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Nonlinear Commutators and Jacobians

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Preface

The remarkable advances in the interpolation theory and the prolific growth of its applications necessitate an introduction to this new study. It is the objective of these lectures to give an account of applications of the recurring phenomenon of cancellation in some nonlinear differential and integral expressions. We divide this presentation into three closely linked lectures, in accordance with the subject matter.

The first lecture takes up the cancellation phenomenon in various types of linear and nonlinear commutators. It is not our intention here to give an account of the main lines of recent developments; instead we confine ourselves to basic commutator results and $BMO - H^1$ duality.

The importance of the commutator estimates is visible in several applications presented in the second lecture, among them in proving the higher order integrability of Jacobians and wedge products and in establishing the compensated compactness principle.

The third lecture is dedicated to the recent advances and applications of the theory of the Jacobians. Unfortunately, I will not be able to include numerous generalizations and contributions, but some will be recognized here by way of digression.

For the sake of brevity, I will need to paraphrase some of the well-known definitions and results.

I hope that these lectures, although not complete, will give to students material worthy of further exploration.

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Lecture 1

1. The Cancellation Phenomena

1.1 Singular Integrals

One of the essential aspects of Fourier Analysis is the cancellation occurring in various integrals. The simplest example is a Calderón-Zygmund type singular integral operator

$$Tf(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} k(x-y)f(y)dy \tag{1.1}$$

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where we emphasize that the non-absolutely convergent integral exists only via a limiting process, due to cancellation of large positive and negative terms.

The building blocks of many singular integrals are the Riesz transforms

$$R_i : L^p(R^n) \to L^p(R^n)\ , \quad i = 1, 2, ..., n, \quad 1 < p < \infty$$

The second order Riesz transforms $R_{ij} = R_i \circ R_j$ are characterized by the symbolic equation

$$\frac{\partial^2}{\partial x_i \partial x_j} = -R_{ij} \circ \Delta \tag{1.2}$$

connecting the second order partials with the Laplacian. The vector-valued Riesz transform

$$R = (R_1, ..., R_n) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n, \mathbb{R}^n) \tag{1.3}$$

carries scalar functions $f \in L^p(\mathbb{R}^n)$ into the gradient fields $\nabla u = Rf = (R_1 f, ..., R_n f)$ where u belongs to the Sobolev class $W^{1,p}(\mathbb{R}^n)$. Its adjoint, denoted by R^*, acts on vector functions $F = (f^1, ..., f^n)$ by the rule $R^* F = R_1 f^1 + ... + R_n f^n$, and thus $R^* R = Id : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. It is from these identifies that Riesz transforms and their norms became important for PDEs. For sharp estimates and for a recent account of related singular integrals, we refer the interested reader to [27].

1.2 A Linear Commutator

In studying linear PDEs with measurable coefficients, we are often left with the task of examining the commutator of a singular integral operator T and the operator of multiplication by a function b, that is:

$$[b, T](f) = b(Tf) - T(bf) \tag{1.4}$$

For Hölder continuous coefficients the utility of such commutators was already well understood due to the work of Schauder and of course many others. For L^p-theory of PDEs we need commutators with measurable functions. The case $b \in L^\infty(\mathbb{R}^n)$ is largely uninteresting because we do not gain any cancellation; the trivial estimate reads as

$$\|[b, T](f)\|_p \le C(n, p)\|b\|_\infty \|f\|_p$$

However, Coifman et al. [12] succeeded in proving better inequality

$$\|[b, T](f)\|_p \le C_p(n)\|b\|_{BMO} \|f\|_p \tag{1.5}$$

where $BMO = BMO(\mathbb{R}^n)$ denotes the space of functions with bounded mean oscillation. This uniform estimate led to a definition of the commutator for $b \in BMO(\mathbb{R}^n)$. Somewhat later, Uchiama observed that $[b, T]$ is in fact a compact operator, whenever $b \in VMO(\mathbb{R}^n)$. It is perhaps for this reason that the Coifman-Rochberg-Weiss commutator attracted so much attention among researchers in PDEs. For example, with the aid of the compactness of $[b, T]$, one easily derives a Fredholm alternative for equations with VMO-coefficients in all L^p-spaces, $1 < p < \infty$ [30]. In the next lecture, we shall demonstrate applications of inequality (1.5) to the integrability theory of the Jacobians and the theory of compensated compactness.

1.3 Maximal Operator

In the other direction, one might consider the relationship between the cancellation effect and a mollifying property of the maximal operator. For $h \in L^1_{loc}(\mathbb{R}^n)$, we recall the Hardy-Littlewood maximal function defined by

$$Mh(x) = \sup\left\{ \fint_Q |h|; \quad x \in Q \right\} \tag{1.6}$$

the supremum being taken over all cubes Q containing x. Here and subsequently, the symbol $\fint_Q h$, also denoted at times by h_Q, stands for the L^1-mean of h over the cube Q. It was Coifman and Rochberg [11] who noticed that maximal functions themselves possess of some regularity. Accordingly, $\log Mh \in BMO(\mathbb{R}^n)$ and we have a uniform bound

$$\| \log Mh \|_{BMO(\mathbb{R}^n)} \leq C(n) \tag{1.7}$$

provided $Mh(x)$ is finite at some (consequently at almost every) point $x \in \mathbb{R}^n$. The interplay between this result and the commutators is both illuminating and useful for the sequel. Therefore, we will take a moment to discuss a new commutator that emerges from this interplay.

1.4 The Rochberg-Weiss Commutator

For a given singular integral operator T, the following expression

$$T^{\log} f = T(f \log |f|) - (Tf) \log |Tf| \tag{1.8}$$

is well defined for all $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Denoting by $h = |f| + |Tf|$ we can write

$$T^{\log} f = T(f \log Mh) - (Tf) \log Mh + T\left[Mh\left(\frac{f}{Mh} \log \frac{|f|}{Mh}\right)\right] - Mh\left(\frac{Tf}{Mh} \log \frac{|Tf|}{Mh}\right)$$

Here we recall the elementary inequality: $|x \log |x|| \leq e^{-1}$ for $|x| \leq 1$, and that $\|Mh\|_p \leq C_p \|f\|_p$. By (1.5) and (1.7) we then conclude with the following estimates

$$\|T^{\log} f\|_p \leq C_p(h) \| \log Mh\|_{BMO} \|f\|_p + C_p(n) \|Mh\|_p \, .$$

Hence,

$$\|T^{\log} f\|_p \leq A_p(n) \|f\|_p \tag{1.9}$$

We emphasize that the individual terms in (1.8) need not belong to $L^p(\mathbb{R}^n)$. But the difference does. This immediately raises the question of whether the latter estimate remains valid for abstract bounded linear operators in Lebesgue spaces $L^s(X, \mu)$. An affirmative answer was given by Rochberg and Weiss [50] as a byproduct of their studies of analytic families of Banach spaces. The core of the proof, however, is the complex method of interpolation originated by Thorin. There has since been more systematic work done by Kalton [35], and Rochberg and Milman [46].

There are, of course, many more examples of nonlinear commutators closely linked with the interpolation theory. Perhaps the most natural one arises in the study of very weak solutions of nonlinear PDEs [22, 29]. Unfortunately, detailed discussion of this topic is beyond the scope of these lectures. In order to avoid unnecessary repetitions we shall now invest a little time for the formulation of the most general and precise commutator results. The reader may also wish to consult the Lipschitz Lectures [23] for more details.

1.5 Complex Method of Interpolation

We shall concern ourselves with a measure space (X, μ) and a separable Hilbert space E. Let $L^p(X, E)$, $1 \leq p \leq \infty$, denote the Lebesgue space of E-valued functions on X and let $T : L^p(X, E) \to L^p(X, E)$ be a bounded linear operator for all p from an interval $[p_1, p_2]$, where $1 \leq p_1 \leq p_2 \leq \infty$. Denote its norm by $\| T \|_p$. Since our arguments strongly depend upon complex method of interpolation, it will be necessary to consider the complexification E_C of E endowed with the natural Hermitian inner product. There is a natural way to extend T to the complex spaces $L^p(X, E_C)$ by setting $T(f + ig) = Tf + iTg$. It is essential that T is linear over the complex numbers. Note too that the p-norms of the extended operator remain unchanged.

The object of our discussion here is the nonlinear commutator of T with the power function $f \to |f|^z f$; that is

$$T^z(f) = T(|f|^z f) - |Tf|^z Tf \tag{1.10}$$

where z is a complex number from the strip

$$\frac{p_1}{p} - 1 \leq Re z \leq \frac{p_2}{p} - 1 . \tag{1.11}$$

We shall make the forthcoming arguments a little easier by restricting the parameter z to a disk

$$B(r) = \{z;\, |z| \leq r\} \tag{1.12}$$

where r is small enough to satisfy

$$p_1 \leq p - rp \leq p + rp \leq p_2 . \tag{1.13}$$

Clearly, T^z is a bounded operator from $L^p(X, E)$ into $L^{\frac{p}{1+Re z}}(X, E)$. It is also continuous, which follows from the inequality:

$$\| T^z f - T^z g \|_{\frac{p}{1+Re z}} \leq C \| f - g \|_p^{(1-r)(1+Re z)} (\| f \|_p^r + \| g \|_p^r)^{1+Re z} \tag{1.14}$$

where $C = C(p_1, p_2)$ depends only on the norms $\| T \|_{p_1}$ and $\| T \|_{p_2}$. This inequality is straightforward and does not exhibit any cancellation. Notice, however, that the commutator T^z depends analytically on z and vanishes at $z = 0$. Basically, by Schwarz's lemma we strengthen the latter estimate as follows:

Theorem 1.
Under the notation given above we have

$$\| T^z f - T^z g \|_{\frac{p}{1+Re z}} \leq \frac{C|z|}{r} \| f - g \|_p^{(1-r)(1+Re z)} (\| f \|_p^r + \| g \|_p^r)^{1+Re z} \tag{1.15}$$

In particular,

$$\| T(f|f|^z) - (Tf)|Tf|^z \|_{\frac{p}{1+Re z}} \leq \frac{C|z|}{r} \| f \|_p^{1+Re z} \tag{1.16}$$

This seemingly simple improvement in (1.15) over (1.14) is extremely useful in applications to nonlinear differential PDEs.

Proof. Because of homogeneity we may assume that

$$\| f - g \|_p^{1-r} (\| f \|_p^r + \| g \|_p^r) = 1 \tag{1.17}$$

Thus, (1.14) reads as

$$\| T^z f - T^z g \|_{\frac{p}{1+Re z}} \leq C = C(p_1, p_2) \tag{1.18}$$

for all $|z| \leq 1$.
The heart of our proof here is the holomorphic function

$$F(z) = \int_X \langle T^z f - T^z g \,, \quad |\varphi|^{p-2-\bar{z}} \varphi \rangle \tag{1.19}$$

defined for $|z| \leq r$, where φ is an arbitrary test function from $L^p(X, E)$ with $\| \varphi \|_p = 1$. This can be easily seen if f, g, and φ are simple functions. Because of the uniform bound (1.15) F is holomorphic in the general case. Clearly, $F(0) = 0$. On the other hand, by Hölder's inequality and (1.18) we have the following uniform bound:

$$|F(z)| \leq \| T^z f - T^z g \|_{\frac{p}{1+Rez}} \| \varphi \|_p^{p-1-Rez} \leq C$$

Now the Schwarz lemma yields $|F(z)| \leq \frac{C|z|}{r}$, and the final step of the proof is immediate:

$$\| T^z f - T^z g \|_{\frac{p}{1+Rez}} = \sup_{\|\varphi\|_p=1} \frac{|F(z)|}{\| |\varphi|^{p-2-\bar{z}} \varphi \|_{\frac{p}{p-1-Rez}}} = |F(z)| \leq \frac{C|z|}{r}$$

as desired. □

1.6 Some Nonlinear Commutators

Suppose z is a pure imaginary number, say $z = it$ with $t \in [-r, r]$. The commutator $T^{it} : L^p(X, E) \to L^p(X, E)$ takes the form: $T^{it} f = T(f|f|^{it}) - (Tf)|Tf|^{it}$, and inequality (1.15) reads as follows.

Corollary 1.
We have

$$\|T^{it} f - T^{it} g\|_p \leq C\frac{|t|}{r}\|f - g\|_p^{1-r}(\|f\|_p^r + \|g\|_p^r) \tag{1.20}$$

An interesting situation arises after dividing (1.20) by t and letting t go to zero. A passage to the limit yields nothing but the Rochberg-Weiss commutator

$$T^{\log} f = \lim_{t\to 0} \frac{T^{it} f}{it} = -i\frac{d}{dt}(T^{it} f)_{t=0} = T(f \log|f|) - (Tf)\log|Tf|\,.$$

In this way we show that $T^{\log} : L^p(X, E) \to L^p(X, E)$ is not only bounded but also a continuous operator. More precisely, we have

$$\|T^{\log} f - T^{\log} g\|_p \leq \frac{C}{r}\|f - g\|_p^{1-r}(\|f\|_p^r + \|g\|_p^r)\,. \tag{1.21}$$

We are still at liberty to select the number r within the conditions (1.13). The obvious choice is to minimize the right-hand side of (1.21). This leads to a qualitatively best possible estimate for the modulus of continuity of $T^{\log}$.

Corollary 2.
We have

$$\|T^{\log} f - T^{\log} g\|_p \leq C\|f - g\|_p \log\left(e + \frac{\|f\|_p + \|g\|_p}{\|f - g\|_p}\right)\,. \tag{1.22}$$

Now suppose that z is real, say $z = -\epsilon$, where $1 - \frac{p_2}{p} \leq \epsilon \leq 1 - \frac{p_1}{p}$. Then, Theorem 1 reduces to:

Corollary 3.
For all $0 < \delta \leq 1$ *and* $f, g \in L^p(X, E)$, *we have*

$$\|T^{-\epsilon} f - T^{-\epsilon} g\|_{\frac{p}{1-\epsilon}} \leq C\frac{|\epsilon|}{\delta}\|f - g\|_p^{1-\delta}(\|f\|_p + \|g\|_p)^{\delta-\epsilon} \tag{1.23}$$

In our applications we shall always have $f \in \ker T$ and, sometimes, $g = 0$. Hence,

$$\|T(|f|^{-\epsilon} f)_{\frac{p}{1-\epsilon}} \leq C|\epsilon| \, \|f\|_p^{1-\epsilon} .$$

Of course, the significance of this inequality is due to the factor $|\epsilon|$ in the right-hand side. As a matter of fact, this is the inequality that was first established for Riesz transforms in [22] by using rather complicated arguments. Within the framework of complex interpolation, generalizations to the abstract operators became straightforward somewhat later. There has since been systematic work on commutators in Orlicz spaces. We shall report on these advances briefly.

An elementary device of averaging inequality (1.23) leads to new nonlinear commutators and their estimates of the following type

$$\|T\Omega(f) - \Omega(Tf)\|_{L^\Psi} \leq C(\|f\|_{L^\Phi}) \tag{1.24}$$

where $\Omega : E \to E$ is a mapping of the form $\Omega(x) = xA(|x|)$ and $\Phi, \Psi : \mathbb{R}_+ \to \mathbb{R}_+$ are Young functions. Because of nonlinearity, C is not a constant here but a function on $\mathbb{R}_+$ which depends on Ω, Φ, and Ψ. Only in the case $\Omega(x) = x \log|x|$, C stands for the multiplication by a constant. The quotient

$$L(t) = \frac{tA(t)}{\Psi^{-1}(\Phi(t))} \qquad t > 0 \tag{1.25}$$

tells us how much degree of integrability we gain in the commutator due to the internal cancellation in (1.24). We gain nothing if A grows (or decays) as a power function, say $A(t) = t^\epsilon$ with $\epsilon \neq 0$; $L(t)$ is simply bounded at infinity. Nevertheless, in this case we still observe a cancellation effect as in Theorem 1. But the improvement of the degree of integrability occurs only when Ω grows almost linearly. Instead of getting into technicalities, we shall give two examples which illustrate quite well the cancellation phenomenon.

Example 1. Letting $A(t) = \log^\alpha(e+t)$ and $\alpha \in (0, 1]$ we obtain

$$\|T[f \log^\alpha(e + |f|)] - (Tf)\log^\alpha(e + |Tf|)\,\|_{L^\Psi} \leq C(\|f\|_{L^\Phi}) . \tag{1.26}$$

Of course, for this inequality to be true, the Young functions Φ and Ψ must be in some relation. For instance, if $\alpha = 1$, we recover the Rochberg-Weiss commutator, in which case the relation is very simple: $\Phi = \Psi$.

With a little work we modify (1.26) to arrive at the following

$$\|T(f \log|f|) - (Tf)\log|Tf|\,\|_{L^\Phi} \leq C_\Phi \|f\|_{L^\Phi} . \tag{1.27}$$

Inequality (1.26) remains valid for the pairs

$$\Phi(t) = t^p \log^{\alpha-1}(e+t)\,, \quad \Psi(t) = t^p \log^{(1-\alpha)(p-1)}(e+t)$$

with any $1 < p < \infty$. The improvement quotient is always $L(t) = \log(e+t)$.

Example 2. In order to gain more than a logarithm one must take Ω closer to the identity. We choose, as an illustration, $\Omega(x) = x \log\log(e + |x|)$. The commutator estimates take the form

$$\|T[f \log\log(e + |f|)] - (Tf)\log\log(e + |Tf|)\|_{L^\Psi} \leq C(\|f\|_{L^\Phi}) .$$

Here we can take the following pair of Young functions:

$$\Phi(t) = t^p \log^{-1}(e+t) \quad \text{and} \quad \Psi(t) = t^p \log^{p-1}(e+t)$$

The improvement quotient is $L(t) = \log(e+t)\log\log(e+t)$. All the above results are qualitatively optimal and amount to the principle:

The further Ω is from the identity, the less cancellation occurs in the commutator.

In most recent years, the nonlinear commutators have been subjected to a great deal of investigation, see for example: [1, 8, 19, 23, 34, 35, 46]. Unfortunately, we cannot develop this point any further. The remaining time of this lecture will be devoted to the BMO-$\mathcal{H}^1$ duality.

1.7 The Hardy Space

In some ways the cancellation phenomenon also enters into the functions of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. It is this delicate cancellation property that makes some divergent integrals meaningful.

A measurable function $a = a(x)$, which is supported in a ball $B \in \mathbb{R}^n$, is said to be an $\mathcal{H}^1$-atom if it satisfies both

$$|a(x)| \le \frac{1}{|B|} \qquad \text{for a.e.} \qquad x \in \mathbb{R}^n, \quad \text{(magnitude property)} \tag{1.28}$$

and

$$a_B = ⨍_B a(x)dx = 0 \quad \text{(cancellation condition)}\,. \tag{1.29}$$

Now, a function $f \in L^1(\mathbb{R}^n)$ belongs to $\mathcal{H}^1(\mathbb{R}^n)$ if and only if it can be written as a (infinite) linear combination of $\mathcal{H}^1$-atoms

$$f = \sum_{k=1}^{\infty} \lambda_k a_k \qquad \text{with} \qquad \sum_{k=1}^{\infty} |\lambda_k| < \infty\,. \tag{1.30}$$

Its norm is then defined by

$$\|f\|_{\mathcal{H}^1} = \inf \left\{ \sum |\lambda_k| \quad ; \quad f = \sum \lambda_k a_k \right\} \tag{1.31}$$

the infimum being taken over all atomic decompositions of f. Note that f must necessarily satisfy the moment condition

$$\int_{\mathbb{R}^n} f(x)dx = 0\,. \tag{1.32}$$

Bounded functions with compact support and with zero L^1-mean form a dense subspace of $\mathcal{H}^1(\mathbb{R}^n)$ since they are in fact finite linear combinations of $\mathcal{H}^1$-atoms. If f is such a function, then

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \le C(n)\|f\|_{\mathcal{H}^1} \|g\|_{BMO}$$

for every $g \in BMO(\mathbb{R}^n)$. Thus, each $g \in BMO(\mathbb{R}^n)$ defines a bounded linear functional on $\mathcal{H}^1(\mathbb{R}^n)$. The converse, due to C. Fefferman, is much deeper and constitutes a central conclusion of the $\mathcal{H}^1$-theory. It states that:

$$BMO(\mathbb{R}^n) \quad \text{is the dual space to} \quad \mathcal{H}^1(\mathbb{R}^n)\,. \tag{1.33}$$

For the action of $g \in BMO(\mathbb{R}^n)$ on an element $f \in \mathcal{H}^1(\mathbb{R}^n)$ we use the symbol $\langle g, f \rangle$ or, sometimes, $\int fg$ if no confusion is possible.

As observed by Sarason [53], this pairing also allows one to realize $\mathcal{H}^1(\mathbb{R}^n)$ as the dual of $VMO(\mathbb{R}^n)$:

$$\mathcal{H}^1(\mathbb{R}^n) \quad \text{is the dual space to} \quad VMO(\mathbb{R}^n) \tag{1.34}$$

see also Coifman and Weiss [13]. Having disposed with these preliminaries, we shall close this lecture with two problems.

Problem 1. Let B be a ball in $\mathbb{R}^n$ of radius R and let $\varphi \in C_o^\infty(B)$. Show that

$$\|\varphi f\|_{BMO} \le C(n)R\|\nabla\varphi\|_\infty \left(\|f\|_{BMO} + |f_B|\right) \tag{1.35}$$

for every $f \in BMO(\mathbb{R}^n)$. Using this (rather technical) result, try the following more challenging problem:

Problem 2. Given $f \in \mathcal{H}^1(\mathbb{R}^n)$ consider the Schwartz distributions $f_t \in \mathcal{D}'(\mathbb{R}^n)$, defined for $t \in \mathbb{R}$ by the formula

$$(f_t, \varphi) = \int_{\mathbb{R}^n} \varphi |f|^{it} f \qquad \varphi \in C_o^\infty(\mathbb{R}^n) \tag{1.36}$$

Show that the function $t \to (f_t, \varphi)$ is differentiable at zero and that the distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ given by

$$(F, \varphi) = \left. \frac{d(f_t, \varphi)}{dt} \right|_{t=0} \tag{1.37}$$

is of order 1. This means that to each bounded region $\Omega \in \mathbb{R}^n$ there corresponds a constant C_Ω such that

$$|(F, \varphi)| \le C_\Omega \|\nabla\varphi\|_\infty \|f\|_{\mathcal{H}^1} \log(e + \|f\|_1) \tag{1.38}$$

where $\varphi \in C_o^\infty(\Omega)$. As a hint, show that

$$(F, \varphi) = i \int_{\mathbb{R}^n} \varphi f \log \frac{|f|}{\mathcal{M}f} + i\langle \varphi \log \mathcal{M}f, f\rangle := i \int_{\mathbb{R}^n} \varphi f \log |f| \tag{1.39}$$

where $\mathcal{M}f = \left[M\left(|f|^{\frac{1}{2}}\right)\right]^2 \in L^1(\mathbb{R}^n)$, see also Section 3.1.

We then conclude:

For $f \in \mathcal{H}^1(\mathbb{R}^n)$ the function $f \log |f|$ can be interpreted as a Schwartz distribution.

Lecture 2

2. Jacobians and Wedge Products

2.1 Brief Overview

Let Ω be an open subset of $\mathbb{R}^n$ and let $f = (f^1, ..., f^n) : \Omega \to \mathbb{R}^n$ be a mapping whose distributional differential $Df \in \mathcal{D}'(\Omega, \mathbb{R}^{n\times n})$ is locally integrable. In particular, its Jacobian determinant

$$J(x, f) = \det\ Df(x) = \det\left(\frac{\partial f^i}{\partial x_j}\right) \tag{2.1}$$

is defined point-wise at almost every point $x \in \Omega$. Using the language of differential forms we write

$$J(x, f)dx = df^1 \wedge ... \wedge df^n \tag{2.2}$$

Jacobians occur in many different contexts, but their distinctive property is that they often appear under an integral sign, for example in the formula for the change of the variables in a multiple integral. To ensure that the Jacobian is integrable one first naturally assumes that f belongs to the Sobolev class $W^{1,n}(\Omega, \mathbb{R}^n)$. By Hölder's inequality

$$\int_\Omega |J(x, f)|dx \le \|df^1\|_n ... \|df^n\|_n \le \int_\Omega |Df(x)|^n dx \tag{2.3}$$

More generally, if $f^i \in W^{1,p_i}(\Omega)$, where $(p_1, ..., p_n)$ are Hölder conjugate exponents, then

$$\int_\Omega |J(x, f)|dx \le \|df^1\|_{p_1} ... \|df^n\|_{p_n} \tag{2.4}$$

Later on we shall do much better by exploiting an internal cancellation property of the wedge products. The starting point will always be integration by parts and Hodge decomposition.

Some, rather special, properties of Jacobians had already been recognized in the late 1960s when Gehring and Reshetnyak laid down the foundation of multidimensional quasiconformal analysis. It seems appropriate here to mention the paper [15] in which Gehring proved that the differential Df of a quasiconformal mapping $f : \Omega \to \mathbb{R}^n$ belongs to $L^{n+\epsilon}_{loc}(\Omega)$. As a matter of fact, his first approach (though not mentioned in his paper) was the point-wise estimate of maximal functions: $M(|Df|^n) \leq c(n)|M(Df)|^n \in L^1$, yielding the $L \log L$-integrability of the Jacobians, see [24]. In another direction, the so-called weak continuity of the Jacobians was already known to Morrey and successfully applied by Reshetnyak in [47]. Finally, we should mention Wente [56], whose ideas and results are closely linked with the fact that the Jacobian of a mapping from $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ actually belongs to the Hardy space $H^1(\mathbb{R}^n)$. Today, the theory of the Jacobians has a life of its own [4, 17, 18, 28, 37, 42, 43, 57] and it is our aim here to relate some of these advances.

As we have already said, the Jacobian can be viewed as a wedge product of exact 1-forms; also forms of higher degree. Namely,

$$J(x, f)dx = df^1 \wedge \ldots \wedge df^n = \varphi^1 \wedge \ldots \wedge \varphi^m \tag{2.5}$$

where each factor φ here is the wedge product of some number of the differentials, say $\varphi = df^{i_1} \wedge \ldots \wedge df^{i_l}$. This is the view we shall adopt and extend to wedge products of forms of arbitrary degree on a Riemannian manifold. It has many roots in the past, see [49] and [25]. This more general setting we wish to present here not only provides a natural framework for better understanding the Jacobians, but also unifies earlier work in many other type of nonlinear differential situations. But first we need to review some facts associated with differential forms.

2.2 Differential Forms

Here and in the sequel X will be a compact C^∞-smooth oriented Riemannian manifold of dimension $n \geq 2$, with or without boundary. For an integer $0 \leq l \leq n$ $\Lambda^l X$ will indicate the bundle of l-covectors. Each fiber $\Lambda^l_a X$ at $a \in X$ is furnished with an inner product induced by the metric tensor on X, which we denote by $\langle \xi, \zeta \rangle$ for $\xi, \zeta \in \Lambda^l_a X$. The exterior algebra bundle over X is the Whitney sum $\Lambda X = \oplus_{l=0}^n \Lambda^l X$ equipped with the wedge product $\alpha \wedge \beta \in \Lambda^{l+k} X$ for $\alpha \in \Lambda^l X$ and $\beta \in \Lambda^k X$. The orientation and the inner product on ΛX give rise to a duality between the l-covectors and the $(n-l)$-covectors. This duality is expressed by the Hodge star operator

$$\star : \Lambda^l X \to \Lambda^{n-l} X$$

Note that $\star\star = (-1)^{ln-l} : \Lambda^l X \to \Lambda^l X$. Sections of $\Lambda^l X$, denoted by $\Gamma(\Lambda^l X)$, are the l-forms on the manifold X. The wedge product, Hodge star, and inner product extend pointwise to the l-forms. The 0-forms are simply functions on X. The measure on X will be the one induced by the volume form

$$dx = \star 1 \in \Gamma(\Lambda^n X)$$

where $1 \in \Gamma(\Lambda^0 X)$ stands for the constant function equal to 1. By the duality $\star dx = 1$. Note that

$$\langle \xi, \zeta \rangle dx = \zeta \wedge \star\xi = \xi \wedge \star\zeta$$

for $\xi, \zeta \in \Gamma(\Lambda^l X)$. We denote by $L^p(\Lambda^l X)$ the Lebesgue space of all measurable sections $\varphi \in \Gamma(\Lambda^l X)$ for which

$$\|\varphi\|_p = \left(\int_X |\varphi|^p dx\right)^{\frac{1}{p}} < \infty$$

If $1 \leq p, q \leq \infty$ is a Hölder conjugate pair, then the scalar product of $\alpha \in L^p(\Lambda^l X)$ and $\beta \in L^q(\Lambda^l X)$ is defined by

$$(\alpha, \beta) = \int_X \alpha \wedge \star\beta = \int_X \beta \wedge \star\alpha = \int_X \langle \alpha, \beta \rangle dx$$

The Sobolev space $W^{1,p}(\Lambda^l X)$ of l-forms on X is defined in the usual fashion by using local coordinates. Since Meyers-Serrin approximation applies here, $C^\infty(\Lambda^l X)$ is dense in $W^{1,p}(\Lambda^l X)$, for all $1 \leq p < \infty$. Then $W_o^{1,p}(\Lambda^l X)$ denotes the closure of $C_o^\infty(\Lambda^l X)$. We say that such forms vanish on ∂X in the distributional sense. Basic differential operators on X are the exterior derivative

$$d : C^\infty(\Lambda^l X) \to C^\infty(\Lambda^{l+1} X)$$

and its formal adjoint

$$d^\star : C^\infty(\Lambda^{l+1} X) \to C^\infty(\Lambda^l X) .$$

Each of these operators applied twice gives zero. The duality between these operators is emphasized by the formula of integration by parts

$$\int_X \langle d\alpha, \beta \rangle - \int_X \langle \alpha, d^\star \beta \rangle = \int_{\partial X} \alpha \wedge \star\beta$$

for $\alpha \in C^\infty(\Lambda^l X)$ and $\beta \in C^\infty(\Lambda^{l+1} X)$. It follows from what we have said that for $\alpha \in W^{1,p}(\Lambda^l X)$ and $\beta \in W^{1,q}(\Lambda^{l+1} X)$

$$\int_X \langle d\alpha, \beta \rangle = \int_X \langle \alpha, d^\star \beta \rangle \tag{2.6}$$

provided one of the forms α or β vanishes on ∂X in the distributional sense. The point to make now is that identity (2.6) remains valid under weaker boundary conditions. Without getting into technicalities, we give the following .

Definition 1.

A differential form $\alpha \in W^{1,p}(\Lambda^l X)$ ($\beta \in W^{1,q}(\Lambda^{l+1} X)$, respectively) is said to have vanishing tangential (normal) component on ∂X in case (2.6) holds for all test forms $\beta \in W^{1,q}(\Lambda^{l+1} X)$ ($\alpha \in W^{1,p}(\Lambda^l X)$). We then write $\alpha_T = 0$ and $\beta_N = 0$, respectively.

The linear space of such forms will be denoted by $W_T^{1,p}(\Lambda^l X)$ and $W_N^{1,q}(\Lambda^{l+1} X)$, respectively. Of course, the Hodge star operator permutes these spaces. Precisely, we have $\star W_T^{1,p}(\Lambda^l X) = W_N^{1,p}(\Lambda^{n-l} X)$. Forms $h \in L^p(\Lambda^l X)$, which are both closed and coclosed, that is: $dh = d^\star h = 0$ in the sense of distributions, will be called harmonic fields of degree l. They are known to be C^∞-smooth in the interior of X. We denote by $H^p(\Lambda^l X)$ the space of those harmonic fields which are L^p-integrable. The other two Banach spaces of harmonic fields of concern to us are $H_T(\Lambda^l X)$ and $H_N(\Lambda^l X)$. These spaces consist of forms with vanishing tangential and normal component on ∂X, respectively. We regard it as well known that these two spaces are finite dimensional. Indeed, they represent the relative de Rham cohomology groups of X.

2.3 Hodge Decompositions

The essence of the L^p-cohomology theory is that every differential form $\omega \in L^p(\Lambda^l X)$, $1 < p < \infty$, splits uniquely into exact, coexact, and harmonic components

$$\omega = d\alpha + d^\star \beta + h \tag{2.7}$$

in accordance with one of the following three possible decompositions of $L^p(\Lambda^l X)$

$$\begin{aligned} L^p(\Lambda^l X) &= dW_T^{1,p}(\Lambda^{l-1}X) \oplus d^\star W_N^{1,p}(\Lambda^{l+l}X) \oplus H^p(\Lambda^l X) \\ &= dW_T^{1,p}(\Lambda^{l-1}X) \oplus d^\star W^{1,p}(\Lambda^{l+l}X) \oplus H_T(\Lambda^l X) \\ &= dW^{1,p}(\Lambda^{l-1}X) \oplus d^\star W_N^{1,p}(\Lambda^{l+l}X) \oplus H_N(\Lambda^l X) \end{aligned} \tag{2.8}$$

If ω is closed, we notice that the coexact component $d^\star\beta$ is not present. In other words, closed forms are exact modulo harmonic fields. Associated with the Hodge decompositions are the projection operators from $L^p(\Lambda^l X)$ onto respective subspaces. Indeed, using local coordinates all the operators involved are linear expressions of either Riesz transforms of second order or Riesz potentials in case of the projections onto harmonic component. We shall finish this subsection with some explicit calculation in the special case of $X = \Omega \subset \mathbb{R}^n$. Using local coordinate functions $x_1, ..., x_n$, every l-form on Ω can be written as

$$\omega = \sum_{1\le i_1<...<i_l\le n} \omega_{i_1...i_l}(x)dx_{i_1} \wedge ... \wedge dx_{i_l}$$

If near a boundary point we choose the system in such a way that $x_n = 0$ on ∂X and the x_n curves are orthogonal to ∂X, then the tangential component of ω is:

$$\omega_T = \sum_{1\le i_1<...<i_l<n} \omega_{i_1...i_l}(x)dx_{i_1} \wedge ... \wedge dx_{i_l} \tag{2.9}$$

while the normal component is:

$$\omega_N = \sum_{1\le i_1<...<i_l=n} \omega_{i_1...i_l}(x)dx_{i_1} \wedge ... \wedge dx_{i_l} \tag{2.10}$$

Using standard coordinates in $\mathbb{R}^n$ we often identify a 1-form

$$\omega = A^1(x)dx_1 + ... + A^n(x)dx_n \tag{2.11}$$

with the vector field $A = (A^1, ..., A^n) : \Omega \to \mathbb{R}^n$ Then we find that

$$-d^\star\omega = div\, A = \frac{\partial A^1}{\partial x_1} + ... + \frac{\partial A^n}{\partial x_n} \tag{2.12}$$

and

$$d\omega = \text{curl } A = \sum_{1\le i<j\le n} \left(\frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}\right) dx_i \wedge dx_j \tag{2.13}$$

which is why engineers might profit from knowing some differential forms.

2.4 Weak Wedge Products

A far-reaching generalization of the differential $df = (df^1, ..., df^n)$ of a mapping $f = (f^1, ...f^n)$ is the m-tuple of differential forms

$$\Phi = (\varphi^1, ..., \varphi^m) \qquad \varphi^i \in \Gamma(\Lambda^{l_i}X)\,, \qquad i = 1, ..., m$$

Continuing this analogy, we introduce the Jacobian of Φ by

$$\mathcal{J}\Phi = \varphi^1 \wedge ... \wedge \varphi^m \in \Gamma(\Lambda^l X)\,, \qquad l = l_1 + ... + l_m \tag{2.14}$$

Hadamard's inequality yields $|\mathcal{J}\Phi| \leq C(n)|\varphi^1|...|\varphi^m|$ and a generalization of (2.4) follows readily by Hölder's inequality

$$\int_X |\mathcal{J}\Phi| \leq C(n)\|\varphi^1\|_{p_1}...\|\varphi^m\|_{p_m} \tag{2.15}$$

where $\frac{1}{p_1} + ... + \frac{1}{p_m} = 1$.

For the duration of these lectures $\mathbf{p} = (p_1, ..., p_m)$ will always stand for a multiexponent of Hölder conjugate numbers $1 < p_1, ..., p_m < \infty$, that is: $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = 1$. Let $\mathbf{l} = (l_1, ..., l_m)$ be an m-tuple of positive integers such that $l = l_1 + ...l_m \leq n$. We denote by $L^{\mathbf{p}}(\Lambda^{\mathbf{l}}X)$ the space of m-tuples (2.14) with $\varphi^i \in L^{p_i}(\Lambda^{l_i}X)$, for $i = 1, ..., m$. Thus, (2.15) states that the Jacobian operator

$$\mathcal{J} : L^{\mathbf{p}}(\Lambda^{\mathbf{l}}X) \to L^1(\Lambda^l X)$$

is bounded, while using the telescoping sum

$$\mathcal{J}(\Phi) - \mathcal{J}(\Psi) = \sum_{k=1}^{m} \varphi^1 \wedge ... \wedge \varphi^{k-1}(\varphi^k - \psi^k) \wedge \psi^{k+1} \wedge ... \wedge \psi^m \tag{2.16}$$

we obtain

$$\int_X |\mathcal{J}(\Phi) - \mathcal{J}(\Psi)| \leq C_{\mathbf{p}} \sum_{k=1}^{m} \left[\|\varphi^k - \psi^k\|_{p_k} \prod_{i \neq k} (\|\phi^i\|_{p_i} + \|\psi^i\|_{p_i}) \right] . \tag{2.17}$$

This shows, in particular, that $\mathcal{J}$ is actually a continuous operator.

In order to obtain something deeper than that, we need to integrate by parts, which requires a certain degree of integrability of $d\Phi = (d\varphi^1, \ldots, d\varphi^m)$. For the sake of clarity we begin our discussion with an exact form $\Phi = (d\alpha^1, ..., d\alpha^m)$, where $\alpha^i \in C^\infty(\Lambda^{l_i - 1}X)$. Thus $\mathcal{J}(\Phi) = d(\alpha^1 \wedge d\alpha^2 \wedge \ldots \wedge d\alpha^m)$ is also exact. Fix an m-tuple $\mathbf{s} = (s_1, ..., s_m)$ of Sobolev conjugate numbers. This means that $1 \leq s_1, ..., s_m \leq \infty$ and $\frac{1}{s_1} + ... + \frac{1}{s_m} = 1 + \frac{1}{n}$. Certainly, one of them is less than n, say $1 \leq s_1 < n$. For each test form $\eta \in C_o^\infty(\Lambda^l X)$ we can write

$$\begin{aligned}\int_X \langle \eta, d\alpha^1 \wedge ... \wedge d\alpha^m \rangle &= \int_X \langle d^*\eta, \alpha^1 \wedge d\alpha^2 \wedge ... \wedge d\alpha^m \rangle \\ &\leq C_{\mathbf{s}} \|d^*\eta\|_\infty \|\alpha^1\|_{\frac{ns_1}{n-s_1}} \|d\alpha^2\|_{s_2} ... \|d\alpha^m\|_{s_m}\end{aligned}$$

Note that $d\alpha^1$ is not affected if we add to α^1 any closed form. Thus, by Poincaré-Sobolev inequality for forms, see [23] and [31], we may assume that

$$\|\alpha^1\|_{\frac{ns_1}{n-s_1}} \leq C(n, s_1)\|d\alpha^1\|_{s_1} .$$

This yields

$$\left| \int_X \langle \eta, \mathcal{J}\Phi \rangle \right| \leq C_{\mathbf{s}} \|d^*\eta\|_\infty \|\varphi^1\|_{s_1} ... \|\varphi^m\|_{s_m} . \tag{2.18}$$

More radical attempts reveal that (2.18) actually holds for closed forms. To see this we introduce the following notation for the space of closed forms:

$$\mathcal{L}^{\mathbf{p}}(\Lambda^{\mathbf{l}}X) = L^{\mathbf{p}}(\Lambda^{\mathbf{l}}X) \cap \ker\, d . \tag{2.19}$$

The integration by parts is no longer so simple, the reason being that for manifolds with nontrivial cohomology the closed forms need not be exact. It is at this point that the regularity of harmonic fields comes to play. Recall that each closed form $\Phi \in \mathcal{L}^{\mathbf{p}}(\Lambda^{\mathbf{l}}X)$ is exact modulo a harmonic field.

And here is the vital point: harmonic fields, being smooth, do not harm our previous estimates. In much the same way as in (2.18), we then conclude with the following inequality:

$$\int_X \langle \eta, \mathcal{J}(\Phi) - \mathcal{J}(\Psi) \rangle \leq$$

$$C_s(X)(\|\eta\|_\infty + \|d^*\eta\|_\infty) \sum_{k=1}^{m} \|\varphi^k - \psi^k\|_{s_k} \prod_{i \neq k} (\|\varphi^i\|_{s_i} + \|\psi^i\|_{s_i}) \tag{2.20}$$

for each test form $\eta \in C^\infty(\Lambda^l X)$. Here we assume that either $\eta_N = 0$ or $(\Phi - \Psi)_T = 0$ on ∂X. Now, on first sight one might think that (2.20) remains valid for the forms Φ and Ψ in class $\mathcal{L}^{\mathbf{s}}(\Lambda^l X)$ whose determinants are locally integrable—far from it. Although the L^p-norms of Φ and Ψ do not enter into this inequality, the L^p-integrability of these forms is still required to secure convergence of the integrals occurring in the proof of (2.20). Nevertheless, by an approximation argument, (2.20) allows one to define $\mathcal{J}(\Phi)$ for all forms of class $\mathcal{L}^{\mathbf{s}}(\Lambda^{\mathbf{l}} X)$ as a Schwartz distribution, namely

$$(\mathcal{J}\Phi, \eta) = \lim_{j\to\infty} (\mathcal{J}\Phi_j, \eta), \qquad \eta \in C_o^\infty(\Lambda^l X) \tag{2.21}$$

where $\{\Phi_j\}$ is any sequence of forms $\Phi_j \in \mathcal{L}^{\mathbf{p}}(\Lambda^{\mathbf{l}} X)$ converging to Φ in $\mathcal{L}^{\mathbf{s}}(\Lambda^{\mathbf{l}} X)$. This is what we call the *weak wedge product,* or weak Jacobian.

As a corollary, we can define the weak Jacobians of mappings $f \in W_{loc}^{1,\frac{n^2}{n+1}}(\Omega, \mathbb{R}^n)$ by taking $s_1 = \dots = s_n = \frac{n^2}{n+1}$.

2.5 Compensated Compactness

One further step toward extensions of the idea of integration by parts is to relax the hypothesis that Φ is a closed form. Instead, we may only impose a certain degree of integrability of $d\Phi$.

Suppose $\Phi = (\varphi^1, \dots, \varphi^m) \in L^{\mathbf{p}}(\Lambda^{\mathbf{l}} X)$ and $d\Phi = (d\varphi^1, \dots, d\varphi^m) \in L^{\mathbf{q}}(\Lambda^{\mathbf{l}+1} X)$, where

$$\mathbf{q} = (q_1, \dots, q_m), \quad q_i > \max\left\{1, \frac{np_i}{n + p_i}\right\} \quad \text{for} \quad i = 1, \dots, m \tag{2.22}$$

and $\mathbf{l} + \mathbf{1} = (l_1 + 1, \dots, l_m + 1)$. As before, we decompose

$$\varphi^i = d\alpha^i + d^*\beta^i + h^i \qquad i = 1, \dots, m$$

where this time the coexact component $d^*\beta^i$ is no longer zero, though its normal part vanishes on the boundary of X. Fortunately, both the harmonic field h^i and the coexact term are still harmless due to their higher integrability property. To see this we invoke the Dirac operator $d + d^* : \mathcal{D}'(\Lambda X) \to \mathcal{D}'(\Lambda X)$ acting on forms of all degree. We have

$$(d + d^*)d^*\beta^i = dd^*\beta^i = d\varphi^i \in L^{q_i}(\Lambda^{l_i+1} X).$$

Since the Dirac operator is elliptic, we gain one derivative for the coexact term $d^*\beta^i$, namely

$$d^*\beta^i \in W_N^{1,q_i}(\Lambda^{l_i} X).$$

By virtue of the compactness of the imbeddings $W^{1,q_i}(\Lambda^{l_i} X) \subset L^{p_i}(\Lambda^{l_i} X)$, $i = 1, \dots, m$, we eventually arrive at what is the most general form of compensated compactness principle.

Theorem 2. (Compensated Compactness)

Under the notation stated above suppose that $\{\Phi_k\}$ converges to Φ weakly in $L^{\mathbf{p}}(\Lambda^{\mathbf{l}}X)$ while the exterior derivatives $\{d\Phi_k\}$ stay bounded in $L^{\mathbf{q}}(\Lambda^{\mathbf{l}+1}X)$. Then for each test form $\eta \in C^\infty(\Lambda^l X)$ we have

$$\lim_{k\to\infty} \int_X \langle \eta, \mathcal{J}\Phi_k\rangle = \int_X \langle \eta, \mathcal{J}\Phi\rangle \tag{2.23}$$

provided either $\eta_N = 0$ or $(\Phi_k - \Phi)_T = 0$ for all k.

Roughly speaking, the lack of compactness of the sequence $\{\Phi_k\}$ in $L^{\mathbf{p}}(\Lambda^{\mathbf{l}}X)$ is compensated by the $L^{\mathbf{q}}$-boundedness of some first order derivatives of $\{\Phi_k\}$.

Theorem 2 is a far-reaching extension of the ideas of Murat [44] and Tartar [55], originated from the celebrated Div-Curl Lemma. This lemma is a particular case of Theorem 2 in dimension $n = 3$ applied to differential 1-forms, confront with (2.11). Theorem 2 also implies the familiar weak continuity property of the Jacobian.

Corollary 4.

Suppose that the mappings $f_k : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ converge weakly to f in the Sobolev space $W^{1,n}(\Omega, \mathbb{R}^n)$. Then $J(x, f_k)$ converges to $J(x, f)$ in $\mathcal{D}'(\Omega)$; that is

$$\int_\Omega \eta(x) J(x, f_k)dx \to \int_\Omega \eta(x) J(x, f)dx \tag{2.24}$$

for every $\eta \in C_o^\infty(\Omega)$.

Lecture 3

3. Some Advances and Applications

3.1 $\mathcal{H}^1$-Theory of the Jacobians

In this section, we make use of the Coifman-Rochberg-Weiss commutator. We shall show that the l-forms $\mathcal{J}\Phi = \varphi^1 \wedge ... \wedge \varphi^m$ for

$$\Phi = (\varphi^1, ..., \varphi^m) \in \mathcal{L}^{\mathbf{p}}(\Lambda^{\mathbf{l}}X)$$

actually belong to the Hardy space $\mathcal{H}^1_{loc}(\Lambda^l X)$. This observation was first made in the Euclidean space by Coifman, et al. in 1989. We refer the reader to [10] for related topics. Some generalizations are in [9]. In order to avoid the necessary prerequisites concerning localized Hardy spaces, we assume here that $X = \mathbb{R}^n$. This is still the most interesting case.

Theorem 3.

The Jacobian operator acts continuously from $\mathcal{L}^{\mathbf{p}}(\Lambda^{\mathbf{l}}\mathbb{R}^n)$ to the Hardy space $\mathcal{H}^1(\Lambda^{\mathbf{l}}\mathbb{R}^n)$, $l = l_1 + ... + l_m$. For $\Phi = (\varphi^1, ..., \varphi^m)$ and $\Psi = (\psi^1, ..., \psi^m)$ the precise inequality reads as

$$\|\mathcal{J}\Phi - \mathcal{J}\Psi\|_{\mathcal{H}^1} \le C_{\mathbf{p}}(n) \sum_{k=1}^m \|\varphi^k - \psi^k\|_{p_k} \prod_{i\neq k} (\|\varphi^i\|_{p_i} + \|\psi^i\|_{p_i}) \tag{3.1}$$

Proof. It involves no loss of generality in assuming that $\Psi = 0$. Indeed, writing $\mathcal{J}\Phi - \mathcal{J}\Psi$ as a telescoping sum (2.16) reduces the problem to the case when $\Psi = 0$. Fix an arbitrary test l-form $\eta \in C_o^\infty(\Lambda^l\mathbb{R}^n)$, $l = l_1 + ... + l_n$. Our aim is to estimate the integral

$$\int_{\mathbb{R}^n} \langle \eta, \varphi^1 \wedge ... \wedge \varphi^m\rangle dx = \int_{\mathbb{R}^n} \varphi^1 \wedge ... \wedge \varphi^{m-1} (\varphi^m \wedge *\eta) \tag{3.2}$$

by means of $\|\varphi^i\|_{p_i}$, $i = 1, ..., m$, and the BMO-norm of η. To this effect, we split the l_m-form $\varphi^m \wedge *\eta \in L^{p_m}(\Lambda^{l_m}\mathbb{R}^n)$ into exact and coexact component

$$\varphi^m \wedge *\eta = d\alpha + d^*\beta . \tag{3.3}$$

Observe that the only harmonic field in $\mathbb{R}^n$ which is L^s-integrable for some $s \geq 1$ is the zero form. Since $\varphi^1, ..., \varphi^{m-1}$ are closed, the l-form $\varphi^1 \wedge ... \wedge \varphi^m \wedge d\alpha$ is exact, and by Stokes' theorem we obtain

$$\int_{\mathbb{R}^n} \varphi^1 \wedge ... \wedge \varphi^{m-1} \wedge d\alpha = \pm \int_{\mathbb{R}^n} d(\varphi^1 \wedge ... \wedge \varphi^{m-1} \wedge \alpha) = 0 .$$

Hence, (3.2) yields

$$\int_{\mathbb{R}^n} \langle \eta, \varphi^1 \wedge ... \wedge \varphi^m \rangle dx = \int_{\mathbb{R}^n} \varphi^1 \wedge ... \wedge \varphi^{m-1} \wedge d^*\beta$$
$$\leq C_{\mathbf{p}} \|\varphi^1\|_{p_1} ... \|\varphi^{m-1}\|_{p_{m-1}} \|d^*\beta\|_{p_m} . \tag{3.4}$$

It remains to estimate $d^*\beta$ in L^{p_m}-norm. This term can be expressed by means of a singular integral operator of the left-hand side of (3.3). Precisely, we have:

$$d^*\beta = T(\varphi^m \wedge *\eta) \tag{3.5}$$

where the operator T acts on forms of all degree, respecting the grading in the exterior algebra $\Lambda\mathbb{R}^n = \oplus_{l=0}^n \Lambda^l \mathbb{R}^n$, namely

$$T : \oplus_{l=0}^n L^s(\Lambda^l\mathbb{R}^n) \to \oplus_{l=0}^n L^s(\Lambda^l\mathbb{R}^n) , \qquad 1 < s < \infty$$

As a matter of fact, T is a linear expression of second order Riesz transforms. The important point to make here is that T vanishes on closed forms. This is an immediate consequence of the uniqueness in the Hodge decomposition; there is no coexact component of a closed form in $\mathbb{R}^n$. As $T\varphi^m = 0$, we can express (3.5) in the form of a commutator

$$d^*\beta = T(\varphi^m \wedge *\eta) - (T\varphi^m) \wedge *\eta$$

and conclude from Coifman-Rochberg-Weiss inequality that

$$\|d^*\beta\|_{p_m} \leq C\|\eta\|_{BMO} \|\varphi^m\|_{p_m} .$$

Our final estimate reads as:

$$\int_{\mathbb{R}^n} \langle \eta, \varphi^1 \wedge ... \wedge \varphi^m \rangle \leq C_{\mathbf{p}}(n) \|\eta\|_{BMO} \|\varphi^1\|_{p_1} ... \|\varphi^m\|_{p_m} \tag{3.6}$$

which also holds for $\eta \in VMO(\mathbb{R}^n)$ by an approximation. The duality argument (1.34) then shows that $\mathcal{J}\Phi \in \mathcal{H}^1(\mathbb{R}^n)$. Finally, the trick with the telescoping sum (2.16) gives estimate (3.1). □

Theorem 3 combined with Problem 2 leads us naturally to study the operator

$$\mathcal{J} \log |\mathcal{J}| : \mathcal{L}^{\mathbf{p}}(\Lambda^{\mathbf{l}}\mathbb{R}^n) \to \mathcal{D}'(\Lambda^l\mathbb{R}^n) \tag{3.7}$$

which carries a given form $\Phi = (\varphi^1, ..., \varphi^m) \in \mathcal{L}^{\mathbf{p}}(\Lambda^{\mathbf{l}}\mathbb{R}^n)$ into the l-form $\varphi^1 \wedge ... \wedge \varphi^m \log|\varphi^1 \wedge ... \wedge \varphi^m|$. In general, this form need not be locally integrable. A vehicle for defining it as a Schwartz distribution is the nonlinear commutator $T^{it}(f) = T(f|f|^{it}) - (Tf)|Tf|^{it}$, where $t \in \mathbb{R}$. Estimate (1.20) ensures the existence of the derivative $i\frac{d}{dt}(|\mathcal{J}\Phi|^{it}\mathcal{J}\Phi)$ at $t = 0$, and this derivative is one of the possible interpretations of $(\mathcal{J}\Phi)\log|\mathcal{J}\Phi| \in \mathcal{D}'(\Lambda^l X)$. Moreover, the operator (3.7) is continuous.

Inspired by this, Anna Verde and I were elaborating such operators in a setting of greater generality. Consider the nonlinear map $\mathcal{Z} = \mathcal{H}^1(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ which takes given $h \in \mathcal{H}^1(\mathbb{R}^n)$ into the function

$$\mathcal{Z}h = h \log\left(e + \frac{|h|}{\|h\|_1}\right) .$$

Notice that $\mathcal{Z}h \in L^1(\mathbb{R}^n)$ amounts to saying that h belongs to the Zygmund space $L\log(\mathbb{R}^n)$. This space is equipped with the norm

$$\|h\|_{L\log(\mathbb{R}^n)} = \|\mathcal{Z}h\|_1$$

where proving triangle inequality is a nontrivial exercise. For a measurable subset $E \subset \mathbb{R}^n$ we define $\|h\|_{L\log L(E)} = \|\mathcal{Z}(\chi_E h)\|_1$.

Although in general $\mathcal{Z}h$ need not be in $L^1_{\text{loc}}(\mathbb{R}^n)$, we were able to give meaning to $\mathcal{Z}h$ as a Schwartz distribution of order 1. For a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ of order 1 we can introduce its seminorms $[F]_R = \sup\{|F\varphi|; \quad \varphi \in C_0^\infty(B), \|\nabla\varphi\|_\infty = 1\}$, where supremum is taken over all balls $B \subset \mathbb{R}^n$ of radius R. With this notation, we can state our main estimate in [32] quite briefly:

$$[\mathcal{Z}g - \mathcal{Z}h]_R \le C_R(n)\|g-h\|_{\mathcal{H}}^{1} \log\left(e + \frac{\|g\|_{\mathcal{H}}^1 + \|h\|_{\mathcal{H}}^1}{\|g-h\|_{\mathcal{H}}^1}\right) . \tag{3.8}$$

Recall a result of Stein [54]; if $h \in \mathcal{H}^1(\mathbb{R}^n)$ and $h \ge 0$ in an open set $\Omega \subset \mathbb{R}^n$, then $h\log h \in L^1_{\text{loc}}(\Omega)$. Ultimately, inequality (3.8) reveals an interesting fact that deserves a statement of its own.

Corollary 5.

If $h_k \to h$ in $\mathcal{H}^1(\mathbb{R}^n)$ and $h_k \ge 0$ on an open region $\Omega \subset \mathbb{R}^n$, then

$$\lim_{k\to\infty} \|h_k - h\|_{L\log L(E)} = 0$$

for each compact $E \subset \Omega$.

One interesting inference from these studies concerns the Jacobian operator.

Corollary 6.

Suppose that the mappings $f_k : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ converge to f in $W^{l,n}(\Omega, \mathbb{R}^n)$ and $\mathcal{J}(x, f_k) \ge 0$, for almost every $x \in \Omega$ and $k = 1, 2, \ldots$. Then

$$\lim_{k\to\infty} \|\mathcal{J}f_k - \mathcal{J}f\|_{L\log L(E)} = 0$$

for each compact $E \subset \Omega$.

It is worth pointing out that $\mathcal{J}f_k - \mathcal{J}f$ may change sign in Ω. In that sense, Corollary 6 is an extension of the familiar result of Müller [42, 43].

3.2 Does There Exist a Fundamental Solution of the Jacobian Operator?

When developing the theory of a differential operator, it is always natural to ask if the operator has an inverse. Here we shall address this question for the Jacobian operator

$$\mathcal{J} : W^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{H}^p(\mathbb{R}^n) \quad , \quad 1 \le p < \infty \tag{3.9}$$

where

$$\mathcal{H}^p(\mathbb{R}^n) = \begin{cases} \mathcal{H}^1(\mathbb{R}^n) & \text{if } p = 1 \\ L^p(\mathbb{R}^n) & \text{if } p > 1 \end{cases}$$

and $W^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ denotes the Sobolev space of mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ whose distributional differential $Df : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is L^{np}-integrable.

$$\mathcal{J}(f) = \mathcal{J}(x, f) = \det\, Df$$

Definition 2.

A continuous map

$$\mathcal{E} : \mathcal{H}^p(\mathbb{R}^n) \to W^{1,np}(\mathbb{R}^n, \mathbb{R}^n)\,, \quad 1 \le p < \infty \tag{3.10}$$

is said to be a fundamental solution of the Jacobian operator if

$$\mathcal{J} \circ \mathcal{E} = Id : \mathcal{H}^p(\mathbb{R}^n) \to \mathcal{H}^p(\mathbb{R}^n) \tag{3.11}$$

There is enough evidence to believe that:

Conjecture 1.

For each dimension $n \ge 2$ the Jacobian operator has a fundamental solution.

Having disposed of the fundamental solution, one could solve in a canonical way the Jacobian equation

$$\det\, Df(x) = h(x) \qquad \text{a.e.} \tag{3.12}$$

for $f \in W^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$, where h is a given function from $\mathcal{H}^p(\mathbb{R}^n)$. Setting $f = \mathcal{E}h$, we would have

$$\int_{\mathbb{R}^n} |Df(x)|^{np} dx \le C(n, p) \|h\|^p_{\mathcal{H}^p(\mathbb{R}^n)} \tag{3.13}$$

Solving (3.12) is an invitation to absolutely new studies in the theory of genuine nonlinear PDEs, see [10] where the question is formulated in case $p = 1$. Following [14], we can show that (3.12) admits solutions if $h \in C_0^\infty(\mathbb{R}^n)$ with integral zero [dense subspace of $\mathcal{H}^p(\mathbb{R}^n)$]. However, these do not necessarily verify the uniform bound (3.13).

We are now naturally led to the following variational problem. Suppose we are given an h of the form (3.12). Minimize the energy integral

$$I[g] = \int_{\mathbb{R}^n} |Dg(x)|^{np} dx \tag{3.14}$$

subject to the volume constraint $\mathcal{J}(x, g) = h(x)$, a.e.

Problem 3. Show that the minimum of $I[g]$ is attained for some $g = f_o \in W^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$.

It may very well be that the minimizer is unique (up to rotations) and, as such, f_o can be used to define the fundamental solution. With the aid of (3.13) we then could extend $\mathcal{E}$ to the entire space $\mathcal{H}^p(\mathbb{R}^n)$. For the related questions, see [14, 41, 48] and [51].

Equation (3.12) in the complex plane is particularly simple. It takes the form

$$\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 = h \tag{3.15}$$

where h is a given real function from $\mathcal{H}^p(C)$ while f is a complex function from $W^{1,2p}(C)$. We now recall the Beurling-Ahlfors transform

$$T\omega(z) = -\frac{1}{2\pi i} \int\int_C \frac{\omega(\zeta) d\zeta \wedge d\bar{\zeta}}{(z-\zeta)^2}$$

which is also known as the complex Hilbert transform. Its characteristic property is that it permutes the Cauchy-Riemann derivatives, in symbols: $T \circ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z}$. Hence, (3.15) reduces to a singular integral equation

$$|T\omega|^2 - |\omega|^2 = h\,, \quad \text{for} \quad \omega \in L^{2p}(C) \tag{3.16}$$

It appears likely that within the next few years there will be much progress made in elucidating what this equation really means.

Problem 4. Consider the following 1-dimensional analogue of Equation (3.16):

$$(H\phi)^2 - \phi^2 = h \in \mathcal{H}^p(\mathbb{R})\,, \quad 1 \le p < \infty \tag{3.17}$$

for $\phi \in L^{2p}(\mathbb{R})$, where H stands for the Hilbert transform.

Give sufficient and necessary conditions (in terms of h and Hh) for Equation (3.17) to be solvable. Show, in particular, that for $h \ge 0$ and $p > 1$ the equation has unique solution up to the $\pm$ sign.

Hint. Make use of the following definition of $\mathcal{H}^1(\mathbb{R})$: $h \in \mathcal{H}^1(\mathbb{R})$ if and only if $h \in L^1(\mathbb{R})$ and $Hh \in L^1(\mathbb{R})$. Moreover, $\|h\|_{\mathcal{H}}^1 \approx \|h\|_1 + \|Hh\|_1$.

3.3 Degree Formulas

From now on we assume that X and Y are compact C^∞-smooth oriented Riemannian n-manifolds, $n \ge 2$, without boundary. In a very pragmatic sense, the integral $\int_X \mathcal{J}(x, f)\,dx$ encodes some topological invariants of the mapping $f : X \to Y$, including its degree; the number of times Y is covered by $f(X)$. A clear trend has emerged in nonlinear elasticity and geometric PDEs such as Ginzburg-Landau equation, harmonic maps, and so on: the trend to figure out if and how it is possible to define the degree of maps with nonintegrable Jacobian. A fundamental character of our estimates can now be attested by applications to this problem. Before we jump to conclusions, though, let us acquaint ourselves with the estimates concerning power type perturbations of the determinant, say $\varphi^1 \wedge \ldots \wedge \varphi^m \left(|\varphi^1| \ldots |\varphi^m|\right)^{-\varepsilon}$. Here we assume that the form $\Phi = (\varphi^1, \ldots, \varphi^m)$ is closed and belongs to the grand space $L^{\mathbf{p})}(\wedge^{\mathbf{l}} X)$, where $\mathbf{l} = (l_1, \ldots, l_m)$ with $l_1 + \ldots + l_m = n$. The space $L^{p)}(\Lambda^l X)$, $1 < p < \infty$, is furnished with the norm

$$\|\varphi\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_X |\varphi|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}}$$

It then follows that the closure of $C^\infty(\Lambda^l X)$, denoted by $VL^{p)}(\Lambda^l X)$, is made up of l-forms such that

$$\lim_{\varepsilon \downarrow 0} \varepsilon \int_X |\varphi|^{p-\varepsilon} = 0\,.$$

In a manner consistent with the above, we introduce the "grand" spaces $L^{\mathbf{p})}(\Lambda^{\mathbf{l}} X)$ and $VL^{\mathbf{p}}(\Lambda^{\mathbf{l}} X)$. Using inequality (1.23) and the Hodge-DeRham decomposition (2.8), we can approximate $|\varphi_i|^{-\varepsilon}\varphi_i$ by closed forms with enough accuracy to obtain:

$$\overline{\lim_{\varepsilon \downarrow 0}} \int_X \det\Phi\, |\det\Phi|^{-\varepsilon} \qquad < \infty \tag{3.18}$$

Not only that, but the limit actually exists if $\Phi \in VL^{\mathbf{p})}(\wedge^{\mathbf{l}} X) \cap \ker d$ and we may quote the limit as *"weak integral"* of the determinant. This suggests the possibility of extending the notion of the degree to mappings with weakly integrable Jacobians. For a recent account of the degree theory along more classical lines, we refer to Brezis and Nirenberg [5]. In that paper, the authors define and establish basic properties of the degree of VMO-maps $f : X \to Y$. Notice that $VMO(X, Y)$ contains the Sobolev space $W^{1,n}(X, Y)$; the latter presents no difficulty because $C^\infty(X, Y)$ is dense in $W^{1,n}(X, Y)$ and Jacobians are L^1-integrable. We soon realized [20] that (3.18) opens a new route to the degree of mappings in some Orlicz-Sobolev classes weaker than $W^{1,n}(X, Y)$, which in many ways complements the approach of Brezis-Nirenberg.

In our method, the l-th cohomology group $\mathcal{H}^l(Y)$ of the target manifold is required to be nontrivial for at least one integer $1 < l < n$, which unfortunately excludes the spheres. The trouble is that if $\mathcal{H}^l(Y) = 0$, there do not exist closed forms $\alpha \in C^\infty(\wedge^l Y)$ and $\beta \in C^\infty(\wedge^{n-l} Y)$ such that $\int_Y \alpha \wedge \beta = 1$. Indeed, every closed l-form is exact, as is the n-form $\omega = \alpha \wedge \beta$ and thus $\int_Y \omega = 0$. Conversely, if we can find a non-trivial harmonic field $\theta \in \mathcal{H}^l(Y)$ with $\int_Y |\theta|^2 = 1$, then $\omega = \theta \wedge *\theta$ does the job. This is just another way of seeing that $\mathcal{H}^l(Y)$ must be nontrivial.

Now, with the assumptions above, we define the Jacobian of $f : X \to Y$ by the rule: $\mathcal{J}(x, f)\,dx = f^\#\omega = f^\#\alpha \wedge f^\#\beta = \varphi^1 \wedge \varphi^2$, where $f^\#$ stands for the pullback of forms via the mapping f. It is very crucial to understand that φ^1 and φ^2 are closed forms only when f is sufficiently regular. Later on, we dispense with this requirement by an approximation. By virtue of (3.18) it is tempting to introduce the degree of f by

$$\deg(f; X, Y) = \lim_{\varepsilon \downarrow 0} \int_X \frac{\mathcal{J}(x, f)\,dx}{|\mathcal{J}(x, f)|^\varepsilon} \tag{3.19}$$

Certainly this makes sense in the class $W^{1,n}(X, Y)$. Be aware of nuances concerning the Sobolev space $W^{1,p}(X, Y)$; smooth mappings $f : X \to Y$ need not be dense in this space when $1 \le p < n$. This case was settled by Bethuel only five years ago [2], contrast with Problem 6. We may assume that Y is isometrically imbedded in some Euclidean space $\mathbb{R}^N$. This involves no loss of generality due to the familiar theorem of Nash. Now, $f : X \to Y$ is said to belong to the Sobolev class $W^{1,p}(X, Y)$ if $f \in W^{1,p}(X, \mathbb{R}^N)$ and $f(x) \in Y$ for almost every $x \in X$. This space does not depend on the imbedding $Y \hookrightarrow \mathbb{R}^N$. It is a complete metric subspace of $W^{1,p}(X, \mathbb{R}^N)$. Finally, the "grand" Sobolev space $W^{1,p)}(X, Y)$ is equipped with the semimetric

$$\rho(f, g) = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_X |Df(x) - Dg(x)|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} .$$

Now, we have the equipment to do a fair amount of generalizations. If f belongs to the grand Sobolev space $W^{1,n)}(X, Y)$, then $\varphi^1 = f^\sharp\alpha \in L^{\frac{n}{l})}(\wedge^l X) \cap \ker d$ and $\varphi^2 = f^\sharp\beta \in L^{\frac{n}{n-l})}(\wedge^{n-l} X) \cap \ker d$, as is easy to check. Consequently, the integrals in (3.19) stay bounded as ε approaches zero. Note that $W^{1,n)}(X, Y)$ contains the Marcinkiewicz-Sobolev space weak-$W^{1,n}(X, Y)$, that is the mappings $f : X \to Y$ whose generalized differential satisfies the inequality $\mathrm{mes}\{x; \quad |Df(x)| > t\} \le Ct^{-n}$ for all positive t. We do not know if the limit in (3.19) actually exists for mappings in the space weak-$W^{1,n}(X, Y)$.

Next, we denote by $\mathcal{W}^{1,n)}(X, Y)$ the closure of $C^\infty(X, Y)$ in the metric of the grand Sobolev space $W^{1,n)}(X, Y)$.

Theorem 4.

Formula (3.19) is valid for $f \in \mathcal{W}^{1,n)}(X, Y)$. The degree of f is an integer that is invariant under homotopy within the class $\mathcal{W}^{1,n)}(X, Y)$.

As a matter of fact, the degree function $\deg : \mathcal{W}^{1,n)}(X, Y) \to \mathbb{Z}$ is uniformly continuous. This answers one of the questions in [5]. Note too that $\mathcal{W}^{1,n)}(X, Y) \not\subset VMO(X, Y)$. It is not difficult to check that mappings in $\mathcal{W}^{1,n)}(X, Y)$ satisfy

$$\lim_{\varepsilon \downarrow 0} \varepsilon \int_X |Df(x)|^{n-\varepsilon}\, dx = 0 \tag{3.20}$$

Problem 5. Does this condition characterize the space $\mathcal{W}^{1,n)}(X, Y)$?

One more subclass of $W^{1,n)}(X, Y)$ merits mentioning here, and that is the Zygmund-Sobolev space of mappings $f : X \to Y$ such that $\int_X |Df|^n \log^{-1}(e + |Df|) < \infty$. We denote it briefly by $\mathcal{Z}(X, Y)$ in honor of A. Zygmund. That condition (3.20) holds in the Zygmund-Sobolev space is undeniable. But the following question may not be obvious:

Problem 6. Is the space $C^\infty(X, Y)$ dense in $\mathcal{Z}(X, Y)$?

Some evidence for it can be found in the recent work of Bethuel [2]. As a rather interesting inference from Theorem 4, let us assume that f preserves the orientation, that is $J(x, f) \geq 0$ a.e. Passing the limit under the integral sign in (3.19) yields: $\deg(f; X, Y) = \int_X J(x, f)\, dx$, though $|Df|^n$ may not be integrable, confront with [28]. There are also the nagging questions of spheres and manifolds with boundary.

Problem 7. How can we get rid of the unwelcome condition $\mathcal{H}^l(Y) \neq 0$? How can we extend formula (3.19) to noncompact manifolds?

To this latter question the answer can possibly be reached from [6].

Coming to an end, there are many problems and results along the line of these lectures which we shall have to postpone. I believe, however, that the questions addressed here give an idea of the general trend of the investigations. Nonlinear commutators, the Jacobians, and the governing PDEs, as evidenced by current literature, attain the status of an important branch of classical analysis. Many fascinating and useful connections with nonlinear elasticity, calculus of variations, singular integrals, etc. remain yet to be discovered.

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