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BASIC SETS OF BRAUER CHARACTERS OF FINITE GROUPS OF LIE TYPE, III

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Let $G(\mathbb{F}_q)$ be a finite classical group where q is odd and the centre of G is connected. We show that there exists a set of irreducible characters of $G(\mathbb{F}_q)$ such that the corresponding matrix of scalar products with the characters of Kawanaka's generalized Gelfand-Graev representations is square unitriangular. This uses in an essential way Lusztig's theory of character sheaves. As an application we prove that there exists an ordinary basic set of 2-modular Brauer characters and that the decomposition matrix of the principal 2-block of $G(\mathbb{F}_q)$ has a lower unitriangular shape.

1. Introduction

1.1. Let G be a connected reductive group over $\mathbb{F}_q$, and let ℓ be a prime not dividing q. Recall that a basic set of Brauer characters is a set of Brauer characters which is linearly independent and such that every Brauer character of $G(\mathbb{F}_q)$ is an integral linear combination of the elements in that set. We say that a basic set is *ordinary* if it consists of the restrictions of ordinary characters of $G(\mathbb{F}_q)$ to the set of ℓ-regular elements. The main results of [GH], [G1] assert that ordinary basic sets always exist provided that ℓ is good for G and that ℓ does not divide the order of the group of components of the centre of G. In [GH] we already gave examples showing that the situation is different if ℓ is a bad prime. In this paper we study the case where G is of classical type with connected centre and $\ell = 2$, q is odd.

1.2. In Section 2 we will count the number of 2-modular Brauer characters in the various blocks of $G(\mathbb{F}_q)$. The proof uses the results of Broué and Michel [BM] on the block distribution of characters, as well as Broué's theory of perfect isometries [Br]. This counting argument also shows that it is sufficient to consider the problem of existence of basic sets for the unipotent blocks $\mathcal{B}_1$ only. In this case, it yields that the number of irreducible Brauer characters equals the number of unipotent classes of $G(\mathbb{F}_q)$. With each unipotent class in $G(\mathbb{F}_q)$ we associate a character of a projective representation, namely Kawanaka's generalized Gelfand-Graev representation (GGGR for short). In Theorem 2.5 and the remarks following it, we state our main result concerning the scalar products of ordinary characters in $\mathcal{B}_1$ with the characters of the GGGR's. This also yields the desired results about basic sets and the triangular shape of the decomposition matrix.

1.3. In [L6] Lusztig has expressed the character of a GGGR in terms of intersection cohomology complexes of closures of unipotent classes with coefficients in various local systems. In Section 3, we collect those results that we shall need. On the other hand, in [L3] the values of ordinary characters of $G(\mathbb{F}_q)$ at unipotent elements are expressed in terms of characteristic functions of character sheaves. (More generally, such an expression of the character values on any element of $G(\mathbb{F}_q)$, up to scalar multiple, is established by Shoji [S2], [S3].) This provides us with a method of computing scalar products between ordinary characters and characters of GGGR's. The next step then is to relate the ordinary characters of $G(\mathbb{F}_q)$ with the unipotent classes of G. This will be done, in Section 4, in terms of Lusztig's map from the set of special classes in the "dual" group to the set of unipotent classes in G. Finally, in Section 5, the main results can be proved.

We hope to be able to provide the analogous information for the exceptional groups by explicitly computing the necessary character values, in the framework of CHEVIE [Chv]. (Note that, in this case, there are several bad primes for each type but there are only finitely many cases to consider.)

1.4. The results in [L6] on GGGR's are proved under the assumption that G is split and that q is a sufficiently large power of a sufficiently large prime. Therefore, our results are also valid only under this condition. It is very likely, however, that they remain valid whenever q is a power of a good prime and with G being split or non-split.

Moreover, we use those properties of character sheaves concerning their restriction to the unipotent variety of G which are stated in [L1], (1.6), and for which a full proof is given only for G of type B_n.

The methods involved in the proofs of such results are of a quite different nature than the kind of arguments we use here. Therefore, we present the applications to questions about Brauer characters etc. here, taking those results for granted.

1.5. Let us assume that the group $G(\mathbb{F}_q)$ has a cuspidal unipotent character χ_0. If $G(\mathbb{F}_q) = GU_n(q)$ then, by [GHM], Theorem 6.10, the ℓ-modular reduction of χ_0 is an irreducible Brauer character (for any ℓ not dividing q). In order to prove this, we used Harish-Chandra induction of GGGR's and the concept of cuspidal unipotent classes. In [GM], we show that similar methods also work for the cases considered in this paper.

1.6. *Notations.* We always denote by ℓ a prime not dividing the prime power q. If a variety or an algebraic group G is defined over $\mathbb{F}_q$ we denote by F the corresponding Frobenius map and by G^F the set of fixed points.

By a function on a finite group H we always understand a function with values in a sufficiently large cyclotomic field contained in $\bar{\mathbb{Q}}_\ell$, and we denote by $x \mapsto \bar{x}$ an automorphism of that field which takes roots of unity to their inverses. If f, f' are two class functions on H we denote by

$$(f, f')_H := \frac{1}{|H|} \sum_{x \in H} f(x)\overline{f'(x)}$$

their scalar product. If G is an algebraic group over $\mathbb{F}_q$ and $H = G^F$ we also write $(f, f')_G$ instead of $(f, f')_{G^F}$. If $H' \subseteq H$ is an H-invariant subset and f a function on H' we will also regard f as a function on all of H by letting its values be zero outside H'.

2. Counting Brauer characters

2.1. Let G be a connected reductive group over $\mathbb{F}_q$ with a connected centre. Throughout this section, we assume that G has only simple components of type A_n, B_n, C_n or D_n, and that $\ell = 2$, q is odd. Blocks and Brauer characters will always be taken with respect to the prime $\ell = 2$.

2.2. The distribution of characters into blocks is compatible with the distribution into geometric conjugacy classes. In order to describe this, let G^* be a group over $\mathbb{F}_q$ dual to G. For a semisimple element $s \in G^{*F}$, let $\mathcal{E}(G^F, s)$ be the corresponding geometric conjugacy class of characters of G^F (cf. [BM]). Broué and Michel [BM] have shown that, given a semisimple element $s \in G^{*F}$ of order prime to 2, the set

$$\mathcal{B}_s := \bigcup_t \mathcal{E}(G^F, st),$$

where t runs over the set of 2-elements of G^{*F} commuting with s, is a union of blocks of G^F. (These results are valid for any connected reductive G, and ℓ any prime.)

We shall denote by m_s the number of irreducible Brauer characters in $\mathcal{B}_s$. If necessary, we shall also write $\mathcal{B}_s(G)$, $m_s(G)$ to indicate the underlying group. The blocks contained in $\mathcal{B}_1$ will be called the unipotent blocks of G^F.

An and, independently, Cabanes and Enguehard have shown that each set $\mathcal{B}_s$ is in fact a single block of G^F (see [CE] and the references there). We will not need this result here.

2.3. Let $L' := Z_{G^*}(s)$ be the centralizer of s in G^*. The fact that the centre of G is connected implies that L' is connected. Since 2 is the only bad prime for G, it follows that L' is a regular subgroup of G^*, that is, the Levi subgroup of a (not necessarily F-stable) parabolic subgroup of G^* (this is due to Schewe, cf. [GH], Lemma 2.2). In particular, L' has connected centre and root system with components A_n, B_n, C_n, or D_n. Let $L \subset G$ be a Levi subgroup dual to L' and $\hat{s}$ be the linear character of L^F dual to $s \in L^{*F}$. By a result of Broué [Br], the map

$$\rho \mapsto \varepsilon_G \varepsilon_L R_L^G(\hat{s} \cdot \rho)$$

defines a perfect isometry between $\mathcal{B}_1(L)$ and $\mathcal{B}_s(G)$. In particular, these two blocks have the same number of irreducible Brauer characters. Thus, using the notation introduced in (2.2), we have

$$m_s(G) = m_1(L) \quad \text{where } L \text{ is dual to } Z_{G^*}(s).$$

Furthermore, since restriction to the 2-regular elements commutes with twisted induction (see [BM], (3.5)) we see that, if $\mathcal{B}_1(L)$ has an ordinary basic set of Brauer characters then so does $\mathcal{B}_s(G)$.

Proposition 2.4. *With the assumptions in (2.1), the number $m_1(G)$ equals the number of unipotent classes of G^F.*

Proof. By induction we may assume that the proposition is true for all connected reductive groups of dimension strictly smaller than $\dim G$, defined over $\mathbb{F}_q$, with connected centre and root system with components of type A_n, B_n, C_n or D_n.

(a) Assume that G is simple adjoint having indecomposable root system of type A_n, B_n, C_n or D_n. If it has type A_n, that is, if $G = PGL_{n+1}$ then even $\ell = 2$ is a good prime and the result is covered by [GH], Theorem 5.1. Note that in this case, the unipotent characters give rise to an ordinary basic set for $\mathcal{B}_1$, and their number equals the number of partitions of $n+1$ which also is the number of unipotent classes. (The result in this case is also contained in [FS], (8A)). So we may assume that G has type B_n, C_n or D_n. Let $S_{2'}$ be a set of representatives of the conjugacy classes of semisimple $2'$-elements of G^{*F}. For each $s \in S_{2'}$, let $L'_s := Z_{G^*}(s)$ and $L_s \subset G$ a regular subgroup dual to L'_s. By (2.3) the blocks $\mathcal{B}_s(G)$ and $\mathcal{B}_1(L_s)$ have the same number of modular irreducible characters.

In our case, G^* has centre of order 1, 2, or 4. So, if $s \neq 1$ then s is non-central and, hence, $\dim L_s < \dim G$. By induction, $\mathcal{B}_1(L_s)$ contains N_s modular irreducible Brauer characters where N_s denotes the number of unipotent classes of L_s^F. So, in this case, $\mathcal{B}_s(G)$ contains N_s modular irreducible characters.

It is a well-known fact that the total number of irreducible Brauer characters equals the number of 2-regular classes. Combining this with the previous fact shows that the number of 2-regular classes of G^F equals

$$m_1(G) + \sum_s N_s,$$

where s runs over the elements in $S_{2'} \setminus \{1\}$. Now there is a natural bijection, $s \mapsto s^*$, between G^{*F}- and G^F-conjugacy classes of semisimple elements of odd order, such that there is an isomorphism $Z_G(s^*) \cong Z_{G^*}(s)^*$, defined over $\mathbb{F}_q$. (This follows by an argument entirely analogous to the proof of [GH], Proposition 4.2; instead of a single good prime, one considers a set of good primes.) Thus, we conclude that the piece $\sum_s N_s$ in the above sum equals the number of non-unipotent 2-regular classes of G^F. So we must have that $m_1(G)$ equals the number of unipotent classes of G^F.

(b) Assume that G is semisimple adjoint. Then there are two cases.

If $G = G_1 \times G_2$ where both G_1 and G_2 are semisimple adjoint, non-trivial and F-stable, then it is readily checked that the result holds for G^F if and only if it holds for G_1^F and G_2^F. Thus we are done by induction.

If no such factorization exists, then $G = G_1 \times \ldots \times G_n$ with simple adjoint factors G_i permuted cyclicly by F. In this case, we have $G^F \cong {G_1}^{F^n}$ and the result follows by applying (a) with G_1, F^n instead of G, F.

(c) Finally, we consider the general case. If the centre of G is trivial, then G is semisimple adjoint and we are done by case (b). Otherwise, let $\pi : G \to G_{ad}$ be the adjoint quotient of G. By induction, we have that $m_1(G_{ad})$ equals the number of unipotent classes of G_{ad}^F. Now, the passage from G^F to G_{ad}^F preserves unipotent classes. (Note that G has connected centre.) Thus, we are done if we can show that $m_1(G_{ad}) = m_1(G)$. Each irreducible Brauer character in $\mathcal{B}_1(G)$ is a constituent of the reduction of some unipotent character of G^F (see [H2], Theorem 2.9). Since the latter always have the centre in their kernel, the same holds for the Brauer characters in $\mathcal{B}_1(G)$. So we have a bijection between the modular irreducible characters in $\mathcal{B}_1(G)$ and $\mathcal{B}_1(G_{ad})$. This completes the proof. □

Theorem 2.5. *Recall that G has connected centre, that the simple components of G are of type A_n, B_n, C_n or D_n, and that $\ell = 2$, q is odd. Assume, moreover, that G satisfies the conditions mentioned in (1.4).*

(a) *There exists an ordinary basic set of Brauer characters for G^F.*
(b) *The decomposition matrix of the unipotent blocks of G^F has a lower unitriangular shape.*

In order to construct an ordinary basic set of Brauer characters for G^F it will be sufficient, by (2.3), to study the irreducible Brauer characters in $\mathcal{B}_1$. By Proposition 2.4, their number is precisely the number of unipotent classes of G^F. With each unipotent class of G^F there is associated a GGGR. Since these are representations induced from unipotent subgroups, they are projective representations. To proceed, we shall use the following result whose proof is straightforward and will be omitted (cf. [H1]).

Lemma 2.6. *Let $\mathcal{G}$ be a finite group and $\mathcal{B}$ a union of p-blocks of $\mathcal{G}$ (for some prime p) which contains precisely m irreducible Brauer characters. Assume that there exist ordinary characters $\rho_1, \ldots, \rho_m$ in $\mathcal{B}$ and characters $\Phi_1, \ldots, \Phi_m$ of projective modules for $\mathcal{G}$ such that the matrix of scalar products*

$$(\rho_i, \Phi_j)_{\mathcal{G}}, \quad 1 \leq i, j \leq m,$$

has determinant ± 1. Then the restrictions of the characters ρ_i to the p-regular elements of $\mathcal{G}$ form a basic set for $\mathcal{B}$. If, moreover, the above matrix is lower unitriangular, then the decomposition matrix of $\mathcal{B}$ has a lower triangular shape.

We apply this with the characters of the various GGGR's as the projective characters Φ_j, and all we need to do is find suitable ordinary irreducible characters in $\mathcal{B}_1$ which satisfy the conditions of Lemma 2.6. Note that these are statements about the ordinary characters of G^F.

3. On generalized Gelfand-Graev representations

3.1. Let G be a connected reductive group defined over $\mathbb{F}_q$. (We do not assume that the centre of G is connected.) Our aim in this section is to collect from [L6] some results on GGGR's in a form convenient for our purposes. We also restate some of the main results of [*loc. cit.*] in a slightly different form. This reformulation is not completely trivial, so we also include the proofs.

As in [L6], we have to assume that q is a sufficiently large power of a sufficiently large prime, and that G is split (cf. also (1.4)).

3.2. (Cf. [L2], (24.1), (24.2).) Let I be the set of all pairs $(C, \mathcal{E})$ where C is a unipotent class of G and $\mathcal{E}$ is an irreducible, G-equivariant local system on C (given up to isomorphism).

We call C the support of $i = (C, \mathcal{E}) \in I$.

For $i = (C, \mathcal{E})$, $i' = (C', \mathcal{E}')$ we write $i \leq i'$ if C is contained in the Zariski closure of C', and $i \sim i'$ if $C = C'$. We write $i < i'$ if $i \leq i'$ but $C \neq C'$. This defines a preorder on I.

The Frobenius map F acts naturally on I. To each pair $i = (C, \mathcal{E}) \in I^F$, we associate an integer s_i as follows (this will be the same as a_0 in [L2], (24.1.2)). The set I can be naturally partitioned into blocks which are permuted by F. With each block, there is associated a Levi subgroup L of a parabolic subgroup of G and a "cuspidal pair" for L (we shall not need to know this precisely.) Now, if i belongs to the block I_0 with associated Levi subgroup L we define, as in [L2], (24.1),

$$s_i := -\dim C - \dim Z_L^\circ \quad (\text{where } Z_L = \text{centre of } L).$$

(The use of this number will become clear later on.)

3.3. We denote by G_{uni} the unipotent variety of G. Let $i = (C, \mathcal{E}) \in I^F$, and fix an isomorphism $\psi : F^*\mathcal{E} \to \mathcal{E}$ which induces maps of finite order on the stalks at points in C^F. Let Y_i be the function on G_{uni}^F defined by $Y_i(x) = \text{Trace}(\psi, \mathcal{E}_x)$ for $x \in C^F$, and extending by zero outside C^F (see [L2], (24.2.2), (24.2.3)). For $i, i' \in I^F$ let $P_{i',i}$ be the integer determined by the algorithm described in [L2], (24.4), see also (24.10). We have $P_{i',i} = 0$ unless $i' \leq i$ and i, i' belong to the same block; moreover $P_{i',i} = \delta_{i,i'}$ (Kronecker delta) for $i \sim i'$. We define a function X_i on G_{uni}^F by

$$X_i = \sum_{i' \in I^F} P_{i',i} Y_{i'}.$$

Using the above properties of the numbers $P_{i',i}$ we see that $X_i = Y_i +$ linear combination of $Y_{i'}$ with $i' < i$.

The functions Y_i $(i \in I^F)$ and X_i $(i \in I^F)$ each form a basis of the space of G^F-invariant functions on G_{uni}^F. For $i, i' \in I^F$ we shall write

$$\lambda_{i,i'} := (Y_i, Y_{i'})_G \quad \text{and} \quad \omega_{i,i'} := (X_i, X_{i'})_G.$$

We then have

$$\omega_{i,i'} = \sum_{j,j' \in I^F} P_{j,i} P_{j',i'} \lambda_{j,j'}.$$

(Cf. also [L2], (24.3), (24.4), and note that the values of Y_i and X_i are cyclotomic integers, by [L2], (25.6.3), (25.6.4)). The matrix $(\omega_{i,j})$ is invertible; its inverse will be denoted by $(\tilde{\omega}_{i,j})$.

3.4. Let C be an F-stable unipotent class in G. For any element $u \in C$ we denote by $A(u)$ the group of components of the centralizer of u in G. Let $u_1, \ldots, u_d$ be representatives of the G^F-orbits on C^F. With each u_r there is associated a GGGR which we denote by Γ_{u_r}. (Usually, we will identify the GGGR with its character.) Now let $i = (C, \mathcal{E}) \in I^F$, and Y_i the corresponding function on G_{uni}^F as above. As in [L6], (7.5), we define

$$\Gamma_i = \sum_{r=1}^{d} (|A(u_1)|/|A(u_r)^F|) Y_i(u_r) \Gamma_{u_r}.$$

We call this function the twisted GGGR associated with $i \in I^F$. Using the orthogonality relations between the functions Y_i [*loc. cit.*] we see that giving the Γ_{u_r} is equivalent to giving the Γ_i.

In order to perform this transformation, we need to know the values $Y_i(u_r)$. For the following discussion, we refer to [S1], (5.1) and Proposition 5.2. Let us assume that C contains a split element. (This is possible if G is a classical group, for example.) We then choose notation so that this is the representative u_1. Let $1 = a_1, \dots, a_d$ be representatives of the conjugacy classes of $A(u_1)$, so that u_r is obtained from u_1 by twisting with a_r. Now there is a bijective correspondence, $i \mapsto \phi_i$, between those elements $i \in I^F$ with support C and the irreducible characters ϕ_i of $A(u_1)$, and there are choices of the isomorphisms $F^*\mathcal{E} \to \mathcal{E}$ used in the definition of Y_i such that

$$Y_i(u_r) = \phi_i(a_r) \quad \text{for all such } i \text{ and } r.$$

In the following, we denote by D_G the standard duality operation on the character ring of $G(\mathbb{F}_q)$. Recall, in particular, that this operation is an involution and self-adjoint with respect to the inner product introduced in (1.6). Moreover, D_G maps an irreducible character to an irreducible character, up to sign.

Lemma 3.5. *Let $i = (C, \mathcal{E}) \in I^F$ with associated integer s_i, and let Γ_i the corresponding twisted GGGR. Let $j \in I^F$. Then $(D_G(\Gamma_i), X_j)_G = 0$ if $i \not\leq j$ or if i, j belong to different blocks. If $i \sim j$ belong to the same block I_0 then*

$$(D_G(\Gamma_i), X_j)_G = \zeta' |A(u)| q^{(-\dim G - s_i)/2} \delta_{i,j},$$

where $u \in C$ and ζ' is a fourth root of unity which only depends on I_0.

Proof. Let I_0 be the block to which the given i belongs and L the associated Levi subgroup. Let a be the order of the group $A(u)$ for $u \in C$. At first, we shall express the dual of Γ_i in terms of the functions X_{i_1} for $i_1 \in I_0^F$. To simplify notation, we denote by C_j the unipotent class which is the support of the element $j \in I$.

In [L6], (7.5), Γ_i is written as a linear combination of the X_{i_1} ($i_1 \in I_0^F$), and in [L6], (8.3), we find an expression of $D_G(X_{i_1})$ as a scalar multiple of $X_{\hat{\imath}_1}$ for some $\hat{\imath}_1 \in I_0^F$ (cf. also [L6], (5.3)). Substituting the defining equation [L6], (6.7)(d), for $\tilde{\omega}_{i',\hat{\imath}_1}$ we obtain the following expression.

$$D_G(\Gamma_i) = \sum_{i', i_1 \in I_0^F} |G^F| q^{c(i')} \zeta^{-1} a \tilde{\omega}_{i',\hat{\imath}_1} P'_{i,i'} \delta X_{\hat{\imath}_1},$$

where $P'_{i,i'}$ is obtained from $P_{i,i'}$ by *changing q to q^{-1}* (cf. [L6], (6.5)), and

$$c(i') = (\dim Z_L^\circ - \dim G + 2 \dim C_{i'} - \dim C)/2.$$

Moreover, δ is a sign and ζ is a fourth root of unity, and both of them only depend on I_0 (see [L6], (7.2) and (8.4)(a)). (Note that, in the formula [L6], (7.5), the exponent of q depends on two indices; by the above rewriting the dependence of one of them disappears.) We set

$$\zeta' := \zeta^{-1}\delta.$$

Now we compute the scalar product of $D_G(\Gamma_i)$ with the function X_j where $j \in I^F$. If j does not belong to the block I_0 then the scalar product will be zero, by [L6], (6.5). So we may assume that $j \in I_0^F$. Using the formula for the scalar product between X_j and $X_{j'}$ in (3.3) above we obtain

$$\begin{aligned}(D_G(\Gamma_i), X_j)_G &= \sum_{i',i_1} \zeta' a q^{c(i')} P'_{i,i'} \tilde{\omega}_{i',\hat{\imath}_1} (X_{\hat{\imath}_1}, X_j)_G \\ &= \sum_{i'} \zeta' a q^{c(i')} P'_{i,i'} \left(\sum_{i_1} \tilde{\omega}_{i',\hat{\imath}_1} \omega_{\hat{\imath}_1,j} \right) \\ &= \sum_{i'} \zeta' a q^{c(i')} P'_{i,i'} \delta_{i',j} = \zeta' a q^{c(j)} P'_{i,j}.\end{aligned}$$

Let us assume that this is non-zero. Then we must have $P'_{i,j} \neq 0$. Using [L6], (6.5), we conclude that $i \leq j$. Now assume that $i \sim j$. Then $\dim C_i = \dim C_j$, and we obtain

$$(D_G(\Gamma_i), X_j)_G = \zeta' a q^{(-\dim G - s_i)/2} \delta_{i,j}.$$

This completes the proof. □

Corollary 3.6. *With the same notation as in Lemma 3.5 the following hold.*

(a) *The characters of the various GGGR's of G^F form a basis of the space of G^F-invariant functions on G^F_{uni}.*

(b) *Let C' be a unipotent class such that C_i is not contained in the (Zariski) closure of C'. Then $D_G(\Gamma_i)(y) = 0$ for all $y \in C'^F$.*

(The statement in (b) should be compared with [L6], Proposition 6.13.)

Proof. (a) We choose a total order on I which refines the given preorder $\leq$. Using Lemma 3.5 and the selfadjointness of D_G we see that the matrix of scalar products between the Γ_i and the $D_G(X_j)$ has a lower triangular shape with non-zero entries along the diagonal. Now the X_j form a basis of the space of G^F-invariant functions on G^F_{uni}. Since $D_G(X_j)$ is a non-zero scalar multiple of $X_{\hat{\jmath}}$ for some $\hat{\jmath}$, and the map $j \mapsto \hat{\jmath}$ is a bijection (see [L6], (5.4)), we conclude that the Γ_i form such a basis, too. Since every GGGR is a linear combination of the Γ_i, the proof is complete.

(b) Let $\{C'_r\}$ be the set of unipotent classes of G such that, for each r, the class C_i is not contained in the closure of C'_r. Let V be the space of all G^F-invariant functions on G^F_{uni} with support contained in $\bigcup_r C_r'^F$. Let

$I(V)$ be the set of all $j \in I^F$ which have as support one of the classes C'_r. Then the functions Y_j ($j \in I(V)$) form a basis of V (cf. (3.2)). The relation between the X_j and Y_j expressed in (3.2) shows that also $X_j \in V$ for $j \in I(V)$. Hence the functions X_j ($j \in I(V)$) are a basis of V. Consequently, we are done if we can show that $(D_G(\Gamma_i), X_j)_G = 0$ for all $j \in I(V)$.

Now C' is one of the classes C'_r. Let $j = (C', \mathcal{E}') \in I(V)^F$. Since C_i is not contained in the closure of C' we have $i \not\leq j$. Hence the proof is complete by Lemma 3.5. □

3.7. Let ρ' be any irreducible character of G, and let C be the unipotent support of ρ'. (We use the symbol ρ' for the irreducible character in order to keep consistency with [L6], Theorem 11.2.) According to [L6], C is the unique unipotent class in G of maximal possible dimension such that $\sum_{x \in C^F} \rho'(x) \neq 0$. Let ρ be the irreducible character of G^F such that $D_G(\rho) = \pm\rho'$. Then, by [L6], Theorem 11.2, the scalar product

$$(\Gamma_i, \rho)_G, \quad \text{where} \quad i = (C, \mathcal{E}) \in I^F,$$

is "small", that is, bounded above independently of q. In order to evaluate this scalar product explicitly, we use the results of this section in the following way. Using the self-adjointness of D_G we see that the above scalar product equals

$$(\Gamma_i, \pm D_G(\rho'))_G = \pm(D_G(\Gamma_i), \rho')_G.$$

Now we note that $D_G(\Gamma_i)(x)$ is zero unless $x \in G^F$ is unipotent. (This follows from the definition of D_G and the fact that Γ_i itself is zero outside G^F_{uni}.) Thus, it will be sufficient to consider the restriction of ρ' to G^F_{uni}. Since the functions $X_{i'}$ ($i' \in I^F$) form a basis for the space of all G^F-invariant functions on G^F_{uni} we can write uniquely

$$\rho'(x) = \sum_{i'} c_{i'} X_{i'} \quad (x \in G^F_{uni})$$

where the sum is over all $i' \in I^F$ such that $i' \sim i$ or the support of i' has strictly smaller dimension than the support of i (see (3.2), and recall the definition of unipotent support of ρ').

On the other hand, by Lemma 3.5, the scalar product of $D_G(\Gamma_i)$ with $X_{i'}$ is zero unless $i \leq i'$. We deduce that, in order to evaluate $(D_G(\Gamma_i), \rho')_G$, it will be sufficient to consider only those terms $c_{i'}X_{i'}$ where $i \sim i'$. Using once more Lemma 3.5 we see that the scalar product of $D_G(\Gamma_i)$ with such a $X_{i'}$ is zero unless $i' = i$. Thus, we finally have that

$$(D_G(\Gamma_i), \rho')_G = \zeta' a q^{(-\dim G - s_i)/2} \bar{c}_i.$$

We see that we have to study the restriction of ρ' to C^F where C is the unipotent support of ρ'. In the next section, we first study in general the relation between characters and unipotent classes.

4. Relating characters and unipotent classes

4.1. Let G be a connected reductive group over $\mathbb{F}_q$. We assume that the centre of G is connected and that q is a power of a good prime for G. Let $T \subset G$ be an F-stable maximal torus contained in some F-stable Borel subgroup B in G. Let (G^*, T^*) be a pair dual to (G, T). We will then identify the Weyl groups $W = N_G(T)/T$ and $W^* = N_{G^*}(T^*)/T^*$ of G and G^* respectively. We assume that the centre of G is connected in order to be sure that centralizers of semisimple elements in G^* are connected.

In his book [L1], Chapter 13, (see also [L6], §10) Lusztig has defined a canonical map Φ from the set of special conjugacy classes in G^* to the set of unipotent classes of G. This map is known to be surjective. In this section, we shall exhibit a set of distinguished special classes in G^* such that the restriction of Φ to this set is a bijection. (We refer to [L1], (4.1), (13.1), (13.2) for the definition of special representations of Weyl groups and special elements in G, G^*.)

With our applications in mind, we consider only groups G such that the root system of G is a sum of root systems of type A_n, B_n, C_n or D_n, and such that F leaves stable each simple component of G.

4.2. Recall the definition of the set I in (3.2). Now let E be an irreducible representation of W. By the Springer correspondence, there is a unique element $i \in I$ associated with E; we then write $E = E_i$. For example, if $i = (C, \bar{\mathbb{Q}}_\ell)$ and C is the class of the regular unipotent elements (respectively, the trivial class) then E_i is the trivial (respectively, the sign) representation.

Lusztig's map Φ is defined as follows (see [L1], (13.3)). Let $g \in G^*$ be a special element. We write $g = sv = vs$ with s semisimple and v unipotent, and denote by $Z(s)$ the centralizer of s in G^*. Let C' be the class of v in $Z(s)$ and $i' = (C', \bar{\mathbb{Q}}_\ell)$ (a pair in the set I' defined with respect to $Z(s)$). The corresponding representation $E_{i'}$ is a special representation of the Weyl group W_s of $Z(s)$. We then apply the j-operation and obtain an irreducible representation E of W^* (see [L1], (13.3)) which we can also regard as a representation of W. This is of the form $E = E_i$ for a pair $i = (C, \bar{\mathbb{Q}}_\ell) \in I$. The map Φ associates the class C with the class of g in G^*.

For the basic facts about the j-operation, we refer to [C3], §11.2. Note that this operation (as defined in [C3]) can be applied to special representations by [C3], Proposition 11.3.8.

4.3. Let C be a unipotent class in G. As in (3.4), we denote by $A(u)$ the group of components of the centralizer of an element $u \in C$. If u is special there is a canonical quotient $\bar{A}(u)$ of $A(u)$, defined in [L1], (13.1).

If $g \in G^*$ is special we define $\bar{A}(g)$ to be the group $\bar{A}(v)$ defined with respect to $Z(s)$ (where $g = sv = vs$ with s semisimple and v unipotent). Then, by [L1], p.346, we have that

$$|A(u)| = \sup_g |\bar{A}(g)|$$

where the supremum is over all special $g \in G^*$ with $\Phi(g) = (u)$. We wish to find a canonical choice for an element g where the supremum is taken.

4.4. The choice is motivated by the following fact. Let $(X^\vee, R^\vee, X, R)$ be the root datum of G with respect to $T \subset G$ and $\Delta^\vee \subset R^\vee$ a system of simple roots determined by B. Then $(X, R, X^\vee, R^\vee)$ is the root datum of G^*. (Note that we use the symbols R, X for G^*, and not for G, in order to simplify the notation in the sequel.) For any $\alpha \in \Delta$, we define

$$R(\alpha) := \langle (\Delta \setminus \{\alpha\}) \cup \{\alpha_0\} \rangle$$

where α_0 is the highest short root in R. This is a closed subsystem of R. Now let E' be a special representation of the Weyl group of such a subsystem. We apply the j-operation to it, and obtain an irreducible representation E of $W \cong W^*$. By Springer's correspondence, there is a corresponding unipotent class of G. It was shown by Lusztig (see [C3], p.388/389) that all unipotent classes of G are obtained in this way, for various α and various special representations E' of the Weyl group corresponding to $R(\alpha)$. We shall show that such pairs α, E' correspond to special elements in G^* for which the supremum in the formula in (4.3) is taken.

Proposition 4.5. *With the assumptions in (4.1), let $u \in G$ be a unipotent element. (Recall, in particular, that the simple components of G are of classical type.) Then there exists a special element $g = sv = vs \in G^{*F}$ (s semisimple, v unipotent) such that the following conditions are satisfied.*

(a) *We have $\Phi(g) = (u)$ and $|A(u)| = |\bar{A}(g)|$.*
(b) *The centralizer $Z(s)$ has the same semisimple rank as G^*.*
(c) *The group $Z(s)$ is maximally-split (i.e. $T^* \subseteq Z(s)$), and the order of s is a power of 2.*

Proof. We have noted in (4.3) that there exists some special element $g_1 = s_1 v_1 \in G^*$ such that $\Phi(g_1) = (u)$. Let $R_1 \subseteq R$ be the root system of $Z(s_1)$, and let E_1 be the special representation of the Weyl group W_1 of $Z(s_1)$ associated with v_1 by Springer's correspondence. Let E be the irreducible representation of W obtained by applying the j-operation to E_1.

(i) At first we claim that there exists a closed subsystem $R' \subseteq R$ such that R_1 is a parabolic subsystem of R' and R', R have equal rank, and that, moreover, R' occurs as the root system of a centralizer of a semisimple element in T^*. (Recall that R_1 is parabolic in R' if every system of simple roots of R_1 can be extended to a system of simple roots in R').

This can be seen as follows. Let $\bar{R}_1$ be the set of all rational linear combinations of elements in R_1 which are roots in R. Then $\bar{R}_1$ is a closed subsystem of R which is parabolic in R, by [Bo], Proposition VI.24. Now let $\bar{\Delta} \subset R$ be a system of simple roots in R which is obtained by extending a system of simple roots $\bar{\Delta}_1 \subset \bar{R}_1$. We then define R' to be the closed subsystem of R generated by the set Δ' which is the union of a system of simple roots of R_1 and the roots in $\bar{\Delta} \setminus \bar{\Delta}_1$. It is clear that R_1 is parabolic in R'. Moreover, since $R_1, \bar{R}_1$ have the same rank, the same holds for R', R. In order to prove that R' occurs as the root system of a centralizer of a semisimple element in T^* we use the results in [C1]. By [*loc. cit.*], Proposition 10, there exists a subgroup $Y_1 \subseteq X$ such that X/Y_1 has torsion subgroup which is cyclic of order prime to q and such that $Y_1 \cap R = R_1$. Let Δ' as before and define Y' to be the subgroup of X generated by Δ' and Y_1. Then it is readily checked that Y' has the analogous properties as Y_1 above. The proof of our claim is therefore completed by using once more [*loc. cit.*], Proposition 10.

(ii) For the arguments to follow it will be convenient to consider only the case that G^* is simple of simply-connected type (or, equivalently, that G is simple of adjoint type). This will be sufficient in order to prove the proposition. For, assume that G is semisimple of adjoint type. Then G^* is semisimple of simply-connected type, hence is a direct product of simple groups of simply-connected type. If the required semisimple elements exist for each factor, the product of these elements has the required properties in G^*. In the general case, we let $G \to G_{ad}$ be the adjoint quotient (note that G is still assumed to have connected centre). This induces an embedding $G^*_{sc} \to G^*$. Moreover, the centralizer in G^* of a semisimple element in G^*_{sc} is the product of the centralizer in G^*_{sc} and the centre of G^*.

Now let G be simple of adjoint type A_n, B_n, C_n or D_n. We have seen in (i) that R' occurs as the root system of the centralizer of a semisimple element $s \in T^*$. This element will be a candidate for the semisimple part of the required special element $g \in G^*$. Since R', R have the same rank, Table 1 on p.15 in [C2] shows that these root systems must have types given as follows. (Note that, by [C2], a root system of type D_1

corresponds to a group of type A_1.)

Type of R	type of R'	comment
A_n	A_n	$R = R'$ necessarily
C_n	$C_a + C_b$	$a + b = n \geq 2$
B_n	$D_a + B_b$	$a + b = n \geq 3$
D_n	$D_a + D_b$	$a + b = n \geq 4$

The types occurring in the above table imply that the order of any such element $s \in T^*$ is a power of 2.

(iii) According to [C2], Table 2, we can choose the element $s \in T^*$ in (ii) even to be fixed by F, provided that q is sufficiently large. (In the notation of [C2], we choose the "twisting element" $w = 1$; the shape of R' shows that the "critical subgroups" do not occur.)

Now, following the proof of [C1], Proposition 19, in our special situation, we see that, in fact, no condition on q is needed (except for being odd). (In the notation of [C1], p.505/506, we have $\Phi = \bar{\Phi}_1$ and hence $|\Gamma^*| - \bigcup_r |E_r^*| \geq (1/m)|\Gamma^*| > 0$, without condition on q.)

The element $s \in T^{*F}$ will be the semsimple part of the required special element g. The arguments at the end of (ii) show that conditions (b) and (c) in the proposition are satisfied. It remains to construct the unipotent part v of the required element g, and to verify condition (a).

(iv) Let W' be the Weyl group of R'. By construction we have $W_1 \subseteq W' \subseteq W$. We apply the j-operation (with respect to $W_1 \subseteq W'$) to E_1 and obtain an irreducible representation E' of W'. We have seen in (i) that W_1 is a parabolic subgroup of W'. Hence, by [C3], Proposition 11.3.11, the representation E' is special. Using the transitivity of the j-operation (see [C3], Proposition 11.2.4) we find that

$$E = j_{W'}^{W}(E'),$$

where E' is special for W' and W', W have the same rank. Let $D_{E'}(t)$ (t an indeterminate) be the generic degree of E'. As usual (see [L1], (4.1)) we define integers $a(E') \geq 0$ and $f(E') > 0$ by the requirement that

$$D_{E'}(t) = f(E')^{-1} t^{a(E')} + \text{ higher powers of } t.$$

Analogously, let $f(E_1)$ be the number defined in terms of the generic degree for the representation E_1 of W_1. We claim that

$$f(E') \geq f(E_1).$$

But this is true since W_1 is parabolic in W' and E_1, E' are special, see [C3], Proposition 11.3.3.

(v) The representation E' in (iv) corresponds to a special unipotent element $v \in Z(s)$ and, by the definitions, $\Phi(g) = (u)$ where $g = sv$. We claim that $|\bar{A}(g)| = |A(u)|$. This can be seen as follows. We know already that $|\bar{A}(g)| \leq |A(u)| = |\bar{A}(g_1)|$. So it will be sufficient to show $|\bar{A}(g)| \geq |\bar{A}(g_1)|$. Combining [L1], (13.1.3) and (4.14.2), we see that the order of $\bar{A}(g)$ equals $f(E')$. (See (iv) and note that the generic degrees of two representations obtained from each other by tensoring with the sign representation are equal up to a power of t; see [CR], Theorem 71.14.) An analogous statement holds for $\bar{A}(g_1)$ and the number $f(E_1)$ defined in terms of the generic degree of the representation E_1 of W_1. Hence we are reduced to the assertion that $f(E') \geq f(E_1)$, and this was proved in (iv).

(vi) Finally, we note that since $s \in T^{*F}$ the group $Z(s)$ is maximally-split. Hence the unipotent element $v \in Z(s)$ can be chosen to be fixed by F, too. This completes the proof. □

A statement similar to Proposition 4.5 also plays a role in Lusztig's work on cells in affine Weyl groups, [L5], §6. We remark that the results obtained so far in this section depend to some extend on the explicit case-by-case description of Springer's correspondence (see the remarks in [L1], (13.1)). This seems to be the case, in particular, for the statement in (4.3). The proof of Proposition 5.4 below will show that it can be reformulated in terms of a certain "multiplicity 1" property for GGGR's.

4.6. Let us briefly recall the parametrization [L1], (4.23), of the set $\mathrm{Irr}(G^F)$ of irreducible characters of G^F. At first, we have a partition

$$\mathrm{Irr}(G^F) = \coprod_s \mathcal{E}(G^F, s),$$

where s runs over a set of representatives of the conjugacy classes of semisimple elements in G^* and $\mathcal{E}(G^F, s)$ denotes the corresponding geometric class of characters (see [L1], (8.4.4)). We fix a semisimple element $s \in G^{*F}$ and assume that

(a) *the centralizer $Z(s)$ is a split group.*

Let W' be the Weyl group of $Z(s)$. The irreducible representations of W' are partitioned into families $\mathcal{F}$ (see [L1], (4.2)). Moreover, with each such family $\mathcal{F}$ there is associated a finite group $\mathcal{G}_{\mathcal{F}}$ and an embedding of $\mathcal{F}$ into the finite set $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$ (see [L1], (4.14), and (4.3) for the definition of that set). By our assumption (a), the statement of [L1], (4.23), simplifies considerably, and we obtain a bijection

(b) $\mathcal{E}(G^F, s) \leftrightarrow \coprod_{\mathcal{F}} \mathcal{M}(\mathcal{G}_{\mathcal{F}})$,

where $\mathcal{F}$ runs over the families of irreducibles representations of W'.

There is a bijective correspondence between families in W' and special unipotent classes in $Z(s)$. We twist this bijection with tensoring by the sign representation. Then the following hold (see [L1], (13.1.3)). Let $C' \subset Z(s)$ be a special unipotent class such that the representation $E' \otimes$ sign of W' lies in the family $\mathcal{F}$, where E' is the Springer representation corresponding to the pair $(C', \bar{\mathbb{Q}}_\ell)$. Let $v \in C'$ and $g = sv$ (a special element in G^*). Then the group $\bar{A}(g)$ is isomorphic to $\mathcal{G}_{\mathcal{F}}$.

The duality operation leaves invariant each set $\mathcal{E}(G^F, s)$. Moreover, if the irreducible character ρ corresponds to an element in the set associated with the family $\mathcal{F}$ of W' then $\pm D_G(\rho)$ corresponds to an element in the set associated with the family $\{E' \otimes \text{sign} \mid E' \in \mathcal{F}\}$. We can now reformulate (b) in the following way (always assuming (a)). There is a bijection

(c) $$\mathcal{E}(G^F, s) \to \coprod_g \mathcal{M}(\bar{A}(g)),$$

where g runs over a set of representatives of the conjugacy classes of special elements in G^* with semisimple part conjugate to s (cf. [L1], (13.2.1)). Moreover, $\rho \in \mathcal{E}(G^F, s)$ corresponds to an element in $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$ if and only if $\rho' = \pm D_G(\rho)$ corresponds to an element in $\mathcal{M}(\bar{A}(g))$. Finally, in this case, $C := \Phi(g)$ is the unipotent class associated with ρ by the method described in [L6], (11.1). So, [L6], Theorem 11.2, shows that

(d) *C is the unipotent support of the character ρ'.*

This gives us the relation with the considerations in (3.7).

4.7. Let $u \in G$ be unipotent and $g = sv = vs \in G^{*F}$ a special element as in Proposition 4.5. Then we have that

$$\mathcal{M}(\bar{A}(g)) = \mathcal{M}(A(u)).$$

This, in addition to the fact that we consider a group G of classical type, will be the crucial point for the subsequent results.

Namely, let C be the class of u. As pointed out in [S1], (5.1), we can choose u to be a split element. Then F acts trivially on $A(u)$. Moreover, $A(u)$ is an elementary abelian 2-group of order d, say (see [C3], §13.3). The definition of $\mathcal{M}(A(u))$ then shows that this is a set of cardinality d^2, and the corresponding Fourier coefficients are of the form $\pm 1/d$. (See [L1], (4.14.3), for the definition of the Fourier matrix.) Thus, the Fourier coefficients corresponding to the set $\mathcal{M}(\bar{A}(g))$ are given in terms of the unipotent element u and the group $A(u)$. (And this would not hold for any special element in the preimage of the unipotent class (u) under the map Φ.)

5. Restricting character sheaves to unipotent classes

5.1. We keep the assumptions on G and q as in the previous section. In particular, the root system of G is a sum of root systems of type A_n, B_n, C_n or D_n, and F leaves stable each simple component of G. Let $T \subset G$ be a split torus, and G^*, T^* a pair dual to G, T.

Let $\hat{G}$ be the set of character sheaves on G. These are certain irreducible G-equivariant ℓ-adic perverse sheaves on G. (See [L4] for an exposition of the theory.) Again, we will have to use the results in [L6] concerning the restriction of character sheaves to the unipotent variety of G. Whenever necessary, we will then also assume that q is a sufficiently large power of a sufficiently large prime.

5.2. A classification of the character sheaves of G is given in [L2], (17.8.3). With each character sheaf, there is associated a certain local system on T and a two-sided cell in its stabilizer in W. These local systems on T are in bijective correspondence with semisimple elements in G^*. Thus, we have a partition

$$\hat{G} = \coprod_s \hat{G}(s)$$

where s runs over a system of representatives for the semisimple classes in G^*. Furthermore, the two-sided cells are in bijective correspondence with the families in a Weyl group. Thus, we have a bijection

$$\hat{G}(s) \leftrightarrow \coprod_{\mathcal{F}} \hat{G}(s, \mathcal{F})$$

where $\mathcal{F}$ runs over the families in the Weyl group W' of $Z(s)$. By [L2], (17.8.3), the set $\hat{G}(s, \mathcal{F})$ can be parametrized by the finite set $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$. We see that character sheaves of G are classified in a similar way as irreducible characters of G^F were classified in [L1], (4.23).

The Frobenius map F acts on $\hat{G}$. If $A \in \hat{G}$ is F-stable, we always assume that an isomorphism $\phi : F^*A \to A$ is chosen as in [L2], (25.1). The corresponding characteristic function χ_A is a G^F-invariant function on G^F such that $(\chi_A, \chi_A)_G = 1$ (cf. the convention in (1.6)). Note that the choice of such an isomorphism is only determined up to multiplication by a root of unity.

If, in the above classification, the element $s \in G^*$ is F-stable and its centralizer is a split group, then all $A \in \hat{G}(s)$ are F-stable.

5.3. Let us fix a unipotent class $C = (u)$ in G, and let $g = sv = vs \in G^*$ be a special element as in Proposition 4.5. To simplify notation, we let

$$\mathcal{M} := \mathcal{M}(\mathcal{G}_{\mathcal{F}}).$$

Now note that, if $\mathcal{F}$, $\mathcal{F}'$ are two families of a Weyl group which are obtained from each other by tensoring with the sign representation then the corresponding sets $\mathcal{M}$ and $\mathcal{M}'$ have the same cardinality. Using (4.6), (4.7) we therefore have

$$|\mathcal{M}| = |\mathcal{M}'| = |\mathcal{M}(A(u))|.$$

For any $m \in \mathcal{M}$, we denote by A_m the corresponding character sheaf, and by χ_m the characteristic function corresponding to A_m and some choice of isomorphism $F^*A_m \to A_m$ as above. On the other hand, we also have an associated irreducible character ρ_m of G^F, defined by the correspondence in (4.6)(b).

Shoji proves in [S2], [S3] (see also [L7]) that for each χ_m there exists an algebraic number μ_m of absolute value 1 such that $\mu_m \chi_m$ is equal to an almost character in the set $\mathcal{M}$, that is, it can be expressed as a linear combination of the characters $\rho_{m'}$ (for various $m' \in \mathcal{M}$) where the coefficients are the entries of the Fourier transform matrix of $\mathcal{M}$. Combining the previous remarks, we obtain the following formula.

$$(\chi_m, \rho_{m'})_G = \frac{\pm\mu_m}{|A(u)|}, \quad \text{where } m, m' \in \mathcal{M} \quad (\text{or } m, m' \in \mathcal{M}').$$

Proposition 5.4. *With the notation in (5.3), let ρ be an irreducible character of G^F such that $\pm D_G(\rho)$ corresponds to an element in the set $\mathcal{M}$. Then there exists a unique (up to conjugation in G^F) element $u \in C^F$ such that*

$$(\Gamma_u, \rho)_G = 1 \quad \textit{and} \quad (\Gamma_y, \rho)_G = 0$$

for all $y \in C^F$ which are not conjugate to u in G^F.

Proof. Let $u_1, \ldots, u_d \in C^F$ as in (3.4) with u_1 split. We consider the element $i := (C, \bar{\mathbb{Q}}_\ell) \in I^F$. By (3.4), we have $Y_i(u_r) = 1$ for all r. Moreover, $|A(u_r)^F|$ divides $|A(u_1)|$ for all r. (In fact, the cardinalities are equal, but this will turn out automatically.) Hence, the corresponding twisted GGGR

$$\Gamma_i = \sum_{r=1}^{d} (|A(u_1)|/|A(u_r)^F|)\Gamma_{u_r}$$

is an ordinary representation. (This is the same set-up as in the proof of [L6], Theorem 11.2.) It will therefore be sufficient to prove that

$$(\Gamma_i, \rho)_G = 1.$$

But this follows from the proof of [L6], Theorem 11.2. For, let n be the multiplicity of ρ in Γ_i. The arguments up to equation (h) in [*loc. cit.*] show that n is $|A(u)|$ times the scalar product of ρ with the characteristic function $\chi_{m'}$ for some $m' \in \mathcal{M}'$ (notation as in (5.3)). Now, using the formula in (5.3) shows that n must be an algebraic number of absolute value 1. Also being an integer, we conclude that it must be 1. □

5.5. We keep the notation of (5.3). In [L4], (4.6), it is stated that the class C has the property that there exists some $m \in \mathcal{M}$ such that $C \subset \operatorname{supp} A_m$ (support of A_m) and that if C' is a unipotent class in G such that $C' \subset \operatorname{supp} A_{m'}$ for some $m' \in \mathcal{M}$ then $\dim C' \leq \dim C$. (A proof can be extracted from [L6], Theorem 10.7. The fact that there exists some $m \in \mathcal{M}$ with non-trivial restriction to C is expressed in equation (g) in the proof of [*loc. cit.*], (10.7).) We therefore call C the unipotent support of the character sheaves in $\mathcal{M}$.

To proceed we need some more precise information about the restriction of the character sheaves in $\mathcal{M}$ to the unipotent class C.

We shall denote by $\mathcal{X}$ the set of those $m \in \mathcal{M}$ such that the restriction of A_m to C is non-zero. Let us consider the following properties of the elements in $\mathcal{X}$.

(a) If $m \in \mathcal{X}$, the restriction of A_m to C is (up to shift) an irreducible, G-equivariant local system $\mathcal{E}_m$ on C. Thus, we have a map $\mathcal{X} \to I$, $m \mapsto i_m := (C, \mathcal{E}_m)$.

(b) The map $m \mapsto i_m$ in (a) is injective.

(c) The map $\varphi : \mathcal{M} \to \mathbb{Z}$, defined by $\varphi(m) = \operatorname{rank}(\mathcal{E}_m)$ if $m \in \mathcal{X}$, and $\varphi(m) = 0$ otherwise, is a Lagragian subspace in the sense of [L1], (12.10).

It is stated in [L3], (1.6), that these conditions are always satisfied. (In fact, Lusztig has pointed out to me that this will only be true for families corresponding to special elements as in Proposition 4.5). This is proved explicitly in [*loc. cit.*] for groups of type B_n. In the following, we shall assume that (a), (b), (c) hold for all special elements $g \in G^*$ as in Proposition 4.5.

If the conjugacy class of g in G^* is F-stable and the character sheaf $A_m \in \mathcal{X}$ is F-stable then we also have $i_m \in I^F$.

Proposition 5.6. *Let $u \in C$ be a split element, and $u = u_1, \dots, u_d \in C^F$ as in (3.4). We assume that the special element $g = sv \in G^*$ is chosen as in Proposition 4.5. Then there exist irreducible characters $\rho_1, \dots, \rho_d \in \mathcal{E}(G^F, s)$ such that the matrix of scalar products*

$$(\Gamma_{u_{r'}}, \rho_r)_G, \quad 1 \leq r', r \leq d,$$

is the identity matrix.

Proof. (a) At first we note that the set $\mathcal{X}$ in (5.5) has cardinality d. For, the group $\mathcal{M}$ is an $\mathbb{F}_2$-vector space of dimension $2d'$ where $d = 2^{d'}$, by the choice of the special element g. By (5.5)(c) and the definition of a

Lagrangian subspace we see that $\mathcal{X}$ is a subspace of $\mathcal{M}$ of dimension d'. This proves the claim.

Let us write $\mathcal{X} = \{m_r \mid 1 \leq r \leq d\}$. For each r, let $i_r := i_{m_r}$ be the corresponding element in I^F, see (5.5)(a).

(b) We claim that the matrix of scalar products

$$(D_G(\Gamma_{i_{r'}}), \chi_{m_r})_G, \quad 1 \leq r', r \leq d,$$

is diagonal and invertible. This can be seen as follows. In order to evaluate the scalar product we only need to know the restriction of χ_{m_r} to C^F. This follows from Corollary 3.6(b) and the characterization of C in (5.5) (see the analogous arguments in (3.7)). By (5.5)(a) and (3.3) the restriction of χ_{m_r} to C^F equals the restriction of X_{i_r} to C^F times a non-zero constant. Thus, it will be sufficient to consider the scalar products

$$(D_G(\Gamma_{i_{r'}}), X_{i_r})_G, \quad 1 \leq r', r \leq d.$$

The assertion now follows from Lemma 3.5 and the fact that all i_r are distinct, see (5.5)(b).

(c) Now each $D_G(\Gamma_{i_{r'}})$ can be expressed as a linear combination of the duals of the GGGR's associated with the class C (see (3.4)), and each χ_{m_r} is a linear combination of the irreducible characters corresponding to the set $\mathcal{M}$ (see (5.3)). It follows from (b) that the matrix of scalar products of the duals of these GGGR's with these irreducible characters has full rank. Thus, there exist irreducible characters $\rho'_1, \dots, \rho'_d$ parametrized by elements in the set $\mathcal{M}$ such that the matrix of scalar products

$$(D_G(\Gamma_{u_{r'}}), \rho'_r)_G, \quad 1 \leq r', r \leq d,$$

is invertible. Now let us write $\rho_r = \pm D_G(\rho'_r)$, for $r = 1, \dots, d$. Then these characters satisfy the assumption of Proposition 5.4. Using the fact that D_G is an involutory isometry we conclude that the matrix

$$(\Gamma_{u_{r'}}, \rho_r)_G, \quad 1 \leq r', r \leq d,$$

is invertible. On the other hand, by Proposition 5.4, each column of this matrix contains precisely one non-zero entry, and this entry is 1. It follows that the above matrix of scalar products is a permutation matrix. Relabelling the ρ_r gives the desired result. □

Note that the statement in Proposition 5.6 does not involve character sheaves. These are used in an essential way in the proof. I don't know if a more elementary proof exists.

5.7. We can now complete the proof of Theorem 2.5, as follows. Let C_n $(1 \leq n \leq N)$ be the F-stable unipotent classes of G, ordered such that $\dim C_n < \dim C_{n'}$ implies $n < n'$. For each n, let $u_{n,r}$ $(1 \leq r \leq d_n)$

be representatives of the G^F-orbits on C_n^F, and let $\rho_{n,r}$ $(1 \leq r \leq d_n)$ be the irreducible characters of G^F provided by Proposition 5.6.

At first we note that all $\rho_{n,r}$ lie in $\mathcal{B}_1$. This follows from Proposition 4.5(c) and the facts collected in (4.6) which mean that these characters lie in geometric conjugacy classes of the form $\mathcal{E}(G^F, s)$, for various semisimple elements whose order is a power of 2.

Furthermore, Proposition 5.6 and the definition of unipotent support and [L6], Theorem 11.2, show that

$$(\rho_{n,r}, \Gamma_{u_{n',r'}})_G = \begin{cases} 1 & , \text{if } n = n', \ r = r' \\ 0 & , \text{if } n < n' \text{ or } n = n' \text{ and } r \neq r'. \end{cases}$$

Ordering the pairs (n, r) lexicographically yields that the corresponding matrix of scalar products has a block lower unitriangular shape, with one block for each unipotent class C_n. Moreover, the diagonal blocks are identity matrices of sizes $d_1, \ldots, d_N$.

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