theta)]^{\alpha+3} \cos^\alpha\theta \sin\theta\, d\theta = 0\,. \tag{9}$$

Integration by parts, applied to (9), with $u = [f(\theta)]^{\alpha+3}$, $dv = \cos^\alpha\theta \sin\theta\, d\theta$, implies

$$\int_{-\pi/2}^{\pi/2} \cos^{\alpha+1}\theta [f(\theta)]^{\alpha+2} f'(\theta)\, d\theta = 0\,. \tag{10}$$

Since $\cos^{\alpha+1}\theta[f(\theta)]^{\alpha+2} \geqslant 0$ for $\theta \in [-\pi/2, \pi/2]$, (10) implies that $f'(\theta) = 0$ for at least one $\theta \in (-\pi/2, \pi/2)$. This proves the theorem.

Corollary. Let S be a three dimensional convex solid of revolution furnished with a density r^β, $\beta \geqslant 0$ where r denotes the distance to the axis of revolution. Then through the center of gravity of S pass an infinite set of normals (the two normals on the axis of revolution and at least one 'equator' of normals).

Proof. Let S be obtained by rotating the mirror symmetric convex planar set K about the y axis. Place the x axis in such a way that the center of gravity of S is at the origin and K lies in the xy plane.

We have

$$\int_S y\, r^\beta\, dV = 0$$

or, equivalently,

$$\int_{K^+} y\, x^\beta \cdot 2\pi x\, dA = 0\,,$$

where K^+ denotes the portion of K not to the left of the y axis. Theorem 3, with $\alpha = 1 + \beta$, yields the corollary.

Note that the corollary shows that a homogeneous convex 'top' balances not only at its top and bottom.

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