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Varieties with low dimensional dual variety

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In this note we classify the complex projective n-folds $X \subset \mathbb{P}^N$ with dimension equal to the dimension of its dual X^* minus one, $n \leq 2/3N$ and positive defect. We show that such n-folds are hyperplane sections of the $n+1$-folds with $dim\ X = dim\ X^*$ (classified by Ein) and some scrolls over curves. The natural generalization is to consider the set $\mathcal{S}_p$ of positive defect n-folds with $dim(X) = dim(X^*) - (p-1)$ (p a fixed positive integer). We show that the set of possible values of the pair (N, n) with $n \leq 2/3N$ corresponding to nondegenerate n-folds in $\mathcal{S}_p$ which are not scrolls is finite.

1 Introduction

Let X be a complex projective n-fold in the projective space $\mathbb{P}^N$. The dual variety of X, denoted by X^*, is the variety of hyperplanes tangent to X (i.e., that contain the embedded tangent space in a point). For most n-folds their dual variety is a hypersurface. We will call the difference $N - 1 - dim(X^*)$ the *defect* of X. The goal of this paper is to classify n-folds X with positive defect under some additional hypothesis on the difference between the dimension of X and the dimension of its dual.

By Zak's Theorem of tangencies [15] $dim(X) \leq dim(X^*)$. Then the defect will be maximum when the equality holds. The nonlinear positive defect varieties with $dim(X) = dim(X^*)$ and $n \leq \frac{2N}{3}$ have been classified by L. Ein [1] and they are the following (we call them Ein varieties):

(a) the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ in $\mathbb{P}^{2n-1}$.

(b) the Plücker embedding in $\mathbb{P}^9$ of the grassmannian of lines in $\mathbb{P}^4$, $G(2,5)$.

(c) the 10-dimensional spinor variety S_{10}, which parametrizes the $\mathbb{P}^4$'s in a smooth quadric hypersurface of $\mathbb{P}^9$, embedded in $\mathbb{P}^{15}$.

In this paper, following a suggestion of F.L. Zak, we extend this result of Ein and classify the nonlinear positive defect n-folds with the property that $dim(X^*) = dim(X) + 1$ and $n \leq \frac{2N}{3}$. We conclude that these n-folds are:

(a) $\mathbb{P}^{n-1}$-bundles over nonsingular curves such that the fibers are embedded linearly with $N = 2n$.

(b) hyperplane sections of Ein $n+1$-folds.

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We can remark that the complete intersections have defect equal to zero. Then the above lists will be the complete lists of n-folds with the conditions requested if the Hartshorne's conjecture on complete intersections is true.

These classifications of varieties with $dim(X) = dim(X^*)$ and with $dim(X) = dim(X^*)-1$ suggest us the study of the general case $dim(X) = dim(X^*)-(p-1)$ (p a fixed positive integer). Let $\mathcal{S}_p$ be the set of complex nondegenerate n-folds in $\mathbb{P}^N$ with positive defect and $dim(X) = dim(X^*) - (p-1)$. Let us call the $\mathbb{P}^r$-bundles over a m-fold Y such that the fibers are embedded linearly *scrolls over* Y. We show that the set of pairs (N, n) corresponding to elements in $\mathcal{S}_p$ which are not scrolls is finite.

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2 Preliminary Results

Let X be a complex projective irreducible algebraic variety. The *conormal variety* of X, denoted by $\mathcal{P}_X$, is the following variety

$$\mathcal{P}_X = \overline{\{(x, H) : \ x \in Sm(X), \ H \in (\mathbb{P}^N)^*, \ T_{X,x} \subset H\}}.$$

where $Sm(X)$ is the set of nonsingular points of X.

The *dual variety* of X is the image of the projection of $\mathcal{P}_X$ on $(\mathbb{P}^N)^*$. From now on X will be a complex nonlinear n-fold in $\mathbb{P}^N$. We also assume that X is nondegenerate (the degenerate case is easy to study ([1], Prop. 1.1)).

Let H be a general tangent hyperplane to X. The biduality theorem ($\mathcal{P}_X = \mathcal{P}_{X^*}$) implies that the locus of points z such that the embedded tangent space on z is contained in H is a linear space of dimension $N-1-dim(X^*)$ ($= def(X)$). This linear space is known as the *contact locus* L of X and H.

Theorem 2.1. (Zak's Theorem of tangencies) [15] *Let X be a nondegenerate complex projective n-fold in $\mathbb{P}^N$ and H a l-plane ($l \geq n$). Let Y be the set of points in X such that its embedded tangent space is contained in H. Then $dim(Y) \leq l - n$.*

We can apply Zak's Theorem to a general tangent hyperplane H and so $dim(X^*) \geq dim(X)$. Thus, as it was said in the introduction, the defect will be maximum when equality holds (that will be the case when X^* is smooth). These maximal defect n-folds with the additional hypothesis $n \leq \frac{2}{3}N$ have been classified by Ein [1] and they are only the ones pointed out in the previous section.

In order to face the problem of classification of n-folds in the next step (i.e. with $dim(X^*) = dim(X) + 1$) we need to recall some results of [1]. They are, mainly, some properties of the conormal bundle of the contact locus of a general tangent hyperplane H in X.

Theorem 2.2. [1] *Let X be a n-fold as before with positive defect k, H a general tangent hyperplane and L the contact locus of X and H. Let us denote by K_X the canonical bundle of X. It follows that:*

(a) $\mathcal{N}^*_{L|X}(1) \simeq \mathcal{N}_{L|X}$

(b) $K_X|_L = \mathcal{O}(\frac{-n-k-2}{2})$.

(c) If $\frac{n-3k-2}{2} \leq 0$ *and* $K_X = \mathcal{O}(b)$ *(b an integer) then the cohomology of the twists of* $\mathcal{N}^*_{L|X}$ *is:*

$$\begin{aligned} H^0(\mathcal{N}^*_{L|X}(a)) &= 0 \quad \text{for all } a \leq 0 \\ H^k(\mathcal{N}^*_{L|X}(a)) &= 0 \quad \text{for all } a \geq -k \\ H^i(\mathcal{N}^*_{L|X}(a)) &= 0 \quad \text{for all } a \geq \tfrac{n-3k}{2} \quad 0 < i < k \\ H^i(\mathcal{N}^*_{L|X}(a)) &= 0 \quad \text{for all } a \leq \tfrac{k-n}{2} \quad 0 < i < k \end{aligned}$$

And, since $\frac{-n-k-2}{2}$ *has to be an integer,* $n \equiv k(mod\ 2)$.

Scrolls with some conditions on the dimension of the fibers are natural examples of positive defect varieties. There are results that partially characterize them, in the following sense:

Theorem 2.3. [2] *In the same conditions as above we have:*

(a) $k = n-2$ *if and only if* X *is a* $\mathbb{P}^{n-1}$*-bundle over a smooth curve.*

(b) $k = n-4$ *if and only if* X *is a* $\mathbb{P}^{n-2}$*-bundle over a smooth surface* $(n \geq 7)$.

(c) If $k \geq \frac{n}{2}$ *then* X *is a scroll.*

Remark . Grassmannians of lines in $\mathbb{P}^{2r}$, $G(2, 2r+1)$, with r a positive integer are other examples of n-folds with positive defect $(k = 2)$.

We will use the following well known fact:

Lemma 2.4. (first bound) *Let* X *be a nondegenerate complex projective* n*-fold in* $\mathbb{P}^N$ *with positive defect* k*, then* $k \leq n-2$.

Proof : Take H a general hyperplane tangent to X. Then k is the dimension of the singular locus of the section $X \cap H$, so $k \leq n-1$. By the congruence $n \equiv k(mod\ 2)$ we have the lemma.

Since 2.2 gives information about the cohomology of the tensor products of $\mathcal{N}^*_{L|X}$ with the powers of the twisting bundle $\mathcal{O}_L(1)$, we can use the Beilinson spectral sequence (BSS) [13] of $\mathcal{N}^*_{L|X}$ to study it. Then, denoting this sequence by $E_r^{p,q}$, by definition

$$E_1^{p,q} = H^q(\mathcal{N}^*_{L|X}(p)) \otimes \Omega^{-p}(-p)$$

and converges to

$$E_i = \begin{cases} \mathcal{N}^*_{L|X} & i = 0 \\ 0 & \text{other case} \end{cases}$$

Each term of the BSS must have elements different from zero in some of its columns because the normal bundle cannot be zero. This is the key fact to prove the following lemma:

Lemma 2.5. (second bound) [1] *Let* X *be a complex, projective and nondegenerate* n*-fold in* $\mathbb{P}^N$ *with* $K_X = \mathcal{O}(b)$ *(b an integer) and positive defect* k*, then* $k \leq \frac{n-2}{2}$.

In most situations 2.2 implies that the first term of the BSS of $\mathcal{N}^*_{L|X}$ has elements different from zero only in some central columns. The next lemma allows us to extract consequences of this fact.

Lemma 2.6. *Let F be a vector bundle over $\mathbb{P}^n$ and P its Hilbert polynomial, that is $P(m) = \Sigma_{j=0}^{n}(-1)^j dim\ (H^j(F(m)))$. Then it follows that*

$$rk(F) = \Sigma_{j=0}^{n}(-1)^{n+j}\binom{n}{j}P(-n+j).$$

Proof: Let us recall that the rank of a coherent sheaf is the rank of the fiber in a general point. By the convergence ($E_{n+1}^{p,q} = E_{\infty}^{p,q}$) of the BSS of F it follows that the $rk(F)$ must be equal to $\Sigma_{-p=0}^{n}\Sigma_{q=0}^{n}(-1)^{p+q}rk(E_{n+1}^{p,q})$. Since each term of the BSS is constructed from the previous one by taking cohomology, we have:

$$\Sigma_{-p=0}^{n}\Sigma_{q=0}^{n}(-1)^{p+q}rk(E_{j+1}^{p,q}) = \Sigma_{-p=0}^{n}\Sigma_{q=0}^{n}(-1)^{p+q}rk(E_{j}^{p,q})$$

for all positive integer j. We can conclude applying recursively this equality from $j = 1$ to $n+1$ because in the 1-term of the BSS each element $E_1^{p,q}$ has rank $\binom{n}{-p}dim(H^q(F(p)))$.

Another proof of this lemma is based on taking consecutive restrictions of $F(m)$ to the different linear spaces in $\mathbb{P}^n$, that is, $\mathbb{P}^{n-1}$, $\mathbb{P}^{n-2}$ ... $\mathbb{P}^0$ and computing the Hilbert polynomial by means of the corresponding exact sequences:

$$0 \to F(a-1)|_{\mathbb{P}^s} \to F(a)|_{\mathbb{P}^s} \to F(a)|_{\mathbb{P}^{s-1}} \to 0.$$

3 Varieties with $dim(X) = dim(X^*) - 1$.

Let X be a nondegenerate nonlinear complex projective n-fold in $\mathbb{P}^N$ with the property that $dim(X) = dim(X^*) - 1$. We write, as before, $k = def(X)$. We will show that the only positive defect n-folds satisfying these properties and $n \leq 2N/3$ are either scrolls over smooth curves with dimension equal to the codimension or general hyperplane sections of Ein varieties.

By 2.4 n must be greater than or equal to $N/2$. If $n = N/2$ then $k = n-2$. Theorem 2.3 implies that X is a scroll over a nonsingular curve C. We can refer to [7] for a study of the cases of C rational and elliptic. If C has genus bigger than or equal to two no examples are known and it was conjectured in [14] that the genus of C is bounded by one. This conjecture has been proved in dimension two [8], three [15] and four [14] and it is, to the best of our knowledge, still open in the rest of dimensions.

Since n cannot be $\frac{N+1}{2}$ (it would imply $k = n-3$) then $n \geq 1 + \frac{N}{2}$. Barth's Theorem and 2.2 imply that $K_X = \mathcal{O}(-N/2)$ and then N is necessarily even.

By the Second Bound Lemma, $N - n - 2 \leq \frac{n-2}{2}$ and so $n \geq 2/3(N-1)$. We assume that $n \leq \frac{2}{3}N$ and we have $2(N-1)/3 \leq n \leq 2N/3$. Therefore we can distinguish three groups of n-folds:

First group: $N = 6r-2$, $n = 4r-2$, $k = 2r-2$.

Second group: $N = 6r+2$, $n = 4r+1$, $k = 2r-1$.

Third group: $N = 6r$, $n = 4r$, $k = 2r-2$.

with r a positive integer (big enough to make $k > 0$).

First group. $(N, n, k) = (6r-2, 4r-2, 2r-2)$ with r an integer greater than or equal to 2.
Since $\frac{n-3k-2}{2} = -r+1 \leq 0$, 2.2 shows that

$$\begin{aligned} &H^i(\mathcal{N}^*_{L|X}(a)) = 0 \quad 0 < i < k \quad -r \geq a \quad or\ a \geq -r+2 \\ &H^0(\mathcal{N}^*_{L|X}(a)) = 0 \qquad a \leq 0 \\ &H^k(\mathcal{N}^*_{L|X}(a)) = 0 \qquad a \geq -k \end{aligned}$$

So the first term $E_1^{p,q}$ of the BSS of $\mathcal{N}^*_{L|X}$ has only one column ($p = -r+1$) with elements different from zero. Then $E_1^{p,q} = E_\infty^{p,q}$, i.e.,

$$\mathcal{N}^*_{L|X} = H^{r-1}(\mathcal{N}^*_{L|X}(1-r)) \otimes \Omega^{r-1}(r-1).$$

It implies that $rk(\mathcal{N}^*_{L|X}) = 2r \geq \binom{2r-2}{r-1}$ and then $r \leq 3$.

If $r = 2$, $N = 10$, $n = 6$, $k = 2$ *and* $K_X = \mathcal{O}(-5)$.
Since X is non degenerate then $h^0(X, \mathcal{O}(1)) = 11 + a$ with a a positive integer. Then X is the isomorphic projection of a Del Pezzo manifold of degree 6+a, and it does not exist by Fujita's classification [5,6].

If $r = 3$, $n = 10$, $k = 4$, $N = 16$ *and* $K_X = \mathcal{O}(-8)$.
We have $h^0(\mathcal{O}(1)) = 17 + a$ like in the previous case. Then X is the isomorphic projection of a Mukai variety with Picard group free of rank 1 generated by the hyperplane section and sectional genus equals 8+a. It does not exist following the classifications of Mukai of these varieties [11,12].

Second group. $(6r+2, 4r+1, 2r-1)$ with r a positive integer.
Again $\frac{n-3k-2}{2} = -r+1 \leq 0$ so

$$H^i(\mathcal{N}^*_{L|X}(a)) = 0, \quad 0 < i < k, \quad -r+2 \leq a \text{ or } a \leq -r-1.$$

Then the first term of the BSS of $\mathcal{N}^*_{L|X}$ has only two columns ($p = -r$ and $p = -r+1$) with elements different from 0. It implies that $E_2^{p,q} = E_\infty^{p,q}$. Let P be the Hilbert polynomial of $\mathcal{N}^*_{L|X}$, Lemma 2.6 for $\mathcal{N}^*_{L|X}$ shows that

$$rk(\mathcal{N}^*_{L|X}) = \binom{2r-1}{r} |P(-r) - P(-r+1)|.$$

Therefore $2r+2 \geq \binom{2r-1}{r}$, i.e., $r \leq 2$.

Since $n > \frac{2}{3}(N-1)$ X is linearly normal [9].

If $r = 1$, $n = 5$, $N = 8$, $k = 1$, $K_X = \mathcal{O}(-4)$, X is a Del Pezzo variety of degree 5, and by Fujita's classification it is a hyperplane section of the Plücker embedding of $G(2,5)$ in $\mathbb{P}^9$.

If $r = 2$, $n = 9$, $N = 14$, $k = 3$, $K_X = \mathcal{O}(-7)$, X is a Mukai variety of degree 12 and sectional genus 7, so by Mukai's classification X is a hyperplane section of the embedding of the spinor variety S_{10} in $\mathbb{P}^{15}$.

Third group. $(6r, 4r, 2r-2)$ with r an integer greater than or equal to 2.

Again $\frac{n-3k-2}{2} = -r+2 \leq 0$ and so we have all the information of 2.2 about the cohomology of $\mathcal{N}^*_{L|X}(a)$. Then 2.6 and the fact that $P(-r) = P(-r+2)$ show that

$$2r+2 = (-1)^r(2\binom{2r-2}{r}P(-r) - \binom{2r-2}{r-1}P(-r+1))$$

We recall now the result of Nagura [10] which assures that for $m \geq 25$ there always exists a prime number p such that $m < p < (1.2)m$. Now it is easy to check that if $r \geq 7$ or $r = 5$ there is always a prime number p such that $r+2 \leq p \leq 2r-2$. But this is a contradiction with the above equality of ranks because such p divides both combinatorial numbers and does not divide $2(r+1)$. For the case $r = 6$ we can see that 3 makes the role of p and so $r \leq 4$.

If $r = 2$, then $n = 8$, $N = 12$, $k = 2$, $K_X = \mathcal{O}(-6)$ and X is linearly normal. So X is a Mukai variety of degree 10 and sectional genus 6. It does not exist by Mukai's classification.

If $r = 3$, then $n = 12$, $N = 18$, $k = 4$ and $K_X = \mathcal{O}(-9)$.
Since $\mathcal{N}^*_{L|X} \simeq \mathcal{N}_{L|X}(-1)$ their Chern polynomials are equal. By 2.2 and 2.6 we have $P(0) = 0$ and $6P(-2) - 8P(-3) = 8$ respectively. These facts and the Riemann Roch Theorem lead to the following result about the Chern classes of $\mathcal{N}^*_{L|X}$:

$$c_1 = -4h \quad c_2 = ch^2 \quad c_3 = (14-3c)h^3 \quad c_4 = (91-14c+(1/2)c^2)h^4$$

with h the class of the hyperplane section of L and c an integer.

By the exact sequence of the normal bundles

$$0 \to \mathcal{N}^*_X(1)|_L \to \mathcal{O}_L^{\oplus 14} \to \mathcal{N}^*_{L|X}(1) \to 0 \qquad (*)$$

$\mathcal{N}^*_{L|X}(1)$ is spanned by global sections and so $P(1)$, which is exactly the dimension of the global sections of $\mathcal{N}^*_{L|X}(1)$, must be greater than or equal to 9, i.e., $c \leq 10$.

If $c = 10$ then the dimension of the vector space $H^0(\mathcal{N}^*_{L|X}(1))$ equals 10, thus we can construct the exact sequence

$$0 \to F \to \mathcal{O}_L^{\oplus 10} \to \mathcal{N}^*_{L|X}(1) \to 0$$

where F is a bundle of rank 2 on L. This is a contradiction with the fact that the inverse of the Chern polynomial of $\mathcal{N}^*_{L|X}(1)$ has degree 4.

On the other hand c must be greater than or equal to 6 ([3], Cor. 2.12) and equality holds when $\mathcal{N}^*_{L|X}$ splits as a sum of line bundles. Therefore $c = 8$.

Let us restrict $\mathcal{N}^*_{L|X}$ to a projective plane in L. We denote this restriction by $\mathcal{N}^*_{\mathbb{P}^2}$ and the dimension of the cohomology vector spaces H^i by h^i. Using the isomorphism $\mathcal{N}^*_{\mathbb{P}^2} \simeq \mathcal{N}_{\mathbb{P}^2}(-1)$ and the Riemann-Roch Theorem we see that $h^0(\mathcal{N}^*_{\mathbb{P}^2}(-i)) = 0$ for i greater than 0, $h^2(\mathcal{N}^*_{\mathbb{P}^2}(-j)) = 0$ for j greater than -2, $h^1(\mathcal{N}^*_{\mathbb{P}^2}(-1)) = 2$ and the couple $(h^0(\mathcal{N}^*_{\mathbb{P}^2}), h^1(\mathcal{N}^*_{\mathbb{P}^2}))$ can be one of (4,2), (3,1) or (2,0).

We claim that $h^0(\mathcal{N}^*_{\mathbb{P}^2}) < 3$. Let us assume that $h^0(\mathcal{N}^*_{\mathbb{P}^2})$ is bigger than or equal to three, then we can choose three linearly independent global sections s_1,

s_2 and s_3 of $\mathcal{N}^*_{\mathbb{P}^2}$. Since this bundle is a subbundle of a trivial one, these sections give the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{N}^*_{\mathbb{P}^2} \to Q \to 0$$

where Q is a bundle of rank 5 on $\mathbb{P}^2$. Then in any line of this $\mathbb{P}^2$ we have the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 4}(-1) \to Q|_{\mathbb{P}^1} \to 0$$

of the restrictions. Since $\mathcal{O}_{\mathbb{P}^1}(-1)$ has no global sections then $Q|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 4}(-1)$. Thus Q is a uniform vector bundle over $\mathbb{P}^2$ and its splitting type is (0,-1,-1,-1,-1). By the classification in [4], Th. 5.1, we can conclude that $Q^*(-1)$ either splits as a sum of line bundles or is $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \Omega_{\mathbb{P}^2}(1)$. In both cases $h^1(Q) = 0$ and so $h^1(\mathcal{N}^*_{\mathbb{P}^2}) = 0$ which gives a contradiction. Then we have concluded that the pair $(h^0(\mathcal{N}^*_{\mathbb{P}^2}), h^1(\mathcal{N}^*_{\mathbb{P}^2}))$ must be (2,0).

Thus the first term of the BSS of $\mathcal{N}^*_{\mathbb{P}^2}$ has elements different from zero only on the diagonal ($p + q = 0$) and it implies that

$$\mathcal{N}^*_{\mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \Omega_{\mathbb{P}^2}^{\oplus 2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(-1).$$

The sequences

$$0 \to \mathcal{N}^*_{\mathbb{P}^n}(a-1) \to \mathcal{N}^*_{\mathbb{P}^n}(a) \to \mathcal{N}^*_{\mathbb{P}^{n-1}}(a) \to 0$$

and the BSS's of the different restrictions $\mathcal{N}^*_{\mathbb{P}^s}$ ($s = 2, 3$) show that the conormal bundle $\mathcal{N}^*_{L|X}$ is isomorphic to $\Omega_L(1) \oplus \Omega^3_L(3)$ (where the notation Ω^p_L means $\wedge^p \Omega_L$).

We can make a blowing up of X along L. Following the notations of Ein ([1], Lema 4.2) we call $p : \tilde{X} \to X$ this blowing up and denote $\mathcal{O}_{\tilde{X}}(a, b)$ the line bundle $p^*\mathcal{O}_X(a) \otimes \mathcal{O}_{\tilde{X}}(-bE)$ where E is the exceptional divisor.

Let us take the following exact sequence:

$$0 \to \mathcal{O}_{\tilde{X}}(1, 2) \to \mathcal{O}_{\tilde{X}}(1, 1) \to \mathcal{O}_E(1, 1) \to 0$$

Since 1 is greater than $n - 3k/2 = 0$ then ([1], Lema 4.2 c) $H^1(\mathcal{O}_{\tilde{X}}(1, 2))$ is equal to 0. Taking the cohomology sequence of the sequence above we have the inequality $h^0(\mathcal{O}_{\tilde{X}}(1, 1)) \geq h^0(\mathcal{O}_E(1, 1)) = h^0(\mathcal{N}^*_{L|X}(1))$. This is a contradiction because $h^0(\mathcal{O}_{\tilde{X}}(1, 1)) = 14$ and $h^0(\mathcal{N}^*_{L|X}(1)) = 15$.

So r cannot be 3.

If $r = 4$ the way of reasoning the non existence of these varieties is completely analogous to the previous one, we only outline the proof.

Once again the isomorphism $\mathcal{N}^*_{L|X} \simeq \mathcal{N}_{L|X}(-1)$ and 2.6 allow us to compute the Chern classes of the conormal bundle $\mathcal{N}^*_{L|X}$ to obtain, with the same notations as before,

$$c_1 = -5h \quad c_2 = ch^2 \quad c_3 = (30 - 4c)h^3 \quad c_4 = (57 - 9c + \frac{1}{2}c^2)h^4$$

$$c_5 = (-297 + 41c - \frac{3}{2}c^2)h^5 \qquad c_6 = (\frac{802}{3}c - 2370 - \frac{17}{2}c^2 + \frac{1}{6}c^3)h^6$$

We also know that $c \geq \binom{5}{2} = 10$ ([3], Cor. 2.12) and the equality implies that $\mathcal{N}^*_{L|X}$ splits as sum of line bundles. On the other hand $\mathcal{N}^*_{L|X}(1)$ is spanned by global sections and then c must be 12.

Let us restrict to a $\mathbb{P}^2$. Making the same kind of arguments as in the case $r = 3$ we can conclude that

$$\mathcal{N}^*_{\mathbb{P}^2} \simeq \mathcal{O}^{\oplus 3}_{\mathbb{P}^2} \oplus \Omega^{\oplus 2}_{\mathbb{P}^2}(1) \oplus \mathcal{O}^{\oplus 3}_{\mathbb{P}^2}(-1)$$

and that the restriction to a $\mathbb{P}^5$ is

$$\mathcal{N}^*_{\mathbb{P}^5} \simeq \Omega_{\mathbb{P}^5}(1) \oplus \Omega^1_{\mathbb{P}^5}(5) \qquad (**).$$

It is easy to see that there is no vector bundle on $\mathbb{P}^6$ with the required conditions and whose restriction to a $\mathbb{P}^5$ is the one of $(**)$, q.e.d.

We can sum up our results in the following theorem.

Theorem 3.1. *Let X be a nondegenerate complex projective n-fold in $\mathbb{P}^N$ with $n \leq \frac{2}{3}N$, positive defect k and such that $dim(X) = dim(X^*) - 1$, then X must be one of the following n-folds:*

(i) scrolls over smooth curves of dimension n in $\mathbb{P}^{2n}$.

(ii) hyperplane sections of $G(2,5)$ or S_{10}.

4 The General Case

Ein's and our classification suggest the study of positive defect varieties in cases when $dim(X) = dim(X^*) - (p-1)$ with p a fixed positive integer. Let us call the set of complex nondegenerate and nonlinear n-folds in this situation $\mathcal{S}_p$. We show that, excluding scrolls, in each $\mathcal{S}_p$ there are only a finite set of possibilities for the couple (n, N)

Let us assume X is not a scroll. Theorem 2.3.c. implies that $k \leq (n/2) - 1$ and then $n \geq 2(N - p + 1)/3$. Thus for $N \geq 2 + 4p$ Barth's Theorem implies that $Pic(X) \simeq Z$. To apply 2.2.c we need that $(n - 3k - 2)/2 \leq 0$, i.e., $n \leq 3(N - p - (2/3))/4$. By the assumption of the inequality $n \leq 2N/3$, this is true for N greater than or equal to $9p+6$. Thus $H^i(\mathcal{N}^*_{L|X}(a)) = 0$ for $a \geq a_1 = (n-3k)/2$ and $0 < i < k$ and for $a \leq a_2 = (k-n)/2$. But the difference $a_1 - a_2$ is less than or equal to $2p$. This implies that the first term of the BSS of the conormal bundle of a variety with $dim(X) = dim(X^*) - (p-1)$ and $N \geq 9p + 6$ has elements different from zero only in the $2p$ central columns. But $k - 2p \geq \frac{1}{3}N - 3p$ is greater than zero. Then, when $N > 9p$ there are some columns in the BSS with no elements different from zero. So 2.6 gives equalities of the type

$$n - k = \sum_{i=0}^{p-1} \binom{k}{\frac{k}{2} + i} x_i \quad k \text{ even } x_i \in \mathbb{Z}$$

$$n - k = \sum_{i=0}^{p-1} \binom{k}{\frac{k+1}{2} + i} x_i \quad k \text{ odd } x_i \in \mathbb{Z}.$$

We must find a prime number between $\frac{k}{2}+p+1$ (or $\frac{k+1}{2}+p+1$) and k and such that it does not divide $\frac{n-k}{2}$. But this prime number exists by Nagura's result, because $\frac{n-k}{2} < \frac{k}{2}+p+1$ and because it is possible to take N big enough to make k as big as needed ($k \geq \frac{1}{3}N - p$).

We have shown that:

Theorem 4.1. *Let p be a fixed positive integer and $\mathcal{S}_p$ as before. Then the set of pairs (n, N) corresponding to n-folds $X \subset \mathbb{P}^N$ of $\mathcal{S}_p$ with $n \leq \frac{2}{3}N$ and which are not scrolls is finite.*

References

[1] L. Ein. Varieties with small dual varieties I. Inv. Math. **86**, 63-74 (1986)
[2] L. Ein. Varieties with small dual varieties II. Duke Math J. **52**, 895-907 (1985)
[3] G. Elenwajg, O. Forster. Bounding Cohomology Groups of Vector Bundles on P_n. Math. Ann. **246**, 251-270 (1980)
[4] G. Elenwajg. Fibrés Uniformes de Rang Elevé sur P_2. Ann. Ins. Fourier, Grenoble **31**, 4, 89-114 (1981)
[5] T. Fujita. On the Structure of polarized Manifolds with Total Deficiency one I. J. Math. Soc. Jpn. **32-4**, 709-775 (1980)
[6] T. Fujita. On the Structure of Polarized Manifolds with Total Deficiency one II. J. Math. Soc. Jpn. **33-2**, 414-434 (1981)
[7] P.Ionescu. Embedded Projective Varieties of Small Invariants III. in A.J. Sommese, A. Biancofiore (Eds.) Algebraic Geometry, (Proceedings, L'Aquila 1988). Lect. Notes in Math. **1417** Berlin, Heidelberg, New York: Springer 1990
[8] A. Lanteri. On the Existence of Scrolls in $\mathbb{P}^4$. Lincei-Rend. Sc. Fis. Mat. e Nat. **59**, 223-227 (1980)
[9] R. Lazarsfeld, A. Van de Ven. Topics in the Geometry of Projective Space. (Recent Work of F. L. Zak), DMV. Seminar, Band 4, Basel, Boston, Stuttgart: Birkhäuser Verlag 1984
[10] J. Nagura. On the Interval Containing at Least One Prime Number. Proc. Japan Acad. **28**, 177-181 (1952)
[11] S. Mukai. Fano Threefolds in Complex projective Geometry. London. Math. Soc. LNS **179**, 255-263. Cambridge University Press 1992
[12] S. Mukai. Biregular Classification of Fano 3-folds and Manifolds of Coindex 3. Proc. Nat Acad. Sci. USA, 3000-3002 (1989)
[13] C. Okonek, M: Schneider, H. Spindler. Vector Bundles on Complex Projective Spaces. Progress in Math. 3. Basel, Boston, Stuttgart: Birkhäuser, 1980
[14] P. Ionescu, M. Toma. On Very Ample Vector Bundles on Curves. Int. J. Math. 8 (1997) to appear
[15] M. Toma. Three-dimensional scrolls in $\mathbb{P}^6$. Arch. Math. **65**, 444-448 (1995)
[16] F.L.Zak. Tangents and Secants of Algebraic Varieties. Tran. of Math. Monog **127**. AMS Providence, R.I. 1993