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Maximum Principles for Minimal Surfaces in $\mathbb{R}^3$ Having Noncompact Boundary and a Uniqueness Theorem for the Helicoid

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0. Introduction. Let M, N be complete minimal surfaces with boundary immersed in $\mathbb{R}^3$ such that the images of M and N are disjoint. Then, the "maximum principle at infinity" states that

$$d(M,N) \geq \min\{d(M,\partial N),(\partial M,N)\} \tag{0.1}$$

where $d(A,B) = \inf\{|x-y| \mid x \in A,\ y \in B\}$, $A,\ B \subset \mathbb{R}^3$. Till recently, this maximum principle in its so far most general form was proved by Meeks-Rosenberg [6] for properly immersed disjoint minimal surfaces with compact boundary in a complete flat 3-dimensional manifold.

After the first draft of this paper had been written the authors have learned of the doctoral thesis of Marc Soret [11] which contains the above maximum principle for properly embedded minimal surfaces with possibly noncompact boundaries provided that M and N have bounded curvature and that one of the two surfaces is parabolic.

In this paper we prove two versions of the maximum principle for immersed minimal surfaces, also assuming bounded curvature. In Theorem 1.1 we require the existence of an exhaustion of M by compact subdomains M_R $(R \geq R_0)$ such that length$(\partial M_R \setminus \partial M) \leq L(R)$, where $\int^{+\infty} L(R)^{-1}dR = +\infty$. In Theorem 1.2 we assume that M (or N) is "bandlike", i.e. the intrinsic distance of points of M to the boundary ∂M is bounded from above. Our methods are rather different from those of Soret in that we do not use any potential theory. Theorem 1.2 will be derived from the Harnack Inequality for elliptic differential equations [3] and Theorem 1.1 is based on a growth estimate for solutions of such equations, extending a result of Collin-Krust [1]. This growth estimate (Proposition 1.1) might be of independent interest. We shall apply Theorem 1.2 to prove the uniqueness of a stable part of the helicoid spanning a double helix among properly immersed minimal surfaces which have the same boundary, bounded curvature and which satisfy the asymptotic condition

$$\lim_{|p|\to\infty} \frac{x^2+y^2}{z^2} = 0, \quad p=(x,y,z) \in M. \tag{0.2}$$

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This condition excludes the obvious counter-example to uniqueness, namely, the complement of the stable part in the whole helicoid. From the curvature estimates of Schoen [10] we obtain the uniqueness of the stable helicoid among stable surfaces with the same boundary and satisfying (0.2).

Let us remark that (0.1) is a consequence of the statement

$$d(M,\partial N) > 0 \text{ and } d(\partial M, N) > 0 \Longrightarrow d(M,N) > 0, \tag{0.3}$$

where M and N are properly immersed minimal surfaces. Hence, in the sequel we shall only prove the (seemingly) weaker form (0.3) of the maximum principle.

1. General results. If two minimal surfaces come very close then one can be represented as a section in the normal bundle of the other. When identified with a real valued function, such a section satisfies a differential equation of the type (1.1) below (see Lemma 3 in the appendix). Therefore, in Proposition 1.1 we shall prove, under apropriate conditions, growth estimates from below for nontrivial solutions of such differential equations on noncompact surfaces. In particular, these estimates rule out the possibility of bounded non constant solutions and will thereby imply the desired maximum principle of Theorem 1.1 below. In Theorem 1.2 we prove a second maximum principle which uses different assumptions and is tailored for the applications to helicoidal surfaces in section 3. Its proof is based on the Harnack principle for general second order partial differential equations [3] and does not use Proposition 1.1.

For Proposition 1.1 below, the following assumptions will be made: M is a complete noncompact Riemannian surface of class C^3 possibly with boundary ∂M. We consider a nonnegative function h on M, proper, of class $C^{0,1}$, satisfying

$$|\nabla h| \leq H$$

almost everywhere, where H is a positive constant.

We assume that, for almost all R, $h^{-1}(R)$ consists of finitely many rectifiable curves and that the length of $h^{-1}(R) \setminus \partial M$ is bounded from above by a continuous positive function $L(R)$. $F(R)$ will be a primitive of $1/L(R)$, that is, $F'(R) = 1/L(R)$, and setting $M_R = h^{-1}([0,R])$ and $C_R = h^{-1}(R) \setminus \partial M$, we assume that, for almost all R, the homological boundary of M_R is $(\partial M \cap M_R) \cup C_R$, endowed with a suitable orientation

On M we consider a function $u \in C^2(M)$ which satisfies the differential equation

$$\operatorname{div} A(x,u,\nabla u) = b(x,u,\nabla u) \tag{1.1}$$

where A denotes a C^2 vector field such that

$$A(x,u,\nabla u) \neq 0 \text{ if } \nabla u \neq 0 \tag{1.2}$$

and b a C^1 function on $\mathbb{R}\times TM$.

Using the method of Collin-Krust ([1]) we may then prove the following Phragmen-Lindelöf type result:

Proposition 1.1 *Let $u \in C^2(M)$ be a solution of (1) which is not constant on at least one connected component of M, and suppose that there are R_0, $\sigma \in I\!R$ and $\lambda > 0$ such that*

$$(1.3) \quad 0 \leq \int_{M_R} (< A, \nabla u > + (u-\sigma)b - \lambda |A|^2) dw - \int_{\partial M \cap M_R} (u-\sigma) < A, \nu > ds$$

for all $R \geq R_0$, where $A := A(x, u, \nabla u)$ and ν is the outward unit normal vector to ∂M. Then, setting

$$\mu(R) = sup_{M_R} |u - \sigma|$$

we have

$$(1.4) \qquad \lim_{R \to \infty} \inf \frac{\mu(R)}{\sqrt{F(R)}} > 0.$$

Proof. By the divergence theorem and (1.1) we have

$$\int_{M_R} < A, \nabla u > dw = \int_{M_R} [\mathrm{div}((u-\sigma)A) - (u-\sigma)\mathrm{div}(A)] dw =$$

$$= \int_{M_R} [\mathrm{div}((u-\sigma)A) - (u-\sigma)b]\, dw = \int_{C_R} (u-\sigma) < A, \nu > d\mathcal{H}^1 - \int_{M_R} (u-\sigma) b dw$$

$$= \int_{C_R} (u-\sigma) < A, \nu > d\mathcal{H}^1 + \int_{\partial M \cap M_R} (u-\sigma) < A, \nu > d\mathcal{H}^1 - \int_{M_R} (u-\sigma) b dw$$

so that

$$\int_{M_R} (< A, \nabla u > + (u-\sigma)b) dw - \int_{\partial M \cap M_R} (u-\sigma) < A, \nu > d\mathcal{H}^1$$

$$= \int_{C_R} (u-\sigma) < A, \nu > d\mathcal{H}^1 \leq \mu(R) \int_{C_R} |A| d\mathcal{H}^1 = \mu(R)\eta(R)$$

where

$$(1.5) \qquad \eta(R) = \int_{C_R} |A| d\mathcal{H}^1.$$

Using (1.3) we obtain therefore

$$(1.6) \qquad a_0 + \lambda \int_{M_R \setminus M_{R_0}} |A|^2 dw \leq \mu(R)\eta(R), \quad R \geq R_0$$

with $a_0 := \lambda \int_{M_{R_0}} |A|^2 dw$ and R_0 is chosen big enough such that a_0 is positive, what is possible due to (1.2). By means of the Schwartz inequality and the co-area formula ([12]) we may then conclude from (1.6) that

$$(1.7) \qquad a_0 + C \int_{R_0}^{R} \frac{\eta^2(r)}{L(r)} \leq \mu(R)\eta(R), \quad R \geq R_0$$

with an apropriate constant C. Observing that $\mu(r)$ is a non decreasing function of R, the proof of the proposition now follows in a similar way as the proof of Theorem II of [1]. For the sake of completness, we write below the main steps.

Given $R_1 > R_0$, and setting $B = \sup_{0<r\leq R_1} \mu(r)$, we have by (1.7)

$$a_0 + C\int_{R_0}^{R} \frac{\eta(r)^2}{L(r)}dr \leq B\eta(R) \tag{1.8}$$

for all $R \in [R_0, R_1]$. We see by a direct computation that the differential equation

$$Bz'(r) = C\frac{z(r)^2}{L(r)} \tag{1.9}$$

has a solution

$$\psi(r) = \frac{1}{\frac{2B}{a_0} - \frac{C}{B}F_0(r)}$$

defined in the interval

$$\left[R_0, F_0^{-1}\left(\frac{2B^2}{a_0C}\right)\right)$$

where $F_0(R) := \int_{R_0}^{R}(1/L(r))dr$. Since $\eta(R_0) > a_0/2B = \psi(R_0)$ we can use (1.8) and (1.9) to prove that $\eta(R) > \psi(R)$ for all $R \in [R_0, R_1]$. Since $\psi(R) \to \infty$ as $R \to F_0^{-1}(\frac{2B^2}{a_0C})$ we must have

$$R_1 < F_0^{-1}\left(\frac{2B^2}{a_0C}\right)$$

and therefore

$$B = \mu(r) > \sqrt{\frac{a_0F_0(R_1)}{2C}}$$

proving (1.4) and the proposition.

Theorem 1.1 *Assumptions:*

(i) M and N are two disjoint, complete, noncompact, properly immersed minimal surfaces with or without boundary in $\mathbb{R}^3$, both with bounded curvature

(ii) $\delta := \inf\{d(N, \partial M), d(N, \partial M)\}$ is positive

(iii) there exist $R > 0$, and $\epsilon > 0$ such that, setting

$$M_R = \{x \in M \mid |x| > R\}$$

and

$$N_{R,\epsilon} = \{(x, \nu) \mid x \in M_R \setminus \partial M_R,\ \nu \in T_x(M)^{\perp},\ |\nu| \leq \epsilon\}$$

the map

$$(x, \nu) \mapsto x + \nu$$

is a diffeomorphism of $N_{R,\epsilon}$ onto its image U_ϵ

(iv) There is a proper Lipschitz function $h : N \to [0, +\infty)$ and a continuous function $L : [0, \infty) \to (0, \infty)$ such that $\int_{h^{-1}(R)} d\mathcal{H}^1 \leq L(R)$ for almost all R and $\int_0^\infty (1/L(r))dr = \infty$.

With the above assumptions, it follows that $d(M, N) > 0$.

Remark. Obviously it is sufficient to prove the theorem for $\tilde{M} := \{x \in M \mid d(x, \partial M) > \delta/2\}$ and $\tilde{N} := \{x \in N \mid d(x, \partial N) > \delta/2\}$ instead of M and N; the assumption of bounded curvature would be satisfied for $\tilde{M}$ and $\tilde{N}$ if, for example, M and N are stable [10].

Proof of Theorem 1.1. We may assume that N is oriented because we can if necessary replace N by its double cover. Since $M \setminus M_R$ is compact and M, N are disjoint the number $\delta_1 := d(M \setminus M_R, N)$ is positive. Setting

$$N^\sigma = \{x \in N \mid d(x, M) \leq \sigma\}$$

it follows that $N^\sigma \subset U_\epsilon$ provided that $\sigma < \min\{\epsilon, \delta, \delta_1\}$. Since $d(N^\sigma, \partial N) \geq \delta - \sigma$ and the curvature of N is bounded we may find positive numbers r and ρ, $\rho < \delta - \sigma$, such that for any $x_0 \in N^\sigma$ the map

$$D_r(x_0) \times (-r, r) \ni (x, t) \longmapsto x + t\nu(x) \in \mathbb{R}^3 \tag{1.10}$$

is a differentiable embedding. Here we denote by $D_r(x_0)$ the geodesic disc of radius r around x_0 and by ν a smooth unit normal field of N. We may further choose r so small that the image of the map (1.10) which we denote by $V_r(x_0)$ is contained in U_ϵ. Given any $\eta > 0$ we may on account of the boundedness of the curvature of M and N and also using Lemma 2 (appendix) make r and σ so small that the angles which the tangent planes of $M \cap V_r(x_0)$ as well those of $D_r(x_0) \subset N$ form with the tangent plane of N at x_0 is bounded by η. We may therefore conclude that $M \cap V_r(x_0)$ is the image of exactly one smooth section of the normal bundle of $D_r(x_0)$. Hence there exists a smooth function u defined on a neighbourhood of N^σ in N such that

$$x + u(x)\nu(x) \in M \tag{1.11}$$

and, moreover,

$$m := \sup_{N^\sigma} |\nabla u| < +\infty. \tag{1.12}$$

From the definition of N^σ it follows trivially that $|u| \geq \sigma$ on ∂N^σ and, by replacing N^σ by one of its components we may assume that $u > 0$ on N^σ and $u \geq \sigma$ on ∂N^σ. We shall now apply Proposition 1.1 to $u|_{N_\sigma}$, where

$$N_\sigma := \{x \in N^\sigma \mid u(x) \leq \sigma\}.$$

Clearly, by (i), u satisfies the minimal surface equation

$$\mathrm{div} A(x, u, \nabla u) = b(x, u, \nabla u)$$

on N where $A = \nabla_p f$ and $b = f_u$ are given in Lemma 3 (appendix). From this lemma and also using (1.12) we obtain

$$< A(x,u,\nabla u), \nabla u > +(u-\sigma)b(x,u,\nabla u) \geq \frac{\left(1+\sqrt{|K|}u\right)^2}{\left(1+(1+\sqrt{|K|}u)^2\right)^{1/2}}|\nabla u|^2$$

$$+\frac{K(1+Ku^2)u(u-\sigma)}{2f(x,u,\nabla u)}.$$

Since we may assume that $|Ku^2| \leq 1/17$, the last term in the above inequality is nonnegative and, using also the estimate we arrive at

$$< A(,x,u,\nabla u), \nabla u > +(u-\sigma)b(x,u,\nabla u) \geq \lambda |A(x,u,\nabla u)|^2$$

for some positive constant λ. It follows now from Proposition 1.1 that u must be constant or unbounded. This proves the theorem.

Proposition 1.3 *Let M_1 be a complete and M_2 a properly immersed minimal surface possibly with boundary in $I\!R^3$ and let us assume that the principal curvatures of M_1 and M_2 are bounded in modulus by some constant k. We assume furthermore that $M_1 \cap M_2 = \emptyset$ and that*

$$\min\{d(M_1, \partial M_2), d(\partial M_1, M_2)\} \geq \delta$$

for some $\delta > 0$. Then there exist positive numbers ϵ, R and C, only depending on δ and k such that whenever $P_1 \in M_1$ with $d(P_1, M_2) < \epsilon$ then it follows that

$$\sup_{P \in D_R(P_1)} d(P, M_2) \leq C d(P_1, M_2),$$

where $D_R(P_1)$ denotes the geodesic disc in M_1 with center P_1.

Proof. We may assume that $\delta = k = 1$. If $P_1 \in M_1$ with $d(P_1, M_2) < \epsilon < 1$ then the ball $B_{1-\epsilon}(P_1)$ in R^3 does not meet ∂M_1 and hence, by Lemma 1, $M_1 \cap B_{1-\epsilon}(P_1)$ contains a graph G_{P_1} over a disc $D_\rho(P_1)$ in the tangent plane T_{P_1} of M_1 at P_1, where $\rho = (1/\sqrt{2})(1-\epsilon)$. Let now $P_2 \in M_2$ with $d(P_1, P_2) = d(P_1, M_2)$; P_2 exists since M_2 is properly immersed. Interchanging the rôles of M_1 and M_2 we obtain a graph $G_{P_2} \subset M_2$ with the corresponding properties. It follows that the line through P_1 and P_2 meets G_{P_1} orthogonally. We may then apply Lemma 2 to the two graphs G_{P_1} and G_{P_2}. Requiring that

$$\epsilon/\rho(\epsilon) < \frac{1}{320}$$

it follows from this lemma that the angle α_0 between the tangent planes of G_{P_1} and G_{P_2} at P_1 and P_2, respectively, can be estimated by $\mathrm{tg}\alpha_0 \leq 1/2$, or $\sin\alpha_0 \leq 1/\sqrt{5}$ and, consequently, also G_{P_2} contains a graph over a disc with center P_1 and radius $\frac{1}{\sqrt{2}}(1-\frac{1}{\sqrt{5}})$ in T_{P_1}. Moreover, if we restrict this graph to a smaller disc $D_{\tilde{\rho}}(P_1) \subset T_{P_1}$

with $\tilde{\rho} = 1/\sqrt{10}$, then it follows from Lemma 1, that the angle α of the tangent plane of G_{P_2} over $D_{\tilde{\rho}}(P_1)$ with T_{P_1} can be estimated by

$$\sin\alpha \leq \frac{2}{\sqrt{5}} < 1.$$

In other words, we have found pieces of M_1 and M_2 which may be represented as minimal graphs of corresponding functions u_1 and u_2 which are defined on a disc $D_{\tilde{\rho}}(P_1)$ where they moreover satisfy the inequality

$$|\nabla u_i| \leq \mathrm{tg}\alpha = \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}} \leq 1. \tag{1.13}$$

Clearly, u_1 and u_2 are solutions of the minimal surface equation

$$\sum_{i=1,2} \frac{\partial}{\partial x_i}\left(\frac{\partial u/\partial x_i}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

and, since $M_1 \cap M_2 = \emptyset$ we may assume that $u_1 < u_2$. The function $v := u_2 - u_1$ then satisfies the differential equation

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right) = 0 \tag{1.14}$$

with

$$a_{ij} = \int_0^1 \frac{\partial}{\partial p_j}\frac{p_i}{\sqrt{1+|p|^2}}\Big|_{p=t\nabla u_2+(1-t)\nabla u_1}\, dt.$$

By (1.13), (1.14) is uniformly elliptic in $D_{\tilde{\rho}}(P_1)$ and from the Harnack Inequality [GT, Thm 8.20] we therefore obtain the existence of constants $C = C(\tilde{\rho})$ and $R = \tilde{\rho}/4$ such that

$$\sup_{D_R(P_1)} v \leq Cv(P_1)$$

which clearly implies

$$\begin{aligned}\sup_{P\in D_R(P_1)} d(P, M_2) &\leq Cd(P_1, P_2)\\ &= Cd(P_1, M_2),\end{aligned}$$

where now $D_R(P_1)$ is the geodesic disc in M_1 with center P_1.

Theorem 1.2 *Let M_1 be a complete and M_2 a properly immersed minimal surface in $\mathbb{R}^3$ such that the principal curvatures of M_1 and M_2 are bounded in modulus by some constant k. We assume furthermore that $M_1 \cap M_2 = \emptyset$, $\min\{d(M_1, \partial M_2), d(M_2, \partial M_1)\} \geq \delta > 0$, and that M_1 is "bandlike", i.e., the radius of geodesic discs contained in the interior of M_1 is bounded by some fixed constant L. Then it follows that $d(M_1, M_2) > 0$.*

Proof. Assume that $d(M_1, M_2) = 0$ and let ϵ, R, C be the constants from Proposition 1.3. We may assume that $\epsilon < \delta$ and $C \geq 1$. Let m be a positive integer such that $mR > L$ and let $\sigma > 0$ such that $\sigma C^m < \epsilon$. By assumption there exists a point $P_1 \in M_1$ such that $d(P_1, M_2) < \sigma < \epsilon$. It follows from Proposition 1.3 that

$$\sup_{P \in D_R(P_1)} d(P, M_2) \leq C\sigma < \epsilon.$$

Therefore we may apply this proposition to any point $P_1' \in \partial D_R(P_1)$ instead of P_1 and we get

$$\sup_{P \in D_R(P_1')} d(P, M_2) \leq C^2\sigma < \epsilon.$$

This implies

$$\sup_{P \in D_{2R}(P_1)} d(P, M_2) < C^2\sigma.$$

Inductively we thus get

$$\sup_{P \in D_{mR}(P_1)} d(P, M_2) < C^m\sigma < \epsilon < \delta$$

Therefore, $D_{mR}(P_1)$ is contained in $M_1/\partial M_1$, a contradiction since $mR > L$. Thus $d(M_1, M_2) > 0$ and the theorem is proved.

2. Application

We apply Theorem 1.2, using a one-parameter family of helicoidal minimal surfaces described below "as barriers", to characterize the helicoid as the unique solution, under certain conditions, of Plateau's Problem for a pair of helices.

Helicoidal minimal surfaces in $\mathbb{R}^3$

A helicoidal surface in $\mathbb{R}^3$ is a surface invariant by the screw motion

$$\phi_t(x, y, z) = (x\cos t - y\sin t, x\sin t + y\cos t, z + t) \quad (t \in \mathbb{R}) \tag{2.1}$$

of $\mathbb{R}^3$ around the z-axes. It is known that a helicoidal minimal surface is the conjugated surface of a helicoid with a certain pitch. More precisely, given $\theta \in (-\pi/2, \pi/2)$, and denoting by H_θ the helicoid with pitch $1/\cos\theta$, the conjugated minimal surface M_θ whose second fundamental form is the one of H_θ rotated by θ is invariant by ϕ_t. Using the Weierstrass Representation of a minimal surface we may therefore obtain an explicit conformal parametrization of M_θ, namely:

$$\begin{aligned} x_\theta(u, v) &= \cos(v\cos\theta)\sinh(u\cos\theta) - \mathrm{tg}\theta\sin(v\cos\theta)\cosh(u\cos\theta)) \\ y_\theta(u, v) &= \sin(v\cos\theta)\sinh(u\cos\theta) + \mathrm{tg}\theta\cos(v\cos\theta)\cosh(u\cos\theta)) \\ z_\theta(u, v) &= -u\sin\theta - v\cos\theta \end{aligned} \tag{2.2}$$

Since any orbit of the screw motion ϕ_t intersects one and only one time the (x, y)-plane, any helicoidal surface is generated by a curve in this plane. Suppose

that $\gamma_\theta = (x_\theta, y_\theta)$ is the generating curve of the surface M_θ. By taking $z_\theta = 0$ in (2.2) we obtain $v\cos\theta = -u\sin\theta$ providing the following parametrization of γ_θ:

$$(2.3)\qquad \begin{aligned} x_\theta(u) &= \cos(u\sin\theta)\sinh(u\cos\theta) + \mathrm{tg}\theta\sin(u\sin\theta)\cosh(u\cos\theta) \\ y_\theta(u) &= -\sin(u\sin\theta)\sinh(u\cos\theta) + \mathrm{tg}\theta\cos(u\sin\theta)\cosh(u\cos\theta) \end{aligned}$$

Let h_i, $i = 1, 2$ be the helices

$$(2.4)\qquad h_i(t) = (a_i\cos t, -a_i\sin t, t)$$

around the z-axes, where a_i, $i = 1, 2$, are constants satisfying

$$(2.5)\qquad \max\{|a_1|,\ |a_2|\} \le \epsilon_0$$

where $\epsilon_0 = \sinh(u_0) = 1.50888...$ and u_0 is the positive root of the equation

$$\cosh(u) - u\sinh(u) = 0.$$

Remark. The number ϵ_0 limits the range of stability of certain pieces of the helicoid. Precisely, it is proved in ([8], p.101) that given any line segment S through the origin in the (x, y)-plane with endpoints p_1 and p_2, if $\max\{|p_1|, |p_2|\} \le \epsilon_0$ then the helicoidal domain generated by S under the standard screw motion (2.1) is stable.

Let us remark that the number ϵ_0 also comes up in a different context: it is shown in [7] (see also [2]) that the cone $K_{\epsilon_0} := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < \epsilon_0^2 z^2,\ \pm x_3 > 0\}$ is the largest circular cone of $\mathbb{R}^3$ with vertex at the origin satisfying the following "nonexistence property": there is no connected, compact minimal C^2 surface M in $\mathbb{R}^3$ with $\partial M \subset K_{\epsilon_0}$ such that ∂M has nonempty intersection with both connected components of K_{ϵ_0}.

We shall give now the following description of the surfaces M_θ.

Proposition 2.1 *The family $\Gamma = \{M_\theta\}$ of helicoidal minimal surfaces, $\theta \in (-\pi/2, \pi/2)$ is a continuous one-parameter family such that:*

(a) any M_θ is complete and properly immersed. The surface M_0 is the usual helicoid with pitch 1, and M_θ is a helicoid if and only if $\theta = 0$

(b) the generating curve $\gamma_{-\theta}$ of $M_{-\theta}$ is the reflection on the x-axis of the generating curve γ_θ of M_θ, $\theta \in (-\pi/2, \pi/2)$; each γ_θ is symmetric with respect to the y-axis

(c) given $\theta \in (0, \pi/2)$, M_θ and $M_{-\theta}$ do not intersect the interior of the cylinder around the z-axis with radius $\mathrm{tg}\theta$

(d) the family $\{M_\theta\}_{\theta\in(0,\pi/2)}$ covers one connected component of $\mathbb{R}^3 \setminus M_0$ and $\{M_\theta\}_{\theta\in(-\pi/2,0)}$ covers the other one

(e) if $\theta \ne 0$, then $d(M_\theta, h_1 \cup h_2) > 0$

(f) for $\theta \neq 0$, if U_θ denotes the connected component of $\mathbb{R}^3 \setminus (M_\theta \cup M_{-\theta})$ which contains the z-axis, then $D_\theta := \partial \overline{U}_\theta \cap M_\theta$ is a bandlike subdomain of M_θ with bounded curvature. Furthermore, ∂D_θ consists of two helices contained in a cylinder around the z-axis with radius strictly bigger than ϵ_0.

Proof. It is known that, as conjugated surfaces of the helicoid, the surfaces M_θ are complete and properly immersed. By construction, M_0 is the only helicoid in the family $\{M_\theta\}$, and this proves (a). It is immediate to see from (2.3) that $\gamma_{-\theta} = (x_{-\theta}, y_{-\theta}) = (x_\theta, -y_\theta)$, $x_\theta(-u) = -x_\theta(u)$, and $y_\theta(-u) = y_\theta(u)$, and this proves (b).

We now prove (c). A computation shows that $r_\theta := \sqrt{x_\theta^2 + y_\theta^2}$ has a global minimum at $u = 0$ whose value is $r_\theta(0) = |\mathrm{tg}\theta|$ so that γ_θ does not intersect the interior of the circle centered at the origin with radius $\mathrm{tg}\theta$. Since such a circle generates, under the screw motion, a cylinder C around the z-axes with the same radius, it follows that M_θ does not intersect the interior of C. From (b) we also see that $M_{-\theta}$ does not intersect the interior of C and therefore (c) is proved.

To prove (d), given $\theta \in (0, \pi/2)$ let c_θ be the connected piece of γ_θ containing $\gamma_\theta(0)$ and contained in the half plane $y > 0$. We will prove that the family $\{c_\theta\}_{\theta \in (0,\pi/2)}$ covers the half plane $y > 0$ and $\{c_\theta\}_{\theta \in (-\pi/2,0)}$ covers $y < 0$, and this of course implies (d). We first prove that c_θ is embedded.

Let $\beta_\theta(u)$ be the angle between the position vector $c_\theta(u)$ and the vector $(0,1)$. Since $r_\theta(u)$ has just one critical point which is a point of minimum, c_θ will be embedded if $\beta_\theta(u)$ is a non decreasing function of u. Assume that u_0 is a local maximum of β_θ. Then γ_θ and the straight line passing through $\gamma_\theta(u_0)$ and the origin are tangent at $\gamma_\theta(u_0)$ and γ_θ lies locally in one side of this straight line. Hence, M_θ and the helicoid generated by this straight line are tangent, with M_θ lying locally in one side of this helicoid. Therefore, by the maximum principle, they have to coincide, that is, M_θ is a helicoid. But then, $\theta = 0$, a contradiction. Therefore, c_θ is embedded.

Now, let p be any point in the half plane $y > 0$. One sees from (2.3) that $y_\theta(0) > 0$ if $\theta > 0$ and since, as one also easily sees from (2.3), y_θ changes sign, γ_θ must intersect the x-axis. Since by (b), γ_θ is symmetric with respect to the y-axis and by (c) does not intersect the interior of disk in the (x,y)-plane centered at the origin with radius $|\mathrm{tg}\theta|$, it follows that the closure $\bar{c}_\theta$ of c_θ has exactly two distinct intersection points with the x-axis if $\theta \neq 0$. Hence, $\bar{c}_\theta$ encloses a compact domain Ω_θ, symmetric with respect to the y-axis, in the half plane $y \geq 0$. Since $d(\partial\Omega_\theta \setminus \{y = 0\}, 0) = r_\theta(0) = \mathrm{tg}\theta$, there is θ close to $\pi/2$ such that $p \in \Omega_\theta$.

In order to prove that there is θ such that $p \in c_\theta$, it is sufficient to prove that c_θ tends uniformly to the x-axis as $\theta \to 0^+$ since $c_\theta = \partial\Omega_\theta \setminus \{y = 0\}$. Choose $\theta \in (0, \pi/2)$ and let $[-a_\theta, a_\theta] \subset \mathbb{R}$ be such that $\gamma_\theta([-a_\theta, a_\theta]) = \bar{c}_\theta$. We have $y_\theta(a_\theta) = y_\theta(-a_\theta) = 0$, $y_\theta(0) = \mathrm{tg}\theta$ and $y_\theta(u) > 0$ for $u \in (-a_\theta, a_\theta)$. Therefore, it is sufficient to prove that y_θ, restricted to $[-a_\theta, a_\theta]$, assumes its maximum at 0. A direct computation using (2.3) shows that the critical points of y_θ are $(n\pi)/\sin\theta$, where n is an arbitrary integer. Since for $n = \pm 1$ we have $y_\theta(\pm\pi/\sin\theta) < 0$, we see

that $[-a_\theta, a_\theta] \subset (-\pi/\sin\theta, \pi/\sin\theta)$ which shows that $u = 0$ is the only critical point of y_θ on $[-a_\theta, a_\theta]$, which, clearly, is a point of maximum. Therefore it is proved that c_θ tends uniformly to the x-axis and therefore there is θ such that $p \in c_\theta$. This proves that the family $\{c_\theta\}_{\theta\in(0,\pi/2)}$ covers the half plane $y > 0$ and this clearly implies that $\{M_\theta\}_{\theta\in(0,\pi/2)}$ covers one connected component of $\mathbb{R}^3 \setminus M_0$. It follows by (b) that $\{M_\theta\}_{\theta\in(-\pi/2,0)}$ covers the other component of $\mathbb{R}^3 \setminus M_0$, proving (d).

To prove (e) we first observe that, by (2.1) and (2.4), the helices h_i are the orbits, under ϕ_t, of the points $h_i(0)$ in the (x, y)-plane. Therefore, in order to prove that $d(M_\theta, h_1 \cup h_2) > 0$ it is enough to prove that $d(\gamma_\theta, \{h_1(0), h_2(0)\}) > 0$. Observe that, according to (2.5), $h_1(0)$ and $h_2(0)$ are points in the x-axis whose distance to the origin is less than or equal to ϵ_0.

We have shown in (c) that $r_\theta(u) := |\gamma_\theta(u)|$ has a global minimum at $u = 0$ whose value is $\mathrm{tg}\theta$. Therefore, given $\theta \neq 0$, by defining $u(\theta)$ as the smallest positive value of u such that $y_\theta(u(\theta)) = 0$, it is enough to prove that $x_\theta(u(\theta)) > \epsilon_0$ since, by (b), we also have $y_\theta(-u(\theta)) = 0$ and $x_\theta(u(\theta)) = -x_\theta(-u(\theta))$. Also by (b), we just need to consider the case $\theta > 0$. Since

$$x_\theta(u(\theta)) = r_\theta(u(\theta)) > r_\theta(0) = \mathrm{tg}\theta,$$

we have just to estimate $x_\theta(u(\theta))$ for $\theta \in (0, \theta_0]$ where $\mathrm{tg}\theta_0 = \epsilon_0$. Since $x_\theta(u(\theta)) > 0$ by (2.3), defining $f : (0, \theta_0] \to \mathbb{R}^+$ by $f(\theta) = x_\theta(u(\theta))$ we first prove that f does not have an extremal value in $(0, \theta_0)$. By contradiction, let us suppose that f assumes an extremal value at a point $\theta \in (0, \theta_0)$. From (2.3), we obtain

$$\mathrm{tg}\theta \cos(u(\theta)\sin\theta)\cosh(u(\theta)\cos\theta) - \sin(u(\theta)\sin\theta)\sinh(u(\theta)\cos\theta) = 0 \tag{2.6}$$

and

$$f(\theta) = \frac{\sinh(u(\theta)\cos\theta)}{\cos(u(\theta)\sin\theta)}. \tag{2.7}$$

A calculation shows that if $\theta \neq 0$ then $\partial y_\theta/\partial u \neq 0$ at $u(\theta)$ so that by the implicit function theorem $u = u(\theta)$ is differentiable. Therefore, by (2.7), a critical point of f must satisfy the equation

$$\cos(u(\theta)\sin\theta)\cosh(u(\theta)\cos\theta)\left(\frac{du}{d\theta}\cos\theta - u(\theta)\sin\theta\right) +$$

$$\sin(u(\theta)\sin\theta)\sinh(u(\theta)\cos\theta)\left(\frac{du}{d\theta}\sin\theta + u(\theta)\cos\theta\right) = 0$$

which together with (2.6) implies that $du/d\theta = 0$. Differentiating implicitly both sides of (2.6) with respect to θ, using $du/d\theta = 0$, we get, after simplifications,

$$\cos\theta\cos(u(\theta)\sin\theta)\cosh(u(\theta)\cos\theta) + \sin\theta\sin(u(\theta)\sin\theta)\sinh(u(\theta)\cos\theta)$$

$$-u(\theta)\cos(u(\theta)\sin\theta)\sinh(u(\theta)\cos\theta) = 0$$

Multiplying both sides by $\sin\theta$ and using (2.6), we get

$$\sin(u(\theta)\sin\theta) - u(\theta)\sin\theta\cos(u(\theta)\sin\theta) = 0$$

or equivalently,

$$\text{tg}(u(\theta)\sin\theta) = u(\theta)\sin\theta \tag{2.8}$$

and this implies

$$u(\theta)\sin\theta = \delta \tag{2.9}$$

where δ is a nonnegative solution of the equation $\text{tg}x = x$. We can not have $\delta = 0$ since $\theta \in (0, \pi/2)$ and therefore $u(\theta)$, $\sin\theta > 0$. It follows that

$$\delta > \pi \tag{2.10}$$

By (2.6) and (2.8) we get

$$\cosh(u(\theta)\cos\theta) - u(\theta)\cos\theta\sinh(u(\theta)\cos\theta) = 0$$

and hence

$$u(\theta)\cos\theta = u_0 \tag{2.11}$$

where u_0 is a positive root of the equation $\cosh(u) - u\sinh(u) = 0$. From (2.9) and (2.11), we obtain

$$\frac{u_0}{\cos\theta} = \frac{\delta}{\sin\theta}$$

and then

$$\delta = u_0\text{tg}\theta \leq u_0\text{tg}\theta_0 = u_0\epsilon_0 = \cosh u_0$$

We have $\cosh u_0 = \sqrt{1+\sinh^2 u_0} < \sqrt{1+(1.6)^2} = 1,8867...$ and hence, by (2.10), $\pi < 1,8867...$, a contradiction.

It follows that f does not have a critical point in $(0, \theta_0]$. If θ_0 is a point of minimum of f then $f(\theta) \geq f(\theta_0) > \text{tg}\theta_0 = \epsilon_0$ for all $\theta \in (0, \theta_0]$ so that $x_\theta(u(\theta)) > \epsilon_0$ for all $\theta \in (0, \pi/2)$ and then (e) is proved. If θ_0 is not a point of minimum of f, then f must assume its global minimum at $\theta = 0$. To estimate this value, we first observe that since f has a limit for $\theta \to 0$ we easily conclude using (2.7) that $u(\theta)$ also has a limit for $\theta \to 0$. To compute $u_1 := \lim_{\theta\to 0} u(\theta)$ we observe that by (2.6), one has

$$\frac{\sin(u(\theta)\sin\theta)}{\sin\theta} = \frac{\cos(u(\theta)\sin\theta)\cosh(u(\theta)\cos\theta)}{\cos\theta\sinh(u(\theta)\cos\theta)}.$$

When θ goes to 0, the left member of this equation tends to u_1 and the right member to $\cosh(u_1)/\sinh(u_1)$ that is, u_1 is the positive root of the equation $\cosh(u) - u\sinh(u) = 0$, or $u_1 = u_0$. We conclude then, using (2.7), that given any $\theta \in (0, \theta_0]$,

$$x_\theta(u(\theta)) = f(\theta) > \lim_{\tilde\theta\to 0} f(\tilde\theta) = \sinh(u_0) = \epsilon_0$$

and therefore $x_\theta(u(\theta)) > \epsilon_0$ for all $\theta \in (0, \pi/2)$, proving (e).

To prove (f), we observe that by (b) and by the proof of (d), D_θ is generated by the segment $\bar{c}_\theta$ which, as we have proved, is compact. Hence, the distance of any point of $\bar{c}_\theta$ to the boundary of D_θ and the curvature of D_θ in a point of $\bar{c}_\theta$ are bounded say, by d and K, respectively. Therefore, given any point $p \in D_\theta$, there exists an isometry ϕ of $\mathbb{R}^3$ belonging to the standard screw motion around the z-axis (2.1), such that $p = \phi(p_0)$ with $p_0 \in \bar{c}_\theta$, so that we obviously we have

$$d(p, \partial D_\theta) = d(\phi(p_0), \phi(\partial D_\theta)) = d(p_0, \partial D_\theta) \leq d$$

and $0 \geq K(p) = K(p_0) \geq K$. This shows that D_θ has bounded curvature and is a "bandlike" domain. It is also clear from the proof of (e) that the distance of any endpoint of $\bar{c}_\theta$ to the origin is exactly $x_\theta(u(\theta))$ which was proved to be strictly bigger than ϵ_0 if $\theta \neq 0$. Since ∂D_θ is generated by the endpoints of $\bar{c}_\theta$, we have proved (f) and Proposition 2.1.

Theorem 1.1. *Let S be a line segment through the origin in the (x, y)-plane with endpoints p_1 and p_2 such that*

$$\max\{|p_1|, |p_2|\} \leq \epsilon_0.$$

Let M be a complete properly immersed minimal surface, with bounded curvature and with boundary $h_1 \cup h_2$, where h_1 and h_2 are the helices generated by p_1 and p_2, and such that

$$\lim_{(x,y,z)\to\infty} \frac{x^2 + y^2}{z^2} = 0, \qquad (x, y, z) \in M. \tag{2.12}$$

Then M coincides with the helicoidal domain generated by S.

Remarks.

1) On the basis of the remark following Theorem 1.2 we may replace the assumption of bounded curvature of M by the stability of M. Hence, according to the remark before Proposition 2.1, we have shown that the helicoidal surface generated by the line segment with endpoints p_1 and p_2 where $|p_1|$, $|p_2| \leq \epsilon_0$ is the only <u>stable</u> properly immersed minimal surface spanning $h_1 \cup h_2$ and also satisfying (2.12).

2) It will be obvious from the proof below that the above uniqueness theorem also holds for any part of the helicoid contained in the cylinder of radius ϵ_0 around the z-axis.

Proof of Theorem 2.1. By Theorem 1 of [4], the restriction to M of the function $d_\epsilon(x, y, z) := x^2 + y^2 - \epsilon z^2$ is subharmonic, for any $0 \leq \epsilon \leq 1$. Therefore, given a ball $B(R)$ in $\mathbb{R}^3$ centered at the origin with radius $R > 0$, the maximum of d_ϵ restricted to $M \cap B(R)$ must be assumed at $\partial(M \cap B(R))$. Given $\epsilon > 0$, by (2.12), we can find $R(\epsilon)$ such that $(x^2 + y^2)/z^2 < \epsilon$ for $(x, y, z) \in M \cap \partial B(R)$ so that $d_\epsilon \leq 0$ on $M \cap \partial B(R)$, for all $R \geq R(\epsilon)$. Since at ∂M we have $d_\epsilon(x, y, z) \leq \epsilon_0^2 - z^2 \leq \epsilon_0^2$ we obtain $d_\epsilon \leq \epsilon_0^2$ on M. By letting $\epsilon \to 0$, we get

$$x^2 + y^2 \leq \epsilon_0^2 \tag{2.13}$$

on M, that is, M is contained in the cylinder with radius ϵ_0 around the z-axes.

By Proposition 2.1, (c), if θ is close to $\pi/2$, then M_θ and $M_{-\theta}$ do not intersect M. By decreasing θ we can find $\theta_1 \in [0, \pi/2)$ and $\theta_2 \in (-\pi/2, 0]$ such that either M_{θ_i} has a first contact point with M for some i, or $M_{\theta_i} \cap M = \emptyset$ but $d(M_{\theta_i}, M) = 0$ for $i = 1, 2$. We will prove that if one of the M_{θ_i} has a first contact with M then M is part of a helicoid and that the other case can not happen. If $\theta_1 = \theta_2 = 0$ then it follows from Proposition 2.1, (d), that M is contained in the standard helicoid and the theorem would be proved. Thus, let us assume $\theta_i \neq 0$ for some i. By Proposition 2.1, (e), $\partial M \cap M_{\theta_i} = \emptyset$. Therefore, if M_{θ_i} has a first contact point with M then it must be in the interior of M and hence, by the maximum principle, M is contained in M_{θ_i}. Since the only surface M_θ that contains the helices $h_1 \cup h_2$ is the helicoid ($\theta = 0$), the theorem is proved in this case, too.

If M and both M_{θ_i} do not have a first contact point we will apply Theorem 1.1 to get a contradiction. Without loss of generality, let us suppose that $\theta_1 \geq -\theta_2$, $\theta_1 > 0$. Then, by definition of θ_2, M is disjoint from $M_{-\theta_1}$, hence M is contained in the connected component U_{θ_1} of $\mathbb{R}^3 \setminus (M_{\theta_1} \cup M_{-\theta_1})$ which contains the z-axis. We obviously have $M \cap D_{\theta_1} = \emptyset$ where $D_{\theta_1} := \partial \overline{U_1} \cap M_{\theta_1}$. By (2.13), M is contained in the cylinder of radius ϵ_0 around the z-axis and it follows therefore by Proposition 2.1, (f), that

$$d(M, \partial D_{\theta_1}) > 0. \tag{2.14}$$

On the other hand, since $d(M, M_{\theta_1}) = 0$ and $M \subset U_{\theta_1}$, we must have

$$d(D_{\theta_1}, M) = 0. \tag{2.15}$$

By Proposition 2.1, (f), we know that D_{θ_1} has bounded curvature and is a "bandlike" domain. Since $\min\{d(M, \partial D_{\theta_1}), d(\partial M, D_{\theta_1})\} > 0$ by (2.14) and Proposition 2.1 (e), we can apply Theorem 2.1 to M and D_{θ_1} to conclude that $d(M, D_{\theta_1}) > 0$, which contradicts (2.15), finishing the proof of Theorem 2.1.

3. Appendix. The proof of the lemma below is completely elementary and therefore omitted.

Lemma 1. *Let M be a complete surface with has no boundary in the open ball $B_R(0) \subset \mathbb{R}^3$, $0 \in M$ and assume that the modulus of the principal curvatures of M is bounded by 1. Let us furthermore suppose that the tangent plane of M at 0 forms an angle $\alpha_0 \in [0, \pi/2)$ with the (x_1, x_2)-plane P. Then M contains the graph of a function u defined on a disc $D_\delta(0) \subset P$ with*

$$\delta = \frac{1}{\sqrt{2}} \min\left\{1 - \sin\alpha_0, \frac{\cos\alpha_0}{\sqrt{1 + \sin\alpha_0}} R\right\}. \tag{3.1}$$

Denoting by $\alpha(x) \in [0, \pi/2)$ the angle between the tangent plane of M at $(x, u(x))$ and the (x_1, x_2)-plane one has

$$\sin\alpha(x) \leq \sin\alpha_0 + \sqrt{2}|x|. \tag{3.2}$$

The function u satisfies

$$|u(x)| \leq 2(tg\alpha_0)|x| + \frac{\sqrt{2}}{\cos\alpha_0}|x|^2. \tag{3.3}$$

Lemma 2. *Let M_1 be a graph defined over a disc $D_R(0)$ in the (x_1, x_2)-plane, $0 < R \leq 1/\sqrt{2}$, and assume that M_1 passes through 0 and has there the (x_1, x_2)-plane as its tangent plane. Similarly, a surface M_2 passes through the point $(0, 0, \epsilon)$, $\epsilon > 0$, where its tangent plane is given by*

$$x_3 = (tg\alpha)x_1 + \epsilon,$$

$\alpha \in [0, \pi/2)$ and M_2 is a Euclidean graph over a disc of radius R in this tangent plane. We also assume that M_1 and M_2 are disjoint and that their principal curvatures are less than or equal to 1 in modulus. Then, if

$$\frac{\epsilon}{R} < \frac{1}{80}$$

it follows that

$$tg\alpha \leq \min\left\{2, 4\sqrt{10}\sqrt{\frac{\epsilon}{R}}\right\}.$$

Proof. We introduce a second retangular coordinate system $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ centered at $(0, 0, \epsilon)$ such that the tangent plane of M_2 at this point becomes $\tilde{x}_3 = 0$, i.e.,

$$\begin{aligned} x_1 &= \cos\alpha\tilde{x}_1 - \sin\alpha\tilde{x}_3 \\ x_2 &= \tilde{x}_2 \\ x_3 &= \epsilon + \sin\alpha\tilde{x}_1 + \cos\alpha\tilde{x}_3. \end{aligned}$$

We may also assume that $R = 1/\sqrt{2}$. We first consider the possibility that $\mathrm{tg}\alpha \geq 2$, i.e., $\cos\alpha \leq (1/2)\sin\alpha$. From Lemma 3, we then obtain the following inequalities on M_2, valid for $-1/\sqrt{2} < \tilde{x}_1 < 0$:

$$x_3 = \epsilon + \sin\alpha\tilde{x}_1 + \cos\alpha\tilde{x}_3 \leq \epsilon + \tilde{x}_1(\sin\alpha + \sqrt{2}\cos\alpha\tilde{x}_1)$$

$$\leq \epsilon + \frac{1}{2}\sin\alpha\tilde{x}_1, \tag{3.4}$$

$$\begin{aligned} x_1^2 = (\cos\alpha\tilde{x}_1 - \sin\alpha\tilde{x}_3)^2 &\leq (\cos\alpha|x_1| + \sin\alpha\sqrt{2}\tilde{x}_1^2)^2 \\ &\leq \tilde{x}_1^2(\cos\alpha + \sin\alpha)^2 \leq 2\tilde{x}_1^2 \end{aligned}$$

Since by assumption the surfaces M_1 and M_2 are disjoint we have that for any two points $P_1 \in M_1$ and $P_2 \in M_2$ such that $x_1(P_1) = x_1(P_2)$, $x_2(P_1) = x_2(P_2) = 0$ the inequality $x_3(P_2) > x_3(P_1)$ must hold which using (3.3) and (3.4) implies

$$\epsilon + \frac{1}{2}\sin\alpha\tilde{x}_1(P_2) > -\sqrt{2}x_1(P_1)^2 = -\sqrt{2}x_1(P_2)^2 \geq -2\sqrt{2}\tilde{x}_1(P_2)^2.$$

We therefore conclude that

$$\epsilon + \frac{1}{2}\sin\alpha\tilde{x}_1 > -2\sqrt{2}\tilde{x}_1^2$$

for $-(1/2) < \tilde{x}_1 \leq 0$. Choosing $\tilde{x}_1 = -\sqrt{\epsilon/(2\sqrt{2})}$ $(> -(1/2))$ we obtain $(1/2)\sqrt{\epsilon/(2\sqrt{2})}\sin\alpha$ 2ϵ, that is, $\sin\alpha \leq 4\sqrt{2\sqrt{2}}\sqrt{\epsilon}$. From the assumption $\mathrm{tg}\alpha \geq 2$ it follows however that $\sin\alpha \geq \sqrt{(4/5)}$ and hence $\epsilon > 1/(40\sqrt{2})$, in other words, $\epsilon \leq 1/(40\sqrt{2})$ implies $\mathrm{tg}\alpha < 2$. For the remainder of the proof we may therefore suppose that $\mathrm{tg}\alpha < 2$.

Substituting $\tilde{x}_1 = -(\sigma/4)\mathrm{tg}\alpha$, $0 \leq \sigma \leq 1$, again the inequalities (3.4) hold for such values of $\tilde{x}_1$ and by the same reasoning as above we obtain

$$\epsilon - \frac{1}{2}\frac{\sigma}{4}\mathrm{tg}\alpha\sin\alpha \geq -2\sqrt{2}\frac{\sigma^2}{16}\mathrm{tg}^2\alpha$$

and therefore

$$\epsilon \geq \frac{1}{8}\sigma\mathrm{tg}\alpha(\sin\alpha - \sqrt{2}\sigma\mathrm{tg}\alpha).$$

From $\mathrm{tg}\alpha \leq 2$ it follows that

$$\sin\alpha = \frac{\mathrm{tg}\alpha}{\sqrt{1+\mathrm{tg}^2\alpha}} \geq \frac{1}{\sqrt{5}}\mathrm{tg}\alpha$$

and hence

$$\epsilon \geq \frac{1}{8}\sigma\mathrm{tg}^2\alpha\left(\frac{1}{\sqrt{5}} - \sqrt{2}\sigma\right).$$

Choosing e.g. $\sigma = 1/(2\sqrt{10})$ we get

$$\epsilon \geq \frac{1}{160\sqrt{2}}\mathrm{tg}^2\alpha$$

which proves the lemma.

Lemma 3. *Let $N \subset \mathbb{R}^3$ be an immersed minimal surface with unit normal ν and let ρ be a real C^1-function on N. Let $d\sigma$ denote the area element of N and κ_i $(i = 1, 2)$ its principal curvatures with corresponding orthonormal principal directions X_1, X_2; $K = \kappa_1\kappa_2$. Then the area of the surface*

$$x \in N \mapsto x + \rho(x)\nu(x) \tag{3.5}$$

is given by the integral

$$\int_N f(x, \rho(x), \nabla\rho(x))d\sigma(x) \tag{3.6}$$

with

$$f(x, \rho, p) = \tag{3.7}$$

$$= \left[(1-K\rho^2)|p|^2 + 2\rho(\kappa_1 < p, X_2 >^2 + \kappa_2 < p, X_1 >^2) + (1+K\rho^2)^2\right]^{(1/2)}.$$

Consequently we have

$$\nabla_p f(x,\rho,p) = \tag{3.8}$$

$$= f(x,\rho,p)^{-1}\left[(1-K\rho^2)p + 2\rho(\kappa_1 < p, X_2 > X_2 + \kappa_2 < p, X_1 > X_1)\right]$$

and

$$\frac{\partial}{\partial\rho} f(x,\rho,p) = \tag{3.9}$$

$$= f(x,\rho,p)^{-1}\left[-K\rho|p|^2 + \kappa_1 < p, X_2 >^2 + \kappa_2 < p, X_1 >^2 + 2K(1+K\rho^2)\rho\right].$$

Proof. By definition

$$d\nu < X_i >= \kappa_i X_i, \quad < X_i, X_j >= \delta_{ij}$$

and therefore we get for for the first fundamental form g_{ij} of (3.6) with respect to the basis X_1, X_2:

$$g_{ij} = \delta_{ij}(1 + (\kappa_i + \kappa_j)\rho + \kappa_i\kappa_j\rho^2) + d\rho < X_i > d\rho < X_j > .$$

The expression (3.8) for $f = (g_{11}g_{22} - g_{12}^2)^{(1/2)}$ follows by direct calculation, using the minimality of N, i.e., $\kappa_1 + \kappa_2 = 0$. (3.8) and (3.9) are obvious from (3.7).

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