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Isometries of JB-algebras

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A bijective linear mapping between two JB-algebras A and B is an isometry if and only if it commutes with the Jordan triple products of A and B. Other algebraic characterizations of isometries between JB-algebras are given. Derivations on a JB-algebra A are those bounded linear operators on A with zero numerical range. For JB-algebras of selfadjoint operators we have: If H and K are left Hilbert spaces of dimension ≥ 3 over the same field $\mathbf{F}$ (the real, complex, or quaternion numbers), then every surjective real linear isometry f from $S(H)$ onto $S(K)$ is of the form $f(x) = U \circ x \circ U^{-1}$ for x in $S(H)$, where τ is a real-linear automorphism of $\mathbf{F}$ and U is a real linear isometry from H onto K with $U(\lambda h) = \tau(\lambda)U(h)$ for λ in $\mathbf{F}$ and h in H.

0. Introduction

In recent years, isometries of certain norm-closed real subtriples of JB*-triples have been deeply studied (see [5] and [6]). In [5], the group of surjective linear isometries of a real C*-algebra A was studied. It was proved that every such an isometry is an automorphism of the Jordan triple structure associated to A. As a consequence, every (surjective linear) isometry extends uniquely by complex linearity to an isometry of the C*-algebra complexification of A. These results made it possible to compute the group of isometries for a type I factor A.

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In the Jordan algebra context, isometries of JB-algebras begun to be studied in [18]. Here, the Jordan analogous to the above results are obtained though our methods do not depend on [5]. More precisely, in section 1 the following is shown (th. 1.9) : If A and B are two JB-algebras, then every surjective linear isometry $F: A \to B$ is of the form $F(x) = b \cdot \Phi(x)$ for all x in A, where b is a central symmetry in $M(B)$ (the algebra of multipliers of B) and $\Phi: A \to B$ is a surjective algebra isomorphism. It follows that linearly isometric JB-algebras are isomorphic as Jordan algebras. The converse is indeed true because, as it is well known, the norm of a JB-algebra is algebraically determined and therefore surjective algebra isomorphisms between JB-algebras are actually linear isometries. We prove (th. 1.9) that, if we fix the vector space of a JB-algebra, then anyone of the mappings $x \mapsto \|x\|$, $x \mapsto x^3$ and $(x\,y\,z) \mapsto \{x\,y\,z\}$ (the Jordan triple product) determines uniquely the other two. Every surjective linear isometry between two JB-algebras A and B extends uniquely by complex linearity to an isometry between the corresponding JB*-algebra complexifications of A and B.

In section 2, isometries $F: A \to B$ between two JBW-factors are considered. In this case (cor. 1.5), F is of the form $F(x) = \varepsilon \cdot \Phi(x)$ for all x in A, where $\varepsilon \in \{1, -1\}$ and $\Phi: A \to B$ is an algebra isomorphism. Thus the determination of all surjective linear isometries between JBW-factors is equivalent to the determination of all surjective algebra isomorphisms between them, a problem that we solve for all JBW-factors of type I except for the JB-algebra $H_3(\mathbb{O})$ of all hermitian 3×3 matrices over the algebra $\mathbb{O}$ of the Cayley numbers. Notice that among JBW-factors of type I are the JB-algebras $S(X)$, where X is a left Hilbert space over the field $\mathbb{F}$ (which is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the division algebra of Hamilton's quaternions) and $S(X)$ denotes the space of bounded selfadjoint operators $x: X \to X$. Let H and K be left Hilbert spaces of dimensions ≥ 3 over $\mathbb{F}$ and let $F: S(H) \to S(K)$ be a surjective real-linear isometry. We prove (th. 2.2) that there are a number $\varepsilon \in \{-1, 1\}$, a real-linear automorphism τ of $\mathbb{F}$, and a surjective real-linear isometry $U: H \to K$ with $U(\lambda h) = \tau(\lambda)U(h)$ for all λ in $\mathbb{F}$ and all h in H, such that $F(x) = \varepsilon\, U \circ x \circ U^{-1}$ for all x in $S(H)$. This result is also valid if $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ and H, K have dimension 2 over $\mathbb{F}$, but it fails to be true for two-dimensional spaces over $\mathbb{H}$. However, without any restriction on the dimensions or the choice of $\mathbb{F}$, if the spaces $S(H)$ and $S(K)$ are isometric, then H and K are $\mathbb{F}$-linearly isometric (cor. 2.5).

1. Algebraic characterization of isometries

A JB-algebra is a real Jordan algebra J endowed with a complete norm $\|\cdot\|$ satisfying $\|a\,b\| \leq \|a\|\,\|b\|$ and $\|a\|^2 \leq \|a^2 + b^2\|$ for all a, b in J. The basic reference for JB-algebras is [9]. The following theorem was first proved in [18] (see also [4] for an alternative proof).

1.1 Theorem *The unit preserving surjective linear isometries between unital JB-algebras are exactly the surjective algebra isomorphisms.*

An element a in a unital JB-algebra is said to be a *symmetry* if $a^2 = 1$. The proof of (1.1) in [18] relies on a lemma asserting that if J is a unital JB-algebra, then the extreme points of the set $\{a \in J : 0 \leq a \leq 1\}$ are the idempotents of J. Since the closed unit ball B_J of J is affinely isomorphic to $\{a \in J : 0 \leq a \leq 1\}$ via the mapping $x \mapsto \frac{1}{2}(x+1)$, the next lemma follows:

1.2 Lemma *The extreme points of the closed unit ball of a unital JB-algebra are the symmetries in J.*

For a unital JB-algebra J, we determine the isolated points of the set S_J of all symmetries in J. For a JB-algebra J, the *Jordan triple product* $\{\cdots\}$ of J is defined by

$$\{x\,y\,z\} := x \cdot (z \cdot y) + z \cdot (x \cdot y) - (x \cdot z) \cdot y$$

for all x, y, z in J.

1.3 Proposition *A symmetry in a unital JB-algebra J is an isolated point of the set of all symmetries in J if and only if it is central.*

Proof If a belongs to S_J, then $(\mathbf{1} - a)^2 = 2(\mathbf{1} - a)$, hence $\|\mathbf{1} - a\| \geq 2$ whenever $a \neq \mathbf{1}$. Therefore, $\mathbf{1}$ is an isolated point of S_J. But if a is a central symmetry in J, then $\phi : x \mapsto a \cdot x$ is a homeomorphism of S_J onto S_J with $\phi(\mathbf{1}) = a$. It follows that all central symmetries in J are isolated points of S_J. Conversely, let a be an isolated point of S_J. For every continuous derivation D on J and every real number λ, $\exp(\lambda D)(a)$ lies in S_J. Since the mapping $f : \lambda \mapsto \exp(\lambda D)(a)$ is continuous with $f(0) = a$, and a is an isolated point of S_J, it follows the existence of a neighbourhood of zero in $\mathbb{R}$ on which f is constant, and so $D(a) = f'(0) = 0$. In particular, taking the derivation given by $D(x) := (a \cdot x) \cdot y - a \cdot (a \cdot y)$ for all x in J, where $y \in J$, we have $y - a \cdot (a \cdot y) = 0$ for all $y \in J$. Therefore $e \cdot y = \{e\,y\,e\}$ for all $y \in J$, where $e := \frac{1}{2}(a + \mathbf{1})$ is an idempotent in J. By ([9], lemma 2.5.5), e (hence also a) is central. □

1.4 Theorem *If A and B are unital JB-algebras, then the linear isometries from A onto B are the mappings of the form $x \mapsto b \cdot \Phi(x)$ where b is a central symmetry in B and Φ is an algebra isomorphism from A onto B.*

Proof Let $F: A \to B$ be a surjective linear isometry. By (1.2) and (1.3), $b := F(\mathbf{1})$ is a central symmetry in B. Therefore the mapping $\Phi: A \to B$ given by $\Phi(x) := b \cdot F(x)$ for all $x \in J$ is a unit preserving surjective linear isometry, hence by (1.1) it is an algebra isomorphism. □

Let J be a JB-algebra. We recall [9] that J is said to be *monotone complete* if each bounded increasing net (a_α) in J has a least upper bound a in J. A bounded linear functional ρ in J is said to be *normal* if $\rho(a_\alpha) \to \rho(a)$ for each net (a_α) as above, and J is said to be a *JBW-algebra* if J is monotone complete and has a separating set of positive normal bounded linear functionals. This is equivalent to requiring J to be a dual Banach space ([9], th. 4.4.16). Every JBW-algebra has a unit. The centre Z of a JBW-algebra J is clearly a JBW-subalgebra of J. If Z consists only of scalars multiples of the identity, then J is called a *JBW-factor.*

1.5 Corollary *If A and B are JBW-factors, then the linear isometries of A onto B are the mappings of the form $\varepsilon\,\Phi$, where $\varepsilon \in \{1, -1\}$ and Φ is an algebra isomorphism from A onto B.*

In what follows we obtain other interesting algebraic characterizations of surjective linear isometries between non necessarily unital JB-algebras. The bidual J^{**} of a JB-algebra J is an (automatically unital) JBW-algebra under the *Arens product* ([9], th. 4.4.3). Moreover, the set $M(J) := \{x \in J^{**} : x \cdot J \subseteq J\}$ is a norm-closed subalgebra of J^{**} [8] called the algebra of *multipliers* of J. Note that $M(J)$ always contains the unit element of J^{**}, and that $M(J)$ is the largest subalgebra of J^{**} containing J as an ideal. Of course, $M(J) = J$ whenever J has a unit element.

1.6 Proposition *If A and B are JB-algebras, then every linear isometry from A onto B extends to a linear isometry from $M(A)$ onto $M(B)$.*

Proof Let F be a linear isometry from A onto B. By (1.4) there are a central symmetry b in B^{**} and an algebra isomorphism Φ from A^{**} onto B^{**} such that $F^{**}(x) = b \cdot \Phi(x)$ for all x in A^{**}. As a consequence we have $F^{**}(x \cdot (y \cdot z)) = F^{**}(x) \cdot (F^{**}(y) \cdot F^{**}(z))$ for all x, y, z in A^{**}. Let x be in $M(A)$ and y be in A. Then $x \cdot y^2$ lies in A, and therefore $F^{**}(x) \cdot (F(y))^2 (= F^{**}(x) \cdot (F^{**}(y))^2 = F^{**}(x \cdot y^2) = F(x \cdot y^2))$ belongs to B. It follows that $F^{**}(x) \cdot u^2$ lies in B for all u in B. Since B is the linear hull of the set $\{u^2 : u \in B\}$ (a consequence of [9], 3.2.4), we have $F^{**}(x) \cdot B \subseteq B$, hence $F^{**}(x)$ belongs to $M(B)$. In this way we have proved that $F^{**}(M(A)) \subseteq M(B)$. By symmetry, we actually have $F^{**}(M(A)) = M(B)$, and the mapping $x \mapsto F^{**}(x)$ from $M(A)$ to $M(B)$ is a

surjective linear isometry extending F. □

Some other auxiliary ideas for our goal are collected in the next two remarks.

1.7 Remark Let X be a Banach space and denote by $B(X)$ the Banach algebra of all bounded linear operators on X. For T in $B(X)$, write $r(T) := \lim\{ \| T^n \|^{1/n} \}$. The real variant of the famous Gelfand-Beurling formula gives

$$r(T) = \text{Max}\{ |\alpha + i\beta| : \alpha, \beta \in \mathbb{R} \text{ with } (T - \alpha \mathbf{1})^2 + \beta^2 \mathbf{1} \text{ not invertible in } B(X)\}$$

(see [1]). But, denoting by $L(X)$ the associative algebra of all (possibly not bounded) linear operators on X, by the Banach isomorphism theorem, the invertible elements in $B(X)$ are those elements in $B(X)$ that are invertible in $L(X)$. It follows

$$r(T) = \text{Max}\{ |\alpha + i\beta| : \alpha, \beta \in \mathbb{R} \text{ with } (T - \alpha \mathbf{1})^2 + \beta^2 \mathbf{1} \text{ not invertible in } L(X)\}.$$

1.8 Remark Let J be a JB-algebra, let x be in J and let T_x denote the (bounded linear) operator on J given by $T_x(y) := \frac{1}{3}(2x \cdot (x \cdot y) + x^2 \cdot y)$. Then for every natural number n we have

$$\|x\|^{2n+1} = \|x^{2n+1}\| = \|T_x^n(x)\| \leq \|T_x^n\| \cdot \|x\| \leq \|T_x\|^n \cdot \|x\| \leq \|x\|^{2n+1}$$

hence $\|T_x^n\| = \|x\|^{2n}$. Therefore $r(T_x) = \|x\|^2$.

1.9 Theorem *If A and B are JB-algebras and $F: A \to B$ is a (non necessarily continuous) bijective linear mapping, then the following statements are equivalent:*

(i) *F is an isometry,*

(ii) *There are a central symmetry b in $M(B)$ and a surjective algebra isomorphism $\Phi: A \to B$ such that $F(x) = b \cdot \Phi(x)$ for all x in A,*

(iii) *There are two mutually complementary ideals P and Q in A such that $F_{|P}$ and $-F_{|Q}$ are (possibly not surjective) algebra isomorphisms,*

(iv) *$F(x \cdot (y \cdot z)) = F(x) \cdot (F(y) \cdot F(z))$ for all x, y, z in A,*

(v) *$F(\{x\, y\, z\}) = \{F(x)\, F(y)\, F(z)\}$ for all x, y, z in A,*

(vi) *$F(\{x\, x\, z\}) = \{F(x)\, F(x)\, F(z)\}$ for all x, z in A,*

(vii) *$F(x^3) = (F(x))^3$ for all x in A,*

(viii) *$F(\langle x\, y\, z\rangle) = \langle F(x)\, F(y)\, F(z)\rangle$ for all x, y, z in A, where for u, v, w all either in A or in B, $\langle u\, v\, w\rangle := \frac{1}{3}(u \cdot (v \cdot w) + v \cdot (u \cdot w) + w \cdot (u \cdot v))$,*

(ix) *$F(\langle x\, x\, y\rangle) = \langle F(x)\, F(x)\, F(y)\rangle$ for all x, y in A.*

Proof "$i \Rightarrow ii$". By (1.6), F extends to a linear isometry G from $M(A)$ onto $M(B)$, and by (1.4) there are a central symmetry b in $M(B)$ and an algebra

isomorphism Ψ from $M(A)$ onto $M(B)$ such that $G(x) = b \cdot \Psi(x)$ for all x in $M(A)$. The proof is concluded by observing the almost obvious equality $\Psi(A) = B$.

"*ii* $\Rightarrow$ *iii*". Φ extends to an algebra isomorphism Ψ from $M(A)$ onto $M(B)$. Then $a := \Psi^{-1}(b)$ is a central symmetry in $M(A)$ such that $F(x) = \Phi(a \cdot x)$ for all x in A. Now $e := \frac{1}{2}(1 + a)$ is a central idempotent in $M(A)$, hence $e \cdot A$ and $(1 - e) \cdot A$ are mutually complementary ideals in A and we see easily that $F_{|e \cdot A} = \Phi_{|e \cdot A}$ and $F_{|(1-e) \cdot A} = -\Phi_{|(1-e) \cdot A}$.

"*iii* $\Rightarrow$ *iv*". By observing that $F(P)$ and $F(Q)$ are (mutually complementary) ideals of B, the result follows after a straightforward calculation.

"*iv* $\Rightarrow$ *v* $\Rightarrow$ *vi* $\Rightarrow$ *vii*". These implications are clear.

"*vii* $\Rightarrow$ *viii*". Note that the mapping $\langle \cdots \rangle$ is symmetric trilinear with $\langle x\, x\, x \rangle = x^3$.

"*viii* $\Rightarrow$ *ix*". This is clear.

"*ix* $\Rightarrow$ *i*". With the notation in (1.8) we have $T_u(v) = \langle u\, u\, v \rangle$ with u, v both either in A or in B, hence, if we assume (*ix*), then the equality $F \circ T_x \circ F^{-1} = T_{F(x)}$ holds for all x in A. But the mapping $S \mapsto F \circ S \circ F^{-1}$ from $L(A)$ to $L(B)$ is a surjective algebra isomorphism. By (1.7) we have $r(T_{F(x)}) = r(T_x)$ and, by (1.8), $\|F(x)\| = \|x\|$ for all x in A. □

A direct consequence of the implication "*i* $\Rightarrow$ *ii*" in the theorem is the next corollary

1.10 Corollary *Linearly isometric JB-algebras are (isometrically) isomorphic.*

We remark that linearly isometric JB*-algebras need not be isomorphic as Jordan algebras (see [3] Example 5.7 and [12] Cor. 2.10). We recall that a JB*-algebra is a complex Jordan algebra C endowed with a conjugate-linear algebra involution * and a complete norm $\|\cdot\|$ satisfying $\|u \cdot v\| \leq \|u\|\|v\|$ and $\|\{u\, u\, u\}\| = \|u^3\|$ for all u, v in C, where the Jordan triple product $\{\cdots\}$ is now defined by

$$\{u\, v\, w\} := u \cdot (v^* \cdot w) - v^* \cdot (w \cdot u) + w \cdot (u \cdot v^*)$$

for all u, v, w in C. The involution * of a JB*-algebra is automatically isometric ([19], lemma 4). If J is a JB-algebra, then the Jordan algebra complexification of J, endowed with the conjugate-linear algebra involution * given by $(x + iy)^* := x - iy$ and a suitable (uniquely determined) norm, becomes a JB*-algebra called the *JB*-complexification* of J and denoted by $J_{\mathbb{C}}$ (see [9] th. 3.3.9, and [17]).

1.11 Corollary *Surjective linear isometries between JB-algebras extend to surjective linear isometries of their respective JB*-complexifications.*

Proof Let A and B be JB-algebras, and $F: A \to B$ a surjective linear isometry. Extending F by complex linearity to a mapping $F_{\mathbb{C}}$ from $A_{\mathbb{C}}$ into $B_{\mathbb{C}}$, and using the implication "$i \Rightarrow ii$" in (1.9), $F_{\mathbb{C}}$ becomes a bicontinuous bijective mapping that preserves the Jordan triple product. Therefore

$$\|F_{\mathbb{C}}(u)\|^3 = \|\{F_{\mathbb{C}}(u)\, F_{\mathbb{C}}(u)\, F_{\mathbb{C}}(u)\}\| = \|F_{\mathbb{C}}(\{u\,u\,u\})\|$$
$$\leq \|F_{\mathbb{C}}\| \cdot \|\{u\,u\,u\}\| = \|F_{\mathbb{C}}\| \cdot \|u\|^3$$

for all u in $A_{\mathbb{C}}$, hence $F_{\mathbb{C}}$ is a contractive mapping. By symmetry, $F_{\mathbb{C}}$ is actually an isometric mapping. □

We conclude this section by obtaining a geometric characterization of algebra derivations on JB-algebras (see [15] and [16] for a study of derivations on JB-algebras).

1.12 Proposition *If J is a JB-algebra, then the connected component of the identity in the group of all linear isometries from J onto J consists only of algebra automorphisms.*

Proof Let C denote the connected component of the identity in the group of all linear isometries from J onto J, and let $\mathbf{1}$ be the unit of J^{**}. Then $\{F^{**}(\mathbf{1}): F \in C\}$ is a connected subset of J^{**}. By (1.2) and (1.3), the set $\{F^{**}(\mathbf{1}): F \in C\}$ has discrete topology. It follows that $F^{**}(\mathbf{1}) = \mathbf{1}$ for every F in C, and the proof is concluded by applying (1.1) □

Let X be a real Banach space, and let T be a bounded linear operator on X. Then the following statements are equivalent (see [2]):

(i) $\|\exp(\lambda T)\| = 1$ for every λ in $\mathbb{R}$,

(ii) $f(T(x)) = 0$ for all x in X and all f in X^* with $f(x) = \|x\| = \|f\| = 1$,

(iii) $g(T) = 0$ for every g in $B(X)^*$ satisfying $g(\mathbf{1}) = \|g\| = 1$, where $\mathbf{1}$ denotes the identity operator on X.

(iv) $\lim\limits_{\substack{\alpha \to 0 \\ \alpha \in \mathbb{R}\setminus\{0\}}} \frac{1}{\alpha}\big(\|\mathbf{1} + \alpha T\| - 1\big) = 0$.

Whenever T satisfies the above conditions, we say that T has *zero numerical range.*

1.13 Corollary *The algebra derivations on a JB-algebra J are the bounded linear operators on J having zero numerical range.*

Proof If T is an algebra derivation on J, then (as observed by Youngson [20]) T is automatically bounded, hence for every λ in $\mathbb{R}$ we may consider $\exp(\lambda T)$

which is an algebra automorphism of J, and so $\| \exp(\lambda T) \| = 1$. It follows that the algebra derivations on J are bounded and have zero numerical range. Conversely, let T be a bounded linear operator on J with zero numerical range. Then $\lambda \mapsto \exp(\lambda T)$ is a continuous one-parameter group of linear isometries of J, hence, for all λ in $\mathbb{R}$, the operator $\exp(\lambda T)$ must lie in the connected component of the identity in the group of all isometries from J onto J. By (1.12), $\exp(\lambda T)$ is an algebra automorphism and, by taking the derivative at zero, T is an algebra derivation. □

2. Spacializing isometries on the space of selfadjoint operators

Let J be a JBW-algebra with centre Z. If p is a projection in J, then its *central support* $c(p)$ is the smallest projection in Z majorizing p. A projection p in J is said to be *abelian* if the JBW-subalgebra $J_p := \{\{p\,x\,p\} : x \in J\}$ is associative. J is said to be of *type I* if there is an abelian projection p in J with $c(p) = 1$. Among JBW-algebras, those of type I are well understood. They may be quite different from von Neumann algebras; however a complete classification of them is known. Namely, by ([9], 5.3.8, 7.5.10, 7.5.11, 6.1.8) the JBW-factors of type I are the JB-algebra $H_3(\mathbb{O})$ of all hermitian 3×3 matrices over the algebra $\mathbb{O}$ of the Cayley numbers, the spin factors, and the JB-algebras $S(H)$ of all selfadjoint (bounded) linear operators on a real, complex, or quaternionic Hilbert space H. In view of (1.5), the determination of surjective linear isometries between JBW-factors reduces to that of (automatically isometric) surjective algebra isomorphisms. Let J be a spin factor. Then $J = \mathcal{H} \oplus \mathbb{R}\mathbf{1}$, where $\mathcal{H}$ is a real Hilbert space of dimension ≥ 2, and the Jordan product in J is given by

$$(u + \lambda\mathbf{1}) \cdot (v + \mu\mathbf{1}) := (\mu\, u + \lambda\, v) + ((u \mid v) + \lambda\mu)\mathbf{1}$$

for λ, μ in $\mathbb{R}$ and u, v in $\mathcal{H}$. From the intrinsic algebraic characterization of $\mathcal{H}$ in J given by the equality $\mathcal{H} = \{0\} \cup \{x \in J \backslash \mathbb{R}\mathbf{1} : x^2 \in \mathbb{R}\mathbf{1}\}$ it can be easily obtained that the algebra automorphisms of J are the mappings of the form $u + \lambda\mathbf{1} \mapsto \phi(u) + \lambda\mathbf{1}$, where ϕ is a unitary operator on $\mathcal{H}$.

Now let H be a Hilbert space over $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the algebra of Hamilton's quaternions). If the dimension of H over $\mathbb{F}$ is 2, then $S(H)$ is a spin factor. Our main goal in this section is to spacialize the algebra isomorphisms of $S(H)$ when the dimension of H over $\mathbb{F}$ is ≥ 3. For a Hilbert space L over $\mathbb{F}(= \mathbb{R}, \mathbb{C}, \text{or } \mathbb{H})$ we denote by $B(L)$ the real Banach algebra of all bounded $\mathbb{F}$-linear operators on L.

2.1 Proposition *Let H and K be left Hilbert spaces over $\mathbb{F}$ (= $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$), and let $\Phi: B(H) \to B(K)$ be a real-linear surjective algebra isomorphism. Then there are a real linear automorphism τ of $\mathbb{F}$ and a bounded bijective real-linear mapping $U: H \to K$ satisfying $U(\lambda h) = \tau(\lambda)U(h)$ for all λ in $\mathbb{F}$ and all h in H, such that $\Phi(x) = U \circ x \circ U^{-1}$ for all x in $B(H)$. If Φ is actually a *-isomorphism, then U can be chosen isometric.*

Proof For a left Hilbert space L over $\mathbb{F}$, consider the right vector space L' over $\mathbb{F}$ whose additive group is that of L whereas the product of a vector l in L' by a scalar λ in $\mathbb{F}$ is defined by $l\lambda := \overline{\lambda} l$, where $^-$ denotes the standard involution on $\mathbb{F}$. Then the pair (L, L'), endowed with the mapping $\langle \cdot, \cdot \rangle: L \times L' \mapsto \mathbb{F}$ given by $\langle l, l' \rangle := (l \mid l')$, becomes a pair of dual vector spaces over $\mathbb{F}$ in the sense of ([10], p. 69-70), and $B(L)$ becomes the real algebra of all $\mathbb{F}$-linear operators on L that have an adjoint with respect to the pairing $\langle \cdot, \cdot \rangle$. Now we may apply the Jacobson isomorphism theorem ([10], p. 79) to the real-linear isomorphism Φ from $B(H)$ onto $B(K)$ to obtain the existence of a real-linear automorphism τ of $\mathbb{F}$ and real-linear bijective mappings $U: H \to K$ and $U': K' \to H'$ satisfying $U(\lambda h) = \tau(\lambda)U(h)$ for all λ in $\mathbb{F}$ and all h in H and $\langle U(h), k' \rangle = \tau(\langle h, U'(k') \rangle)$ for all h in H and k' in K', and such that $\Phi(x) = U \circ x \circ U^{-1}$ for all x in $B(H)$. In intrinsic terms on the Hilbert spaces H and K, we have $(U(h) \mid k) = \tau((h \mid U'(k))$ for all h in H and k in K, and therefore $U'(\lambda k) = \tau^{-1}(\lambda)U'(k)$ for all λ in $\mathbb{F}$ and k in K. It follows that U and U' are bounded and, even more, if U and U' are considered as bounded linear operators between the Hilbert-space-realifications of H and K, then the equality $U' = U^*$ holds, and $U^* \circ U$ is actually $\mathbb{F}$-linear. If Φ is a *-isomorphism, then using that $\Phi(x) = U \circ x \circ U^{-1}$ for all x in $B(H)$, we obtain easily that $U^* \circ U$ is in fact a positive multiple of the identity operator on H, hence (by multiplying U by a positive number if necessary) we may choose U in such a way that it is an isometry. □

2.2 Theorem *Let H and K be left Hilbert spaces over $\mathbb{F}$ (= $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$) of dimension ≥ 3 over $\mathbb{F}$, and let $F: S(H) \to S(K)$ be a surjective algebra isomorphism. Then there are a real-linear automorphism τ of $\mathbb{F}$, and a surjective real-linear isometry $U: H \to K$ satisfying $U(\lambda h) = \tau(\lambda)U(h)$ for all λ in $\mathbb{F}$ and all h in H, such that $F(x) = U \circ x \circ U^{-1}$ for all x in $S(H)$.*

Proof If H is finite-dimensional over $\mathbb{F}$, then there exists a natural number $p \geq 3$ such that the associative algebra with (real-linear) involution $B(H)$ (whose hermitian part is $S(H)$) can be regarded as the algebra with canonical involution $M_p(\mathbb{F})$ of all $p \times p$ matrices with entries in $\mathbb{F}$. If H is infinite-dimensional over $\mathbb{F}$, then H can be identified with the hilbertian sum of three

copies of H, and therefore the algebra with involution $B(H)$ can be regarded as the algebra with canonical involution $M_3(B(H))$. In any case, we may apply a result of W. S. Martindale (see [11], corollary 1, p. 143) to obtain that every Jordan homomorphism from $S(H)$ into a real associative algebra A extends in a unique way to an associative homomorphism from $B(H)$ to A. Therefore we have an associative homomorphism $\Phi: B(H) \to B(K)$ extending F and, by the uniqueness of the extension, Φ is a *-homomorphism. Analogously, we find an associative homomorphism $\Psi: B(K) \to B(H)$ extending F^{-1}. Now $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are automorphisms of $B(H)$ and $B(K)$, respectively, extending the identity operators on their respective hermitian parts, hence, again by the uniqueness of the extensions, $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity operators on $B(H)$ and $B(K)$ respectively. It follows that Φ is a surjective *-isomorphism, and the proof is concluded by applying (2.1) □

2.3 Remark The reference ([11], cor. 1, p. 143) applied in the above proof covers also the 2-dimensional case whenever $\mathbb{F}$ is generated, as real associative algebra, by its hermitian part (namely by $\mathbb{R}\mathbf{1}$), a fact that occurs only for $\mathbb{F} = \mathbb{R}$. As a consequence, (2.2) remains valid for 2-dimensional Hilbert spaces H and K over $\mathbb{R}$. However it fails to be true for 2-dimensional spaces over $\mathbb{H}$ by ([11], exercise 2, p. 144).

In the case $\mathbb{F} = \mathbb{C}$, the following alternative proof of (2.2) can be given avoiding any restriction on dimensions. We have $B(L) = S(L) \oplus i\, S(L)$ for L equal to either H or K, hence we may extend F to a surjective complex-linear Jordan isomorphism $F_{\mathbb{C}}: B(H) \to B(K)$. By ([9], Cor. 7.4.9), $F_{\mathbb{C}}$ is either an isomorphism or an antiisomorphism. If $F_{\mathbb{C}}$ is an isomorphism, then by ([13],theorem 4.10.9) there is a surjective complex-linear isometry $U: B(H) \to B(K)$ satisfying $F_{\mathbb{C}}(x) = U \circ x \circ U^{-1}$ for all x in $B(H)$. If $F_{\mathbb{C}}$ is an antiisomorphism, then by ([14], lemma 3.6) there is a surjective conjugate-linear isometry $U: B(H) \to B(K)$ satisfying $F_{\mathbb{C}}(x) = U \circ x^* \circ U^{-1}$ for all x in $B(H)$. We note that the results derived above from references [13] and [14] can also be easily obtained from (2.1).

2.4 Remark The converse of (2.2) is true without any restriction on the dimensions, namely: If $\mathbb{F}$ denotes either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, if τ is an automorphism of $\mathbb{F}$, if H and K are Hilbert spaces over $\mathbb{F}$, and if $U: H \to K$ is a surjective real-linear isometry satisfying $U(\lambda h) = \tau(\lambda) U(h)$ for all λ in $\mathbb{F}$ and all h in H, then $U \circ x \circ U^{-1}$ belongs to $S(K)$ whenever x is in $S(H)$, and the mapping $x \mapsto U \circ x \circ U^{-1}$ from $S(H)$ into $S(K)$ is a surjective algebra isomorphism.

2.5 Corollary *Let H and K be Hilbert spaces over $\mathbb{F}$, and assume that $S(H)$ and $S(K)$ are linearly isometric. Then H and K are $\mathbb{F}$-linearly isometric.*

Proof By (1.5) we know that $S(H)$ and $S(K)$ are actually isomorphic. Now, except for some cases in low finite dimensions, where the result follows from a straightforward calculation, we may apply (2.2) to obtain that H and K are $\mathbb{R}$-linearly isometric, hence they have the same hilbertian dimension over $\mathbb{F}$. □

2.6 Remark If $\mathbb{F}$ denotes either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, if τ is an (automatically isometric) real-linear automorphism of $\mathbb{F}$, and if H and K are Hilbert spaces over $\mathbb{F}$ with the same hilbertian dimension, then there always exists a surjective real-linear isometry $U: H \to K$ satisfying $U(\lambda h) = \tau(\lambda)U(h)$ for all λ in $\mathbb{F}$ and all h in H. Indeed, if $\{h_i\}_{i\in I}$ and $\{k_i\}_{i\in I}$ are orthonormal bases in H and K, respectively, then the mapping $U: \sum \lambda_i h_i \mapsto \sum \tau(\lambda_i)k_i$ from H to K is such a surjective linear isometry.

2.7 Remark Let τ be a real-linear automorphism of $\mathbb{F}$. Of course, τ is the identity operator if $\mathbb{F} = \mathbb{R}$, whereas τ is the identity operator or the standard involution if $\mathbb{F}$ is $\mathbb{C}$. For the case $\mathbb{F} = \mathbb{H}$ it is known (see [7], p. 215) that, if we denote by S^3 the group of all unimodular elements in $\mathbb{H}$, if T is the group of all real-linear automorphism of $\mathbb{H}$, and if for μ in S^3 we write $\tau_\mu(\lambda) := \mu^{-1}\lambda\mu$ for all λ in $\mathbb{H}$, then the mapping $\mu \mapsto \tau_\mu$ from S^3 to T is a surjective group-homomorphism with kernel $\{1, -1\}$.

For a Hilbert space L over $\mathbb{F}$, we denote by $S_0(L)$ the JB-algebra of all selfadjoint compact $\mathbb{F}$-linear operators on L.

2.8 Corollary *Let H and K be left Hilbert spaces over $\mathbb{F}$, assume that H and K are not 2-dimensional over $\mathbb{F}$ whenever $\mathbb{F} = \mathbb{H}$, and let $F: S_0(H) \to S_0(K)$ be a surjective linear isometry. Then there are a number $\varepsilon \in \{1, -1\}$, a real-linear automorphism τ of $\mathbb{F}$, and a surjective real-linear isometry $U: H \to K$ with $U(\lambda h) = \tau(\lambda)U(h)$ for all λ in $\mathbb{F}$ and all h in H, such that $F(x) = \varepsilon U \circ x \circ U^{-1}$ for all x in $S_0(H)$.*

Proof For every left Hilbert space over $\mathbb{F}$ we have $S_0(L)^{**} = S(L)$, hence F^{**} is a linear isometry from $S(H)$ onto $S(K)$. The result follows by applying Corollary (1.5), Theorem (2.2) and Remarks (2.3). □

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