einziges Beispiel möge die automatische Berechnung von Stundenplänen angeführt werden (siehe [4]). Dort geht es, im Gegensatz zu unserer Puzzle-Aufgabe, nicht darum, alle Lösungen zu finden, sondern man wird im allgemeinen damit zufrieden sein, *eine* Lösung gefunden zu haben, welche mit allen Nebenbedingungen verträglich ist, und dann die Durchmusterung des Baumes abbrechen. Allenfalls können noch Optimierungsforderungen dazu kommen.

Ganz allgemein muss bei kombinatorischen Problemen unterschieden werden zwischen der Frage nach der Anzahl von Lösungen und der Aufgabe, die Lösungen effektiv zu konstruieren. Nun ist es aber so, dass auch in Fällen, da die erste der beiden Aufgaben einigermassen elementar gelöst werden kann, die zweite zum mindesten nicht ganz trivial zu sein braucht, und dass dann die Aufgabe, einen vernünftigen Algorithmus (lies: ein Computerprogramm) aufzustellen, oft sehr reizvoll ist. Eine hübsche Zusammenstellung von Problemen, die es zum Teil verdienen, auch von diesem Standpunkte aus betrachtet zu werden, findet man in [3].

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The Congruence $2^{p-1} \equiv 1 \pmod{p^2}$ and Quadratic Forms with High Density of Primes

(In memory of N. G. W. H. BEEGER, who died October 5, 1965)

There are not too many great mathematicians in the field where number theory and recreational mathematics overlap, and the most recent decennium took two away: Maurice Borisovich KRAITCHIK, who died August 19, 1957, in Brussels, 75 years old, and Nicolaas George Wijnand Henri BEEGER, who died in Amsterdam, over 80 years old. Both were friends and enriched each other and the world with their fruitful work.

In honoring N. G. W. H. BEEGER it may be permitted to digress a little from the subject. Beeger's modesty and unselfishness went so far, that as head of a commission for publishing the prime numbers of the 11th million and as the editor of this work, he didn't even mention himself in the title [2][1]), or, when he found in 1938 the quadratic form $x^2 + x - 53509$ with high density of primes, the smallest prime factor appearing in it being 61, he communicated this pearl to Luigi POLETTI of Pontremoli

[1]) Numbers in brackets refer to References, page 88.

(now of Livorno) and waited patiently for 13 years, until POLETTI published it [13], and that even with the print error – 52509.

But Beeger's fame originated already in 1922 with the discovery of the second Wieferich square of base 2, which is the more remarkable, since the most advanced electronic computers of today have not yet found a third one of this kind. A Wieferich square of base a is the square of a prime, p^2, such that the congruence $a^{p-1} \equiv 1 \pmod{p^2}$ is valid. Though there were known certain small p for $a = 3, 7, 8$ with $p = 11, 5, 3$, respectively, the case $a = 2$ became important, when A. WIEFERICH of Münster demonstrated in 1909 (Crelle's Journal, Vol. 136) that if p is prime and $2^p - 2$ is not divisible by p^2, the equation $x^p + y^p = z^p$ cannot be solved in terms of positive integers which are not multiples of p. W. MEISSNER of Charlottenburg [10] found then that $2^p - 2$ is divisible by p^2 when $p = 1093$ and for no other prime p less than 2000. N. G. W. H. BEEGER of Amsterdam [1] searched through all $p < 10000$ and found $2^{3510} \equiv 1 \pmod{3511^2}$. Carl-Erik FRÖBERG of Lund [6] searched through all $10000 < p < 50000$ and found nothing. The same happened to Sidney KRAVITZ of Picatinny Arsenal [8] for $50000 < p < 100000$, to Hans RIESEL of Stockholm [14] for $100000 < p < 500000$, to Melvin HAUSNER of the Courant Institute [7] for $500000 < p < 1000000$, and to David SACHS of New York [15] for $1000000 < p < 2000000$.

Let us now treat quadratic forms with high density of primes. They originated with Euler's famous polynomials $x^2 + x + q$ with $q = 3, 5, 11, 17$, and 41, which take on only prime values when $0 \leqq x \leqq q - 2$. But beside this peculiarity there exist others:
(1) their factorization for x beyond $q - 2$ yields q the smallest prime appearing in it, (2) their discriminant $d = 1 - 4q$ is a negative prime, (3) q is prime, (4) $q + 2$ is prime, (5) the field $R(\sqrt{d})$ has class number 1. Moreover, Harvey COHN of Tucson [3, p. 156] proves the following theorem: The polynomial $x^2 + x + q$ will assume prime values for $0 \leqq x \leqq q - 2$ if and only if for $d = 1 - 4q$, $R(\sqrt{d})$ has class number 1.

Special attention because of the high density of primes drew the form $x^2 + x + 41$. Luigi POLETTI (Math. Tabl. and other Aids to Comp. *2*, 354 (1947)) exploited this form for primes up to $x = 55102$. The corresponding number of primes is 18667. A. FERRIER of Cusset (now of Ebrevil, Allier) [5, p. 16] substitutes $x - 40$ in $x^2 + x + 41$ obtaining $x^2 - 79x + 1601$, in which the first 80 values are primes. This is the same form which Howard EVES mentions [4, p. 145]. N. R. PEKELHARING [12] writes an article: "The Number 41", and D. H. LEHMER of Berkeley [11] writes several: On the Function $X^2 + X + A$. Sidney KRAVITZ [9] is more interested in the composite numbers. He finds the smallest $f(n) = n^2 - n + 41$ which is divisible by three not necessarily distinct prime factors to be $f(421) = 47 \cdot 53 \cdot 71$ [$= f(420)$ in the x-notation] of which he writes: "This result was found by laborious calculation with a desk computer".

To eliminate similar hardships, the author broke up the composites of $x^2 + x + 41$ into sets and subsets of astonishing permanence and symmetry. Introducing the parameter y we find:

$$f(y_0^2 + 40) = (y_0^2 - y_0 + 41)(y_0^2 + y_0 + 41) \text{ for } y_0 = 0, 1, 2, \ldots$$

$$f(2y_1^2 - y_1 + 81) = (y_1^2 - y_1 + 41)(4y_1^2 + 163) \text{ for } y_1 = 0, 1, -1, 2, -2, \ldots$$

$$f[(k+1)(y_k^2 - y_k + 41) + y_k - 1] = (y_k^2 - y_k + 41)[(k+1)^2 (y_k^2 - y_k + 41) + (k+1)(2y_k - 1) + 1] \text{ for } y_k = 0, 1, -1, 2, -2, \ldots$$

This y-set eliminates all composite numbers up to $x = 243$ and many beyond. Now, introducing the parameter z for the second set, we receive:

$$f(6z_0^2 - z_0 + 244) = (4z_0^2 + 163)(9z_0^2 - 3z_0 + 367) \text{ for } z_0 = 0, 1, -1, 2, -2, \ldots$$

$$f(10z_1^2 - z_1 + 407) = (4z_1^2 + 163)(25z_1^2 - 5z_1 + 1019) \text{ for } z_1 = 0, 1, -1, 2, -2, \ldots$$

which eliminates all composite numbers up to $x = 488$. Hence $f(420) = 47 \cdot 53 \cdot 71$ falls easily out by means of the subsets y_5, y_7, and y_8, since $6y_5^2 - 5y_5 + 245 = 8y_7^2 - 7y_7 + 327 = 9y_8^2 - 8y_8 + 368 = 420$ for $y_5 = -5$, $y_7 = -3$, and $y_8 = -2$. The next $f(x)$ with 3 prime factors, $f(431) = 43 \cdot 61 \cdot 71$, falls out with y_5, y_6, and y_9, since $6y_5^2 - 5y_5 + 245 = 7y_6^2 - 6y_6 + 286 = 10y_9^2 - 9y_9 + 409 = 431$ for $y_5 = 6$, $y_6 = 5$, $y_9 = 2$.

But not only the form $x^2 + x + 41$ has high density of primes. D. H. LEHMER [11] in 1936 by means of a mechanical device found similar ones:

$x^2 + x + 19421$ with smallest $p = 47$,
$x^2 + x + 333491$ with smallest $p = 53$,
$x^2 + x + 601037$ with smallest $p = 61$,
$x^2 + x + 5237651$ with smallest $p = 67$,
$x^2 + x + 9063641$ with smallest $p = 71$,
$x^2 + x + 12899891$ with smallest $p = 73$,
$x^2 + x + 24073871$ with smallest $p = 83$,
$x^2 + x + 28537121$ with smallest $p = 89$,
$x^2 + x + 67374467$ with smallest $p = 107$,
$x^2 + x + 146452961$ with smallest $p = 109$.

It should be mentioned that A is not always prime, for example: $146452961 = 1459 \cdot 100379$, but 67374467 is prime.

Since the device was set up for maximal p disregarding smaller or equal p and A in continuing the search, forms with smallest $p = 43$, 59, 79, 97, 101, and 103 did not occur (as would have for smallest $p = 43$ at $x^2 + x + 55661$). Therefore, one should not wonder that in 1938 N. G. W. H. BEEGER of Amsterdam found $x^2 + x + 27941$ and $x^2 + x + 72491$ with two further smallest $p = 47$. BEEGER was also the first to extend those forms to negative A and found, as mentioned previously, the form $x^2 + x + 53509$ with smallest $p = 61$. Finally, the present writer discovered in the entire range $-300000 < A < 300000$ for $p > 43$, besides the forms already cited:

$x^2 + x - 42739$ with smallest $p = 47$,
$x^2 + x - 98563$ with smallest $p = 47$,
$x^2 + x - 129403$ with smallest $p = 47$,
$x^2 + x - 152839$ with smallest $p = 47$,
$x^2 + x - 244843$ with smallest $p = 47$,
$x^2 + x - 258163$ with smallest $p = 47$,
$x^2 + x - 90073$ with smallest $p = 53$,
$x^2 + x - 169933$ with smallest $p = 59$,
$x^2 + x - 211999$ with smallest $p = 59$,
$x^2 + x - 249439$ with smallest $p = 61$.

It would be rewarding to exploit $x^2 + x + A$ for $A < -300000$, since this would yield counter parts to the large A and p of LEHMER.

Let us define "high density of primes" completely arbitrary as "having at least 60% primes within the first 160 values of $f(x)$". By this definition some new forms $x^2 + x + A$ may be found and some old ones deleted. But, in general, one could guess that no $x^2 + x + A$ with high density of primes exists with, let say, smallest $p < 17$. Unfortunately, this guess is wrong. The present writer found recently a quadratic