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Homogeneity or otherwise for Certain Morphism Spaces

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We examine the homogeneity or otherwise of the space of continuous homomorphisms from one nontrivial compact group to another taken with the compact open topology. If the range group is abelian then pointwise defined addition (not composition, although we are using the compact open topology) gives a topological group which is shown to be totally disconnected. If both compact groups are the same and this one group is nonabelian but is connected (something less will do) then our space is nonhomogeneous.

We note that this work arose from a paper of Professor A. D. Wallace [3] (see Corollary 2 below), and from suggestions made by him and Professor K. N. Sigmon. The author is also grateful to the Mathematics Departments of the University of Florida and the University of Puerto Rico at Mayaguez for the opportunity to present these results at the Symposium on Semigroups and the Multiplicative Structure of Rings held in Mayaguez in March 1970.

We present first our abstract theorem which leads not only to the above mentioned results but also to a generalisation of a result due to J. C. Beidleman and R. H. Cox [1]. We include Wallace's [3] result as a substantial corollary although in fact it was his work which led to the development of our theorem.

Theorem. Let G be a compact group, and suppose T_1 and T_2 are Hausdorff spaces such that T_2 is compact. We further suppose that there is a continuous function $T_1 \times T_2 \to G$ denoted by juxtaposition, such that, for each $s \in T_1$, $s T_2$ is a (compact) group. Then, if $q \in T_1$, and $q T_2 = 1$, it follows that $C T_2 = 1$ (where C is the component of T_1 to which q belongs). (There is an obvious dual).

Proof. We first state two results involving Lie groups. (See Montgomery Zippin [2].)

(1) A compact group is a Lie group, if, and only if, there is some open set about the neutral element which properly contains no closed subgroup other than the trivial one.

(2) If G is a compact group, and if U is any open set about the neutral element, then there is a morphism f (continuous!), of G onto a Lie group with $\ker f \subseteq U$.

We continue with the proof of the theorem.

Let f be any morphism of G onto a Lie group L, and let

$$I(f) = \{t \in T_1 \colon f(tT_2) = 1 \in L\}.$$

$I(f)$ is not empty, since we suppose $qT_2 = 1 \in G$, and then $f(qT_2) = 1 \in L$, since f is a morphism. Let $t_0 \in I(f)$. Then $f(t_0 T_2) = 1$. For V an open set about $1 \in L$ such that V contains no nontrivial closed subgroup of L (possible by (1)), we put $W = f^{-1}(V) \supseteq t_0 T_2$. Then, by the compactness of T_2, $\exists\, W'$, open in T_1, such that $t_0 \in W'$, and $W \supseteq W' T_2$. Now, if $t' \in W'$, then $f(t' T_2) \subseteq V$. But $t' T_2$ is a closed subgroup of G, and so $f(t' T_2)$ is a closed subgroup of L. By our manner of choosing V, we have that $f(t' T_2) = 1$ giving $W' \subseteq I(f)$ and, since t_0 was arbitrary, we have that $I(f)$ is open.

Also,
$$\begin{aligned} f(I(f)^* T_2) &\subseteq f((I(f)\, T_2)^*) \\ &\subseteq [f(I(f)\, T_2)]^* \\ &\subseteq \{1\}^* \\ &= 1. \end{aligned}$$

Thus $I(f)$ os closed, and open, as well as nonempty, and so $C \subseteq I(f)$. Suppose that $x \in C$, and $y \in T_2$, with $xy \neq 1 \in G$. Then $G \setminus \{xy\}$ is an open set in G to which 1 belongs. However, by (2) above, we may choose our f, and L, such that $f^{-1}(1) \subseteq G \setminus \{xy\}$. But then $f(xy) \neq 1 \in L$, and this is a contradiction because $f(xy) \in f(I(f)\, T_2) = \{1\}$.

Corollary 1. Let $(R, +, \cdot)$ be a compact nearring. Then, if we denote the connected component of R which contains the neutral element 0 of R by C, we have $CR = 0$.

Proof. Take T_1, T_2 and G all as R. However, we write additively, so that the 1 of the theorem now appears as 0. The map $R \times R \to R$ is the multiplication of the nearring. 0 plays the role of q. For each $s \in R$, sR is seen to be a compact subsemigroup of $(R, +)$, because $sr_1 + sr_2 = s(r_1 + r_2) = sr$. Then of course since sR is compact it must be a subgroup of $(R, +)$, and we get immediately that $CR = 0$.

The above result generalizes work by J. C. Beidleman and R. H. Cox [1]. The following result of A. D. Wallace [3], which gave rise to our theorem, clearly follows as a corollary to our result and its dual.

Corollary 2. If $(M, +)$ is a compact additive group which is an R-semimodule over the compact semiring $(R, +, \cdot)$, with $(R, +)$ a group, then in both $(R, +)$, and $(M, +)$, denoting the neutral element by 0, and the connected component of R, respectively M, which contains 0, as C, respectively D, we have that

$$\text{(i)} \quad RD = 0$$
$$\text{(ii)} \quad CM = 0.$$

Now we can gain a little more by looking at the theorem a little differently. Let us denote by T the set of algebraic homomorphisms from a compact group G_1 into a compact group G_2 such that the homomorphisms are continuous maps. Let us consider two cases. One case is where G_2 is abelian and the other is where $G_1 = G_2 = G$ is not abelian, but is connected. We consider T as a topological space with the compact-open topology induced by the groups in the standard way. Then $T \times G_1 \to G_2$, by evaluation, is a continuous map.

Corollary 3. In case G_2 is abelian T is totally disconnected.

Proof. The trivial homomorphism t, which takes all of G_1 onto the neutral element of G_2, has the property of q in our Theorem if we identify T_1 with T, T_2 with G_1, G with G_2, and the 0 of G_2 with the 1 of G. Certainly sG_1 is a compact subgroup of G_2. Thus, applying the proposition, we obtain that for each t' in the component of T containing t, we get $t'G_1 = 0$. However, this means that $t' = t$. Thus the component of T containing t is just $\{t\}$. I am indebted to Professor Sigmon for pointing out that that since G_2 is abelian, we may construct a topological group on T using pointwise defined addition, and the compact-open topology. It then follows that T is totally disconnected.

Corollary 4. In case $G_1 = G_2$ is not abelian, but is connected, then T is nonhomogeneous.

Proof. Once again, taking $T_1 = T$, $T_2 = G$, and $G = G$ in our Theorem, we have that the component of T containing the trivial homomorphism t, is $\{t\}$. Now however, by mapping G into T in the following way: $g \to f_g$: $G \to G$, where $f_g(x) = g^{-1}xg$, we produce a continuous image of G in T which contains more than one element, and so we have a connected subset of T with more than one element, which means that T is nonhomogeneous.

References

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