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Autor: Smith, Samuel B.

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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A Based Federer Spectral Sequence and the Rational Homotopy of Function Spaces

Samuel B. Smith
Department of Mathematics
Saint Joseph's University
Philadelphia, PA 19131

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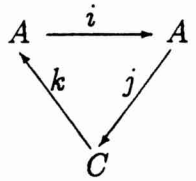
We study the rational homotopy of function spaces within the context of Quillen's minimal models. Our method is to consider a spectral sequence with $E_2^{p,q} = \tilde{H}^q(X, \pi_{p+q}(Y) \otimes \mathbb{Q})$ converging to the rational homotopy groups of components of the based function space $M(X,Y)_*$. Our results include calculations of rational homotopy groups as well as general contributions to the rational classification problem for components of function spaces.

1. An Exact Couple. Let $M(X,Y)$ and $M(X,Y)_*$ denote, respectively, the spaces of free and based continuous functions between two spaces X and Y. As an overriding assumption, all spaces are taken to be simply connected CW complexes. We denote the path components corresponding to a map $f : X \to Y$ by $M_f(X,Y)$ and $M_f(X,Y)_*$.

The based Federer spectral sequence for a map $f : X \to Y$ arises, like the original [1], from an exact couple of the form



Suppose X comes equipped with a fixed CW decomposition so

that for each $q > 1$ there is a cofibration sequence

$$\bigvee_{\alpha} S^{q-1}_{\alpha} \xrightarrow{h_q} X^{q-1} \to X^q,$$

where X^q is the q-skeleton of X and h_q is the wedge of the attaching maps for the q-cells of X. Let $f_q : X^q \to Y$ denote the restriction of f and let $W^{q-1} = \bigvee_\alpha S^{q-1}_\alpha$. We obtain a long exact sequence on homotopy of which a portion is

$$\pi_{p+1}(M_0(W^{q-1}, Y)_*) \xrightarrow{\partial_q} \pi_p(M_{f_q}(X^q, Y)_*) \xrightarrow{(\rho_q)_*}$$

$$\pi_p(M_{f_{q-1}}(X^{q-1}, Y)_*) \xrightarrow{(\overline{h_q})_*} \pi_p(M_0(W^{q-1}, Y)_*).$$

Set $A_{p,q} = \pi_p(M_{f_q}(X^q, Y)_*)$, $C_{p,q} = \pi_{p+1}(M_0(W^{q-1}, Y)_*)$, and let $i = (\rho_q)_*$, $j = (\overline{h_q})_*$ and $k = \partial_q$. By adjointness, $C_{p,q} \cong \bigoplus_\alpha \pi_{p+1}(\Omega^{q-1}(Y)) \cong \tilde{C}^q(X, \pi_{p+q}(Y))$, the reduced cellular q-cochains of X. When X is a finite complex, by [6, Theorem 2.3] we may replace X by any rationally equivalent space without affecting the rational homotopy of $M_f(X, Y)_*$. In particular, we may assume X comes equipped with a minimal CW decomposition with respect to its rational homotopy type. In this case, for degree reasons there are no nontrivial coboundary relations in the cellular cochain complex for X and so $C_{p,q} = \widetilde{H}^q(X, \pi_{p+q}(Y))$. Thus we have

Theorem 1.1 *Let X be finite and $f : X \to Y$ a based map. Then there is a spectral sequence with $E^2_{p,q} = \widetilde{H}^q(X, \pi_{p+q}(Y) \otimes \mathbb{Q})$ converging to $\pi_p(M_f(X, Y)_*) \otimes \mathbb{Q}$.* □

2. Null Components. We prove that the spectral sequence collapses on null components. The key lemma here is

Lemma 2.1 *Let X have dimension n and suppose $\alpha \in \pi_m(X)$ for $m > n$. Then $\Sigma(\alpha)$ is of finite order in $\pi_{m+1}(\Sigma(X))$.*

Proof. Let $f : S^m \to X$ represent α and let T_f denote the mapping cone of f. Since $\Sigma(T_f) \simeq T_{\Sigma f}$ it follows that $\Sigma(T_f)$ admits a minimal CW decomposition for which Σf is the attaching map for the top cell. But $\Sigma(T_f)$ is rationally equivalent to a wedge

of spheres and so the Quillen minimal model for $\Sigma(T_f)$ has trivial differential. By [3, Proposition 8.12] the differential in the Quillen model of $\Sigma(T_f)$ is determined by the attaching maps in a minimal CW decomposition. Thus Σf is rationally trivial. □

Theorem 2.2 *Let X be a finite complex and Y any space. Then*

$$\pi_q(M_0(X,Y)_*) \otimes \mathbb{Q} \cong \bigoplus_{k=2}^{\infty} \widetilde{H}^{k-q}(X, \pi_k(Y) \otimes \mathbb{Q}) \quad \text{and}$$

$$\pi_q(M_0(X,Y)) \otimes \mathbb{Q} \cong \bigoplus_{k=2}^{\infty} H^{k-q}(X, \pi_k(Y) \otimes \mathbb{Q}).$$

Proof. Since $\pi_*(M_0(X,Y)) \cong \pi_*(M_0(X,Y)_*) \oplus \pi_*(Y)$, the second line is a consequence of the first. For the first line, we note that when f is trivial, the map $j = (\overline{h_q})_* : A_{p,q-1} \to C_{p-1,q}$ is given by $j(\beta)(\sigma_i) = \beta \circ \Sigma^p(\alpha_i)$, for $\beta \in A_{p,q-1} \cong [\Sigma^p(X^{q-1}), Y]$, where here the σ_i denote the q-cells of X and the $\alpha_i \in \pi_{q-1}(X^{q-1})$ denote the corresponding homotopy elements. Thus the rational spectral sequence collapses by Lemma 2.1. □

The original Federer spectral sequence rarely collapses for nontrivial components. In fact we have

Theorem 2.3 *Given $f : X \to Y$ with either X finite or Y rationally finite-dimensional, if there exists $\alpha \in \pi_n(X) \otimes \mathbb{Q}$ and $\beta \in \pi_m(Y) \otimes \mathbb{Q}$ with $[f_*(\alpha), \beta]_w \neq 0$ in $\pi_{n+m-1}(Y) \otimes \mathbb{Q}$ then $M_f(X,Y) \not\simeq_{\mathbb{Q}} M_0(X,Y)$.*

Proof. By Theorem 2.2 (or [5, Lemma 3.2] when Y is finite-dimensional) it suffices to show that $\beta \notin \rho_*(\pi_m(M_f(X,Y)) \otimes \mathbb{Q})$ where $\rho : M_f(X,Y) \to Y$ is the evaluation fibration. Let Y_0 denote the rationalization of Y. Then $M_f(X,Y)$ is (weakly) rationally equivalent to $M_f(X,Y_0)$. If $\beta \in \rho_*(\pi_m(M_f(X,Y)) \otimes \mathbb{Q})$ then there exists a map $F : S^m \times X \to Y_0$ with $F(s,*) = g(s)$ and $F(*,x) = f(x)$ where $g : S^m \to Y_0$ represents β. Let $h : S^n \to X$ represent α and define $H : S^m \times S^n \to Y_0$ by $H(s,t) = F(s,h(t))$. Note that H extends $g \vee (f \circ h) : S^m \bigvee S^n \to Y_0$. Thus $[f_*(\alpha), \beta]_w = 0$ in $\pi_{n+m-1}(Y_0)$ contradicting our hypothesis. □

3. The First Differentials. Suppose X has a minimal rational CW decomposition with first nontrivial attaching maps in degree $m+1$. Let $W^m \xrightarrow{h_m} X^m \longrightarrow X^{m+1}$, be the corresponding cofibration sequence with $W^m = \bigvee_{i=1}^{s} S^m$ and $X^m = \bigvee_{j=1}^{t} S^{n_j}$ where each $n_j \leq m$. The first potentially nonzero differentials for $f : X \to Y$ are then the $d_{(m-n_j+1)} : C_{p,n_j} \to C_{p-1,m+1}$ for $j = 1, \ldots, t$. Let $\alpha_1, \ldots, \alpha_s \in \pi_m(X^m)$ be the homotopy classes represented by h_m and let $a_1, \ldots, a_s \in \pi_m(X^m) \otimes \mathbb{Q}$ be the corresponding rational homotopy classes. Let $x \in \pi_p(S^p) \otimes \mathbb{Q}$ and $y_j \in \pi_{n_j}(S^{n_j}) \otimes \mathbb{Q}$ for $j = 1, \ldots, t$ be nontrivial elements. By Hilton's Theorem, each a_i is decomposable into Whitehead products involving the y_j. Use the Jacobi identity in the Whitehead algebra $\pi_*(S^p \vee X^m) \otimes \mathbb{Q}$ to write

$$[x, a_i]_w = \sum_{j=1}^{t} [[x, y_j]_w, g_{ij}]_w, \tag{1}$$

for each i and some $g_{ij} \in \pi_{m-n_j+1}(X^m) \otimes \mathbb{Q}$. Recalling that $C_{p,n_j} \cong \tilde{C}^{n_j}(X, \pi_{p+n_j}(Y_0))$, given $b_j \in \pi_{p+n_j}(Y_0)$ write b_j^* for the cochain which takes the value b_j on the cell corresponding to S^{n_j} and vanishes on the other cells of X. With this notation we prove

Theorem 3.1 *The differentials*

$$d_{(m-n_j+1)} : \tilde{C}^{n_j}(X, \pi_{p+n_j}(Y_0)) \to \tilde{C}^{m+1}(X, \pi_{p+m}(Y_0))$$

are given, for $j = 1, \ldots, t$, *by*

$$d_{(m-n_j+1)}(b_j^*)(\sigma_i) = [b_j, f_*(g_{ij})]_w,$$

where $\sigma_1, \ldots, \sigma_s$ *are the* $m+1$*-cells of* X.

Proof. If $a \in \pi_n(X) \otimes \mathbb{Q}$ then $\overline{a}$ will denote a representative in the Quillen model (L_X, ∂_X) of X for the corresponding element of $\pi_{n-1}(\Omega X) \otimes \mathbb{Q}$. If V is a graded vector space then $F(V)$ will denote the free graded Lie algebra generated by V. The differential $d_{(m-n_j+1)} = (\overline{h_m})_* \circ ((\rho_m)_* \circ \cdots \circ (\rho_{n_j+1})_*)^{-1} \circ \partial_{n_j}$. The proof consists in unravelling this composition.

Write the Quillen model for $S^p \times X_m$ as

$$(L_{S^p \times X_m}, \partial_{S^p \times X_m}) = (F(\mathcal{Q}(\overline{x}) \oplus V_{(m)} \oplus s^p(V_{(m)})), \partial_{S^p \times X_m})$$

where $V_{(m)}$ is the graded vector space generated by the elements $\overline{y}_j$ and $s^p(V_{(m)})$ is the pth suspension of $V_{(m)}$. The differential $\partial_{S^p \times X_m}$ vanishes on $\overline{x}$ and $V_{(m)}$ and is given on $s^p(V_{(m)})$ as $\partial_{S^p \times X_m}(s^p(\overline{y}_j)) = [\overline{x}, \overline{y}_j]$. The element $((\rho_m)_* \circ \cdots \circ (\rho_{n_j+1})_*)^{-1} \circ \partial_{n_j}(b_j^*) \in A_{p,m}$ is represented by the map $F_m : S^p \times X^m \to Y_0$ given on Quillen models by $(F_m)_*(\overline{x}) = 0, (F_m)_*(\overline{y}_i) = (f_m)_*(\overline{y}_i)$ and $(F_m)_*(s^p(\overline{y}_i)) = \delta_{ij} \cdot \overline{b}_j$.

Note that $F_m \circ (1 \times h_m) : S^p \times W^m \to Y_0$ is null on the subspace $S^p \bigvee W^m$ and so induces $G : \Sigma^p(W^m) \to Y_0$. The element $d_{m-n_j+1}(b_j^*)(\sigma_i) \in \pi_{p+m}(Y_0)$ is represented by the restriction of G to the ith sphere of the wedge $\Sigma^p(W^m) = \bigvee_{k=1}^s S^{m+p}$. Write the Quillen model for $S^p \times W^m$ as

$$(L_{S^p \times W^m}, \partial_{S^p \times W^m}) = (F(\mathcal{Q}(\overline{x}) \oplus V_{m-1} \oplus s^p(V_{m-1})), \partial_{S^p \times W^m})$$

where here V_{m-1} is concentrated in degree $m-1$ and generated by elements $\overline{u}_k$ for $k = 1, \ldots, s$. The Quillen model for $\Sigma^p(W^m)$ is just $(F(s^p(V_{m-1})), 0)$, the graded Lie algebra with trivial boundary. Thus $(F_m)_* \circ (1 \times (h_m)_*)(s^p(\overline{u}_i)) \in (L_Y, \partial_Y)$ is a Quillen model representative for $d_{m-n_j+1}(b_j^*)(\sigma_i)$.

Now $1 \times (h_m)_* : (L_{S^p \times W^m}, \partial_{S^p \times W^m}) \to (L_{S^p \times X_m}, \partial_{S^p \times X_m})$ satisfies $(h_m)_*(\overline{u}_i) = \overline{a}_i$ and so

$$\begin{aligned} \partial_{S^p \times X^m}(1 \times (h_m)_*)(s^p(\overline{u}_i)) &= (1 \times (h_m)_*)(\partial_{S^p \times W^m}(s^p(\overline{u}_i)) \\ &= (1 \times (h_m)_*)([\overline{x}, \overline{u}_i]) \\ &= [\overline{x}, \overline{a}_i] \\ &= \textstyle\sum_{l=1}^t (-1)^{p+n_l}[[\overline{x}, \overline{y}_l], \overline{g}_{il}], \end{aligned}$$

where in the last line we have have used (1) translated to Quillen models with appropriate sign change. In the Quillen model for $S^p \times X^m$, $\partial_{S^p \times X^m}([s^p(\overline{y}_l), \overline{g}_{il}]) = [[\overline{x}, \overline{y}_l], \overline{g}_{il}]$ and, for degree reasons, $[s^p(\overline{y}_l), \overline{g}_{il}]$ is the unique element with this property. Thus the element $d_{m-n_j+1}(b_j^*)(\sigma_i) \in \pi_{p+m}(Y_0)$ is the homotopy class represented in the Quillen model of Y by

$$(F_m)_* \left(\sum_{l=1}^t (-1)^{p+n_l}[s^p(\overline{y}_l), \overline{g}_{il}] \right) = (-1)^{p+n_j}[\overline{b}_j, (f_m)_*(\overline{g}_{ij})].$$

The result follows by converting back to Whitehead products. □

4. Consequences and Calculations. We deduce some consequences of Theorem 3.1. The first is a version of Theorem 2.3 for based function spaces. Let $\mathrm{att}(X, \mathbb{Q})$ equal the dimension of the first nontrivially attached cell in a minimal rational CW decomposition for X. (We set $\mathrm{att}(X, \mathbb{Q}) = 0$ if all attaching maps of X are of finite order.) Also let $P_*(Y) = \ker\{\mathrm{ad}_w : \pi_*(X) \to (\mathrm{Der}_w)_*(\pi_*(X))\}$ the kernel of Whitehead adjoint representation. We prove

Theorem 4.1 *Let X be a finite complex with $att(X, \mathbb{Q}) > 0$ and Y a space with $P_n(Y_0) = 0$ for $n \leq att(X, \mathbb{Q})$. If $f : X \to Y$ induces an injection on rational homotopy through degree $att(X, \mathbb{Q})$, then $M_f(X, Y)_* \not\simeq_{\mathbb{Q}} M_0(X, Y)_*$.*

Proof. We show that the rational based Federer spectral sequence does not collapse for f. With notation as in §3, write $[x, a_i]_w = \sum_{j=1}^{t} [[x, y_j]_w, g_{ij}]_w$, for some $g_{ij} \in \pi_{m-n_j+1}(X_m) \otimes \mathbb{Q}$. Since, by hypothesis, $a_i \neq 0$ for some i we know $g_{ij} \neq 0$ for some j. Since f induces an injection on rational homotopy through degree $\mathrm{att}(X, \mathbb{Q})$, $f_*(g_{ij})$ is nontrivial in $\pi_*(Y) \otimes \mathbb{Q}$, as well. Since $P_n(Y_0)$ vanishes for $n \leq \mathrm{att}(X, \mathbb{Q})$ there exists $b_j \in \pi_k(Y) \otimes \mathbb{Q}$ such that $\mathrm{ad}_w(f_*(g_{ij}))(b_j) \neq 0$. Thus by Theorem 3.1, $d_{(m-n_j+1)}(b_j^*)(\sigma_i) = [b_j, f_*(g_{ij})]_w \neq 0$. □

Corollary 4.2 *Let X be a finite complex with $P_n(X_0) = 0$ for $n \leq att(X, \mathbb{Q})$. Then the path components of $M(X, X)_*$ are all of the same rational homotopy type if and only if X is a rational co-H-space.* □

We remark that the hypotheses of Theorem 4.1 and Corollay 4.2 are not superfluous. For example, one proves directly that the components of $M(\mathbb{C}P^n, \mathbb{C}P^m)_*$ are all rationally equivalent.

Theorem 3.1 offers a practical method for computing rational homotopy groups of nontrivial components. To illustrate, we calculate the rational homotopy groups of the space $M_f(X, Y)_*$ when X is a formal space of rational L.S. category two. By the main result of [2], X then occurs as the cofibre of a rationally nontrivial map $h : W_0 \to W_1$ where $W_0 = \bigvee_{i=1}^{s} S^{n_i}$ and

$W_1 = \bigvee_{j=1}^{t} S^{n_j}$. Write $P\widetilde{H}_*(X)$ for the primitive reduced rational homology of X and let $P^c\widetilde{H}_*(X)$ denote a vector space complement in $\widetilde{H}_*(X, \mathbb{Q})$. Then $P\widetilde{H}_*(X) \cong \widetilde{H}_*(W_1)$ and $P^c\widetilde{H}_*(X) \cong s(\widetilde{H}_*(W_0, \mathbb{Q}))$. Let $y_j \in \pi_*(S^{n_j}) \otimes \mathbb{Q}$ be nonzero elements and let $c_1, \ldots, c_t$ be the corresponding basis for $P\widetilde{H}_*(X)$. If $\Delta : X \to X \times X$ is the diagonal map, given $c \in P^c\widetilde{H}_*(X, \mathbb{Q})$ we can write $\Delta_*(c) = 1 \otimes c + c \otimes 1 + \sum_{j,k} q_{jk}(c_j \otimes c_k + (-1)^{n_j n_k} c_k \otimes c_j)$, where $q_{jk} \in \mathbb{Q}$ and $q_{jk} = (-1)^{n_j n_k} q_{kj}$. Given $f : X \to Y$ define

$$D(f)_p : \mathrm{Hom}(P\widetilde{H}_*(X), \pi_{p+*}(Y_0)) \to \mathrm{Hom}(P^c\widetilde{H}_*(X), \pi_{p+*-1}(Y_0))$$

by setting $D(f)_p(b_j^*)(c) = (-1)^{p+1} \sum_{k=1}^{t} 2q_{jk}[b_j, f_*(y_k)]_w$. Here, as before, $b_j^* \in \mathrm{Hom}(P\widetilde{H}_*(X, \mathbb{Q}), \pi_{p+*}(Y_0))$ denotes the map which takes the value $b_j \in \pi_{p+n_j}(Y_0)$ on c_j and vanishes on the other c_k. We prove

Theorem 4.3 *Let X be a finite formal complex of rational L.S. category two. Let $f : X \to Y$ be a given map. Then*

$$\pi_p(M_f(X, Y)_*) \otimes \mathbb{Q} \cong \mathit{coker}(D(f)_{p+1}) \oplus \mathit{ker}(D(f)_p).$$

Proof. We identify $\overline{h}_* : \pi_p(M_{f|W_1}(W_1, Y_0)_*) \to \pi_p(M_0(W_0, Y_0)_*)$ with $D(f)_p$. Since X is formal, [3, Proposition 8.8] implies the Quillen model of X is given explicitly as $L_X = F(s^{-1}\widetilde{H}_*(X, \mathbb{Q}))$ with $\partial_X(s^{-1}(c)) = -\sum_{j,k}(-1)^{n_j} q_{jk}[s^{-1}(c_j), s^{-1}(c_k)]$. Let $c \in P^c\widetilde{H}_*(X, \mathbb{Q})$. By linearity, we may suppose that c corresponds to a nonzero element in $\pi_{n_i}(S^{n_i}) \otimes \mathbb{Q}$. If $a_i \in \pi_{n_i}(W_1) \otimes \mathbb{Q}$ represents $h|S^{n_i}$ then by [3, Proposition 8.12] $a_i = \sum_{j,k} q_{jk}[y_j, y_k]_w$. Let $x \in \pi_p(S^p) \otimes \mathbb{Q}$ be nonzero and use the Jacobi identity to obtain

$$\begin{aligned}[x, a_i]_w &= \textstyle\sum_{j,k}[x, [y_j, y_k]_w]_w \\ &= (-1)^{p+1} \textstyle\sum_{j,k} q_{jk}\left([[x, y_j]_w y_k]_w + (-1)^{n_j n_k}[[x, y_k]_w y_j]_w\right).\end{aligned}$$

Thus, in the notation of §3, $g_{ij} = (-1)^{p+1} \sum_{k=1}^{t} 2q_{jk} y_k$ and so

$$(\overline{h})_*(b_j^*) = (-1)^{p+1} \sum_{k=1}^{t} 2q_{jk}[b_j, f_*(y_k)]_w = D(f)_p(b_j^*). \quad \square$$

Example 4.4. (a) Given $f : \mathbb{C}P^2 \to Y$, we determine the rational homotopy groups of $M_f(\mathbb{C}P^2, Y)_*$. Let $\iota_2 \in \pi_2(\mathbb{C}P^2)$ be nontrivial. By Theorem 4.3 we have

$$\pi_p(M_f(\mathbb{C}P^2, Y)_*) \otimes \mathbb{Q} \cong \operatorname{coker}\{\operatorname{ad}_w f_*(\iota_2) : \pi_{p+1}(Y_0) \to \pi_{p+2}(Y_0)\}$$
$$\oplus \ \ker\{\operatorname{ad}_w f_*(\iota_2) : \pi_p(Y_0) \to \pi_{p+1}(Y_0)\}.$$

(b) Next let $f : S^m \times S^n \to Y$ and let $\iota_m \in \pi_m(S^m)$ and $\iota_n \in \pi_n(S^n)$ be nontrivial. Here Theorem 4.3 implies

$$\pi_p(M_f(S^m \times S^n, Y)_*) \otimes \mathbb{Q} \cong$$

$$\operatorname{coker}\{\operatorname{ad}_w f_*(\iota_n) + \operatorname{ad}_w f_*(\iota_m) : \pi_{p+m+1}(Y_0) \oplus \pi_{p+n+1}(Y_0) \to \pi_{p+m+n}(Y_0)\}$$

$$\oplus \ \ker\{\operatorname{ad}_w f_*(\iota_n) + \operatorname{ad}_w f_*(\iota_m) : \pi_{p+m}(Y_0) \oplus \pi_{p+n}(Y_0) \to \pi_{p+m+n-1}(Y_0)\}.$$

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