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Extended Dyer-Lashof algebras and modular coinvariants

Dedicated to the memory of Jose Adem

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Given an increasing sequence of integers $N = (0, n_1, n_2, \ldots)$, a functor G_N is constructed from the category $\Im$ of based spaces of the homotopy type of CW compleces and based maps to a subcategory $\Im_N$ of $\Im$ in analogy to May's approximation model C. A family of homology operations RN is associated to G_N and its algebraic structure is described in terms of modular coinvariants of parabolic subgroups.

0. Introduction

The Dyer-Lashof algebra R, the algebra of homology operations on QX ($= \lim \Omega^n \Sigma^n X$), being an invariant on the category of infinite loop spaces, was studied extensively in '70 s; its relation with modular invariants was realized around 1975 and described in the early '80 s.

The space QX can be approximated by May's model CX which is constructed using the symmetric groups Σ_n and its homology can easily be described using modular coinvariants. The main advantage of CX is its simpler structure compared with the function space QX. In particular, CS^0 is built successively of $B\Sigma_n$'s and nothing else. Moreover $B\Sigma_\infty$ is homologically equivalently to $(QS^0)_0$, for S^0 the zero sphere.

Since the symmetric group is an important object in algebraic topology, it is interesting to investigate the analogous properties of certain of its subgroups and this is our project. In [15], we defined homology operations on certain families of topological spaces, $G \int X$, for G certain subgroups of Σ_∞. As a main tool, modular coinvariants were used eliminating the complication of Adem relations and questions concerning operations can be reduced to relations between subrings of rings of invariants. It turns out that subrings of rings of invariants of certain families of subgroups of $GL_n(\mathbf{Z}/p\mathbf{Z})$, for $n = 1, 2, \ldots$, induce families of operations in homology on certain spaces, because of their relation with certain families of subgroups of symmetric groups. In this work, we construct certain spaces $G_N X$ related to $G \int X$ above in analogy to May's models CS^0 and $B\Sigma_\infty$. We note here that the difference between CX and $G_N X$ is that $G_N X$ is associated to a family of compatible permutation subgroups, $S = \{G_n \mid \Sigma_{p^n,p} := E_1 \int \cdots \int E_n \leq G_n \leq \Sigma_{p^n},\ n = 1,2,\dots\}$ instead of Σ_m itself, for $m \geq 0$. Here Σ_{p^n} is the symmetric group of all permutations of all elements of V^n, an n dimensional vector space over $\mathbf{Z}/p\mathbf{Z}$ for p a prime number, E_i is the subgroup of the i-th translations, and $\Sigma_{p^n,p}$ is a fixed p-Sylow subgroup of Σ_{p^n}. The basic properties of the new space are studied and homology operations are defined in theorem (1.12). This family of operations admits the structure of an algebra, denoted RN and called **the extended Dyer-Lashof algebra**. The difference between RN and R is that Adem relations are not allowed at certain positions. We examine its algebraic structure and show that its dual is closely related to the rings of invariants of various parabolic subgroups of $GL_n(\mathbf{Z}/p\mathbf{Z})$, theorem (2.22). At the end, we calculate the mod-p structure of $H_*(G_N X)$ which is free with basis a fixed homogeneous basis of $H_*(X,\ \mathbf{Z}/p\mathbf{Z})$ over RN and therefore a modular coinvariant theoretic description can be given, theorem (3.1).

Let $E(x_1, \dots, x_n)$ be an anticommutative exterior algebra on n generators and $P[y_1, \dots, y_n]$ a polynomial algebra on n generators over $\mathbf{Z}/p\mathbf{Z}$ both graded with degrees: $|x_i| = 1$ and $|y_i| = 2$ for $i = 1, \dots, n$. Then $E(x_1, \dots, x_n) \otimes P[y_1, \dots, y_n] \equiv H^*(V^n, \mathbf{Z}/p\mathbf{Z})$ as modules over the Steenrod algebra and there is a natural action of the subgroup G on this algebra, where G is one of the following groups: $U_n \leq B_n \leq P_n(N) \leq GL_n$, (the group of upper triangular matrices with one's along the main diagonal, the Borel subgroup, the parabolic subgroup associated to a sequence of positive integers, and the general linear group, respectively). In section 4 we calculate the ring of invariants $(E(x_1, \dots, x_n) \otimes P[y_1, \dots, y_n])^G$ for G as above.

Sections 2 and 4 are revised forms of the algebra chapter of my Ph. D. thesis [13]. We note that this is a generalization of part of May' work in [8] mainly and later by Huynh [12] and Campbell [4] concerning the modular coinvariants. I would like to thank J. P. May for his encouragement and advice on constructing these families of spaces in section 1, my supervisor H.E.A. Campbell, coadvisor J.McCleary, and S. Kochman for their help and support.

1. Definitions-Notation

Let $(BG,\ EG,\ \pi_G,\ G)$ be the principal G-bundle associated with the group G such that EG is a contractible space. We recall that the functors E and B preserve products. The topological wreath product between G and H is defined by $EG \times_G (BH)^n$, where $G \leq \Sigma_n$ and the definition above is extended to any topological space X: $G \int X = EG \times_G X^n$, (see [18]).

Since maps between subgroups of symmetric groups are important in this work, we shall establish the notation needed. As usual Σ_{p^n} acts on $\mathbf{n} := \{1, \dots, p^n\}$. Let I be a strictly increasing sequence of $\mathbf{k}$ elements from $\mathbf{k+1}$.

(1.1) Let i_I be the monotonic increasing embedding of $\mathbf{k}$ in $\mathbf{k+1}$ described by $I = (i_1, \dots, i_{p^k})$, i.e. $i_I(m) = i_m$. We also use the same symbol i_I for the induced inclusion $i_I : \Sigma_{p^k} \longrightarrow \Sigma_{p^{k+1}}$ which makes the following diagram commutative, and the induced map between $E\Sigma_{p^k}$ and $E\Sigma_{p^{k+1}}$.

$$\begin{array}{ccc} \mathbf{k} & \xrightarrow{i_I} & \mathbf{k+1} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle i_I(\alpha)} \\ \mathbf{k} & \xrightarrow{i_I} & \mathbf{k+1} \end{array}$$

(1.2) Let $\sigma_I : \Sigma_{p^{k+1}} \longrightarrow \Sigma_{p^k}$ be the map associated with i_I which makes the following diagram commutative

$$\begin{array}{ccc} \mathbf{k} & \xrightarrow{i_I} & \mathbf{k+1} \\ \downarrow{\scriptstyle \sigma_I(\alpha)} & & \downarrow{\scriptstyle \alpha} \\ \mathbf{k} & \xrightarrow{i_J} & \mathbf{k+1} \end{array}$$

Here $\alpha \in \Sigma_{p^{k+1}}$ and J is the sequence $\alpha(I)$ obtained by reordering the sequence $(\alpha(i_1), \ldots, \alpha(i_{p^k}))$.

We extend the definition above to the cases: a) from $G_n \int \Sigma_{p^k}$ to $G_n \int \Sigma_{p^{k+1}}$; and b) from $\Sigma_{p^k} \int G_n$ to $\Sigma_{p^{k+1}} \int G_n$.

(1.3) a) Let $\alpha = (\beta; \delta_1, \ldots, \delta_{p^n}) \in G_n \int \Sigma_{p^k}$, then the map $i_I : \Sigma_{p^k} \longrightarrow \Sigma_{p^{k+1}}$ above induces a map (the same notation is used) $i_I : G_n \int \Sigma_{p^k} \longrightarrow G_n \int \Sigma_{p^{k+1}}$ given by $i_I(\alpha) = (\beta; i_I(\delta_1), \ldots, i_I(\delta_{p^n}))$. Respectively, the map $\sigma_I : \Sigma_{p^{k+1}} \longrightarrow \Sigma_{p^k}$ above induces a map (the same notation is also used) $\sigma_I : G_n \int \Sigma_{p^{k+1}} \longrightarrow G_n \int \Sigma_{p^k}$ by $\sigma_I(\beta; \delta_1, \ldots, \delta_{p^n}) = (\beta; \sigma_I(\delta_{i_1}), \ldots, \sigma_I(\delta_{i_{p^n}}))$. Here $\alpha = (\beta; \delta_1, \ldots, \delta_{p^n}) \in G_n \int \Sigma_{p^k}$ acts on (s,t) by $(\beta(s), \delta_s(t))$, where $\{(s,t) | \ s = 1, \ldots, p^n \text{and } t = 1, \ldots, p^{k+1}\}$ is given the lexicographic ordering.

(1.4) b) Let $\alpha = (\beta; \delta_1, \ldots, \delta_{p^n}) \in G_n \int \Sigma_{p^{k+1}}$ then the map $i_I : \Sigma_{p^k} \longrightarrow \Sigma_{p^{k+1}}$ above induces a map $i'_I : \Sigma_{p^k} \int G_n \longrightarrow \Sigma_{p^{k+1}} \int G_n$ given by $i'_I(\beta; \delta_1, \ldots, \delta_{p^{k+1}}) = (i_I(\beta); \delta'_1, \ldots, \delta'_{p^{k+1}})$. where $\delta'_{i_j} = \delta_j$, for $1 \geq j \geq p^k$; and the identity, otherwise. Respectively, the map $\sigma_I : \Sigma_{p^{k+1}} \longrightarrow \Sigma_{p^k}$ above induces a map $\sigma'_I : \Sigma_{p^{k+1}} \int G_n \longrightarrow \Sigma_{p^k} \int G_n$ given by $\sigma'_I(\beta; \delta_1, \ldots, \delta_{p^{k+1}}) = (\sigma_I(\beta); \delta_{i_1}, \ldots, \delta_{i_{p^k}})$.

The notation above is suitable for subgroups given by wreath products i.e. $\Sigma_{p^k} \int \Sigma_{p^n}$. We note that σ_I is onto and satisfies $\sigma_I(\alpha'\alpha) = \sigma_{J'}(\alpha')\sigma_J(\alpha)$. We use the same symbol σ_I for the induced map $\sigma_I : E\Sigma_{p^{k+1}} \longrightarrow E\Sigma_{p^k}$. We avoided to introduce more symbols, because, we believe, they would cause more confusion.

(1.5) We shall study spaces related to the following systems of permutation subgroups (G_n):

a) $G_n = \Sigma_{p^n,p}$.

b) G_n associated to an increasing sequence of non-negative integers $N = (0, n_1, n_2, \ldots)$ such that $G_0 = 1$; $G_1 = \Sigma_p$; $G_i = \Sigma_{p^i}$, if $i \leq n_1$;
or $G_n = \Sigma_{p^{n_1}} \int \cdots \int \Sigma_{p^{n_k}} \int \Sigma_{p^{n-\nu_k}}$, if $\nu_k < n \leq \nu_{k+1}$, where $\nu_k = \sum_1^k n_t$.

c) G_n associated to a reversed order increasing sequence of non-negative integers $N = (\ldots, n_2, n_1, 0)$ such that $G_0 = 1$; $G_1 = \Sigma_p$; $G_i = \Sigma_{p^i}$, if $i \leq n_1$;
or $G_n = \Sigma_{p^{n-\nu_k}} \int \Sigma_{p^{n_k}} \int \cdots \int \Sigma_{p^{n_1}}$, if $\nu_k < n \leq \nu_{k+1}$ where $\nu_k = \sum_1^k n_t$.

d) $G_n = \Sigma_{p^n}$, $n \geq 0$ is associated to $N = \emptyset$.

Note 1.6. We shall use the maps σ_I for families described in b) and σ'_I for those in c). We restrict our study to the cases a), b), and d) above. The case c) can be studied using the same method and we will discuss the results without proofs.

The infinite symmetric group Σ_{p^∞} is defined by $\Sigma_{p^\infty} := lim \Sigma_{p^n}$ (direct limit), with respect to monomorphisms: $i_1 : \Sigma_{p^n} \longrightarrow \Sigma_{p^{n+1}}$ which fixes the last $(p-1)$ blocks. The infinite subgroup G_∞ is defined in the obvious way.

(1.7) Let us define a natural map called γ between appropriate product of total spaces:

$$\gamma : EG_m \times EG_{k_1} \times \cdots \times EG_{k_{p^m}} \longrightarrow EG_{\ell(m+k)}$$

to be the composition induced by $EG_m \times EG_{k_1} \times \cdots \times EG_{k_{p^m}} \xrightarrow{1 \times i_{I_1} \times \dots i_{I_{p^m}}} EG_m \times (EG_k)^{p^m}$, $E(G_m \int G_k) \xrightarrow{i_{I_{p^{m+k}}}} EG_{\ell(m+k)}$, and the obvious map: $EG_m \times (EG_k)^{p^m} \longrightarrow E(G_m \int G_k)$. Here $k = max\{k_1, \dots, k_{p^m}\}$, i_{I_j} is induced by the inclusion which leaves the last $p^k - p^{k_j}$ elements fixed, and $\ell(m+k)$ is an integer such that $G_m \int G_k \leq G_{\ell(m+k)}$ with $\ell(m+k) \geq m+k$.

REMARK. Let $m = n_1 + \cdots + n_{i_m - 1} + m - \nu_{i_m - 1}$ and $k = n_1 + \cdots + n_{i_k - 1} + k - \nu_{i_k - 1}$ such that $m - \nu_{i_m-1} \leq n_{i_m}$ and $k - \nu_{i_k - 1} \leq n_{i_k}$. If $m - \nu_{i_m - 1} < n_{i_m}$, let $i_{t_0} = max\{i_t \mid i_t \leq i_k,\ n_{i_m} - \sum_{j=1}^{i_t} n_j \geq 0\}$. Then we replace $\Sigma_{p^{n_1}} \int \cdots \int \Sigma_{p^{n_{i_{t_0}}}}$ (which is part of G_k) by $\Sigma_{p^{n_{i_m}}}$. If $m - \nu_{i_m - 1} = n_{i_m}$, let $i_{t_0} = max\{i_t \mid i_t \leq i_k,\ n_{i_m+1} - \sum_{j=1}^{i_t} n_j \geq 0\}$. Then we replace $\Sigma_{p^{n_1}} \int \cdots \int \Sigma_{p^{n_{i_{t_0}}}}$ by $\Sigma_{p^{n_{i_m+1}}}$. Next we continue by replacing part of the group $\Sigma_{p^{n_{i_{t_0}+1}}} \int \cdots \int \Sigma_{p^{n_{i_k}}}$ using the method above, until the whole group has been replaced, say, by $G_{\ell(k)}$. Let $G_{\ell(m+k)} = \Sigma_{p^{n_1}} \int \cdots \int \Sigma_{p^{n_{i_m}}} \int G_{\ell(k)}$.
Example: Let $N = (0, 1, 1.2, 2, 3, \dots)$, $G_5 = \Sigma_p \int \Sigma_p \int \Sigma_{p^2} \int \Sigma_p$ and $G_3 = \Sigma_p \int \Sigma_p \int \Sigma_p$, then $G_{\ell(8)} = \Sigma_p \int \Sigma_p \int \Sigma_{p^2} \int \Sigma_{p^2} \int \Sigma_{p^2}$. In any case $\ell(m+k)$ can be $\sum_{j=1}^{i_m + i_k} n_j$, but the faster the sequence increses the biggest the group we get.

The map γ satisfies:
a) $\gamma(g(e); e_1, \dots, e_{k_{p^m}}) = \gamma(e; e_{g^{-1}(1)}, \dots, e_{g^{-1}(k_{p^m})}) i(g)$.
Here g permutes $(\mathbf{k}, \dots, \mathbf{k})$, $i(g)$ is the inclusion of g in G_{m+k}, and $i(g)$ acts on EG_{m+k}.
b) $\gamma(e; g_1 e_1, \dots, g_{k_{p^m}} e_{k_{p^m}}) = \gamma(e; e_1, \dots, e_{k_{p^m}}) i_{I_{p^{m+k}}} (i_{I_1}(g_1)), \dots, i_{I_{p^m}}(g_{k_{p^m}}))$.
Here $i_{I_j}(g_j)$ acts on $\mathbf{k}$ for $1 \geq j \geq p^m$ and $(g_1, \dots, g_{k_{p^m}})$ acts on $EG_m \times (EG_k)^{p^m}$ by the induced action of G_{k_1} on EG_k via the inclusion $G_{k_1} \xrightarrow{i_1} G_k$ which leaves the last $p^k - p^{k_1}$ blocks fixed.

(1.8) Let X be an object from the category $\Im$ of based spaces of the homotopy type of a CW complex and based maps. We denote the base point by "$*$". To any sequence N defined in (1.5) we associate the topological space $G_N X$ constructed as follows:

$$G_N X = \sqcup_{n \geq 0} EG_n \times X^{p^n} \sqcup \{(*, *)\} / (\approx).$$

Here $(\approx)$ denotes the equivalence relation:
i) $(ge;\ x) \approx (e;\ gx)$ for $e \in EG_n$, $g \in G_n$, and $x \in X^{p^n}$;

ii) $(e;\ s_I x) \approx (\sigma_I e;\ x)$ for $e \in EG_n$ and $x \in X^{p^{n-1}}$.
Here $s_I : X^{p^{n-1}} \longrightarrow X^{p^n}$ is given by $s_I(x_j) = (x_{i_j})$ and fills the rest components with base points and g acts on $x = (x_1, \ldots, x_{p^n})$ by $gx = (x_{g^{-1}(1)}, \ldots, x_{g^{-1}(p^n)})$.

For $N = \emptyset$, $G_N X$ is nothing else but CX, May's approximation model for QX (see May [20]). If $X = \{*\}$, then $G_N X = \{*\}$.

We shall use the obvious filtration $F_k(X)$: the image of $\sqcup_{i=0}^{k} EG_n \times X^{p^n} \sqcup \{(*,*)\}$ in $G_N X$ to calculate its mod-p homology in section **3**.

First we discuss the basic properties of this family of spaces and we proceed to the definition of homology operations. The geometric properties as well as their relation with other interesting spaces are under investigation and we shall present our results in a separate work.

A map $f : X \longrightarrow Y$ induces a map $G_N(f) : G_N X \longrightarrow G_N Y$ by $G_N(f)(e; x_1, \ldots, x_{p^n}) = (e; f(x_1), \ldots, f(x_{p^n}))$ for $e \in EG_n$ and $x_i \in X$.

(1.9) A natural map $\mu : G_N(G_N X) \longrightarrow G_N X$ is defined by

$$\mu[e; [e_1, x_1], \ldots, [e_{p^n}; x_{p^n}]] = [\gamma(e; e_1, \ldots, e_{p^n}); x_1, \ldots, x_{p^n}, *, \ldots, *],$$

for $e \in EG_n$, $e_i \in EG_{n_i}$, and $x_i \in X^{p^{k_i}}$.
It is easy to verify that the map μ is well defined. We also define the natural map $\eta : X \longrightarrow G_N X$ by $\eta(x) = (1; x)$, for $x \in X$. With these two maps $G_N X$ becomes a monad in $\Im$, see May [20]. The map μ above plays important role in the definition of the homology operations defined below. In a general setting μ is replaced by the structure maps θ_n as the following definition explains:

Definition 1.10. a) A space X is called a G_N-space, if there exist maps $\theta_n : EG_n \times X^{p^n} \longrightarrow X$ called structure maps that satisfy the following properties:
i) The following diagram is commutative:

$$\begin{array}{ccccc}
EG_n \times EG_{k_1} \times \cdots \times EG_{k_{p^n}} \times X^{(\sum_{i=1}^{p^n} p^{k_i})} & \xrightarrow{\gamma \times s_1} & EG_{n+k} \times X^{p^{n+k}} & & \\
 & & & \searrow^{\theta_{n+k}} & \\
\downarrow 1\times u & & & & X \\
 & & & \nearrow_{\theta_n} & \\
EG_n \times EG_{k_1} \times X^{p^{k_1}} \times \cdots \times EG_{k_{p^n}} \times X^{p^{k_{p^n}}} & \xrightarrow{\Theta} & EG_n \times X^{p^n} & &
\end{array}$$

Here $\Theta = 1 \times \theta_{k_1} \times \cdots \times \theta_{k_{p^n}}$, u is the evident shuffle map, $s_1(x) = (x, *, \ldots, *)$, and $k = max\{k_1, \ldots, k_{p^n}\}$.
ii) $\theta_0(1; x) = x,\ x \in X$.
iii) $\theta_n(ge; x) = \theta_n(e; gx)$, for $e \in EG_n$, $g \in G_n$, and $x \in X^{p^n}$.
b) A map between G_N-spaces is a map which respects the structure maps.

Proposition 1.11. *Let $Y = G_N X$, then Y is a G_N-space and $Y \in \Im$. Moreover;*
a) there is a product $\theta : Y \times Y \longrightarrow Y$ and the base point acts as an identity in homotopy: $({}_*\theta) \simeq 1_Y \simeq \theta_*$, where $({}_*\theta(y)) = (*, y)$ and $\theta_*(y) = (y, *)$.

b) If $G_1 = \Sigma_p$, then θ is homotopy commutative. For $p = 2$, θ is homotopy commutative.
c) If $p > 2$ and $G_1 = \Sigma_p$. then θ is homotopy associative. For $p = 2$ and $G_2 = \Sigma_4$, then θ is also homotopy associative.
d) $\theta_n(1; -, *, \ldots, *) \simeq 1_Y$.

Proof. We define $\theta_n : EG_n \times Y^{p^n} \longrightarrow Y$ to be the composite $\theta_n = \mu \circ \pi_n$ and $\pi_n : EG_n \times Y^{p^n} \longrightarrow G_N Y$ is the obvious map. For the second assertion we refer to May [18] proposition 2.6. For a), let $\theta : Y \times Y \longrightarrow Y$ be the composite

$$Y \times Y \longrightarrow EG_1 \times Y^p \xrightarrow{\pi_1} G_N Y \xrightarrow{\mu} Y.$$

Let $H : I \times Y \longrightarrow EG_1 \times Y^p \xrightarrow{\theta_1} Y$ be given by $H(t, y) = \theta_1(\omega(t); y, *, \ldots, *)$, where $\omega : I \longrightarrow EG_1$ is a path which connects 1 and $\gamma(1; 1, *, \cdots *)$.
Then $H(0, t) = \theta_1(1; y, *, \ldots, *) = \mu[1; [c, x], [*, *], \ldots, [*, *]] = \mu[\sigma_1 1; [c, x]] = \mu[1; [c, x]] = [\gamma(1, c); x] = [c, x]$. Here $y = [c, x]$ for $c \in EG_n$, $x \in X^{p^n}$, $\sigma : EG_1 \longrightarrow EG_0 \equiv E\Sigma_1$, and $\gamma(1; c) = c \in EG_n \equiv E(G_0 \times G_n) \equiv EG_0 \times EG_n$.
$H(1, y) = \theta_1(\gamma(1; 1, *, \ldots, *): y, *, \ldots, *) = \theta(y, *)$.
For b), let $G_1 = \Sigma_p$, then $\theta(y_1, y_2) = \mu[1; [c_1, x_1], [c_2, x_2], [*, *], \ldots, [*, *]] = \mu[\gamma(1; c_1, c_2, *, \ldots, *); x_1. x_2, *, \ldots, *] = \mu[g^{-1}\gamma(1; c_1, c_2, *, \ldots, *); x_2, x_1, *, \ldots, *]$.
Let $\omega : I \longrightarrow E\Sigma_p$ be a path from $g^{-1}\gamma(1; c_1, c_2, *, \ldots, *)$ to $\gamma(1; c_2, c_1, *, \ldots, *)$, then let $H : I \times Y \times Y \longrightarrow E\Sigma_p \times Y^p \xrightarrow{\theta_1} Y$ be given by:
$H(t, y_1, y_2) = \theta_1(\omega(t), y_1, y_2, *, \ldots, *))$.
For c), we use an appropriate path in $E\Sigma_p$.
d) Let $H : I \times Y \longrightarrow EG_1 \times Y \xrightarrow{\theta_0} Y$ be given by $H(t, y) = \theta_0(\omega(t), y)$ and $\omega : I \longrightarrow EG_1$ connects 1 and $\sigma_1(1)$, where $\sigma_1 : EG_n \longrightarrow E\Sigma_1$.

REMARK (1.12). a) The cartesian product $X \times Y$ of G_N-spaces (X, θ') and $(Y, \theta")$ is again a G_N-space, where the structure maps θ_n are given by the composite:

$$EG_n \times (X \times Y)^{p^n} \xrightarrow{\Delta \times u} EG_n \times EG_n \times X^{p^n} \times Y^{p^n} \xrightarrow{1 \times T \times 1}$$
$$\xrightarrow{1 \times T \times 1} EG_n \times X^{p^n} \times EG_n \times Y^{p^n} \xrightarrow{\theta'_n \times \theta_n"} X \times Y.$$

b) The diagonal Δ on a G_N-space X is a map between G_N- spaces X and $X \times X$.

Now we are ready to express the main theorem of this section. It should be pointed out at this point that the functor G_N associated to $G_n = E_1 \int \cdots \int E_n$ is different from G_N to $G_n \Sigma_{p^{n_1}} \int \ldots \Sigma_{p^{n_t}}$ not only because of Adem relations but also of the geometry it assigns to a space X. We use the letter e for homology operations on G_N-spaces for $G_n = E_1 \int \cdots \int E_n$ and Q for the family $G_n \Sigma_{p^{n_1}} \int \ldots \Sigma_{p^{n_t}}$. The basic differences between e and Q are discussed in Proposition (1.15).

Theorem 1.13. *Let X be a G_N-space, then there exist homomorphisms $Q^s : H_*(X) \longrightarrow H_*(X)$, (let $e^s : H_*(X) \longrightarrow H_*(X)$ for the family of groups $G_n = \Sigma_{p^n, p}$ and note that $Q^s = e^s$ in this case) for $s \geq 0$, which satisfy the following* *properties:*

1) *The* Q^s (e^s) *are natural with respect to maps of* G_N*-spaces.*

2) Q^s, (e^s), *raises degree by* $2s(p-1)$ $(2s)$.

3) $Q^s x = 0$, $(e^s = 0)$, *if* $2s < |x|$ $(2s < (p-1)|x|)$.

4) $Q^s x = x^p$, $(e^s = x^p)$, *if* $2s = |x|$ $(2s = (p-1)|x|)$.

5) $Q^s x_0 = 0$, $(e^s x_0 = 0)$, *where* x_0 *coresponds to the base point class.*

6) *i)* $Q^s\ (x \otimes y) = \sum_{i+j=s} Q^i\ x \otimes Q^j\ y$ $(e^s\ (x\otimes y) = \sum_{i+j=s} e^i\ x \otimes e^j\ y)$, , *if* $x \otimes y \in H_*(X \times Y)$.

ii) $Q^s\ (xy) = \sum_{i+j=s} Q^i\ xQ^j\ y$ $(e^s\ (xy) = \sum_{i+j=s} e^i\ xe^j\ y)$, *if* $xy \in H_*(X)$.

iii) $\psi(Q^s\ x) = \sum_{i+j=s} Q^i\ x' \otimes Q^j\ x''$ $(\psi(e^s\ x) = \sum_{i+j=s} e^i\ x' \otimes e^j\ x'')$, *if* $\psi(x) = \sum x' \otimes x''$.

7) *Adem relations hold everywhere except at positions* $\nu_i = \sum_1^i n_j$ *from the left for any element of length* $n_1 + \cdots + n_k$. *(no Adem relations for* $G_n = \Sigma_{p^n,p}$ *or* $G_n = \Sigma_p \int \cdots \int \Sigma_p$*).*

$Q^r Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} Q^{r+s-i} Q^i$, if $r > ps$.

$$Q^r \beta Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s)}{pi-r} \beta Q^{r+s-i} Q^i - \\ - \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} Q^{r+s-i} \beta Q^i, \text{ if } r \geq ps.$$

8) *The Nishida relations hold: Let* P_*^r *be the dual to Steenrod cohomology operation* P^r. *Then*

$$P_*^r Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(s-r)-1}{r-pi} Q^{s-r+i} P_*^i.$$

$$P_*^r e^s = \sum_i (-1)^{r+i} \binom{(p-1)(s-(p-1)r)}{r-pi} e^{s-(p-1)(r-i)} P_*^i.$$

$$P_*^r \beta Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(s-r)-1}{r-pi} \beta Q^{s-r+i} P_*^i + \\ + \sum_i (-1)^{r+i} \binom{(p-1)(s-r)-1}{r-pi-1} Q^{s-r+i} P_*^i \beta.$$

$$P_*^r \beta e^s = \sum_i (-1)^{r+i} \binom{(p-1)(s-(p-1)r)-1}{r-pi} \beta e^{s-(p-1)(r-i)} P_*^i + \\ + \sum_i (-1)^{r+i} \binom{(p-1)(s-(p-1)r)-1}{r-pi-1} e^{s-(p-1)(r-i)} P_*^i \beta.$$

All coefficients are to be reduced mod-p.

Note. The absence of Adem relations at certain positions does effect the action of the operations above. To see this let $p = 2$ and $G_n = \Sigma_{2^n,2}$, hence no Adem relations are allowed. If they were allowed, we would have $Q^3Q^1 = Q^2Q^2$ (here we identify e^i and Q^i). Now let $x_i \in H_1(X)$ and apply Q^3Q^1 and Q^2Q^2 onto x_1. $Q^2Q^2x_1 = 0$, because of the degree, but there is no particular reason for $Q^3Q^1x_1 = 0$.

Proof. The definition of the action has been given in [15], we repeat it here for completeness. The natural inclusion $i_{(\Sigma_{p,p},\Sigma_p)} : B(\Sigma_{p,p}) \longrightarrow B(\Sigma_p)$ induces a coalgebra epimorphism in mod-p homology. The action is defined for the family $G_n = \Sigma_{p^n,p}$ as follows

$$e^s : H_*(X) \longrightarrow H_*(\Sigma_{p,p} \int X) \longrightarrow H_*(X), \text{by}$$

$$e^s x := \begin{cases} (\theta_1)_*(e_{s-(p-1)|x|/2} \otimes x^p), & \text{if } (p-1)|x|/2 \leq s \\ 0, & \text{otherwise.} \end{cases}$$

Here we used the Steenrod decomposition $H_*(\Sigma_{p,p} \int X) \equiv H_*(\Sigma_{p,p}) \otimes P_1 H_*(X) \oplus H_*(\Sigma_{p,p}; M)$, where P_1 is the Steenrod map in homology and $H_*(\Sigma_{p,p}; M)$ the submodule of $H_*(X)^p$ generated by $\{\otimes_1^p x_i \mid x_i$ belongs into a homogeneous basis of $H_*(X)$and $x_{i_s} \neq x_{i_t}$for some sand $t\}$. The definition above extends to any number of operations e^s, see [15] definition 3.3. Let $Q_i \in H_*(\Sigma_p)$ and $e_i \in H_*(\Sigma_{p,p})$ such that $i_*(e_i) = Q_i$ (see May [19], lemma 1.4), we define $Q_i x := (\theta_1)_* i_*(e_i \otimes x^p)$ and we extend to

$$Q^s x := \begin{cases} (-1)^s \nu(|x|) Q_{(2s-|x|)(p-1)} x, & \text{if } 2s \leq |x| \\ 0, & \text{otherwise.} \end{cases}$$

Here $\nu(|x|) = (-1)^{|x|(|x|-1)(p-1)/4}((p-1)!/2)^{|x|}$, see [15] definition 3.6. All properties of the Q^s's have been verified in [15] theorem 3.8 for G_n-spaces except the fifth which is a consequence of property a) in the definition of a G_N-space. For $G_N X$ spaces proofs are similar and are omitted.

The following proposition describes fundamental properties between operations.

Proposition 1.14. *Let β be the Bockstein operation in homology.*
a) The $\beta^\epsilon e^i$ are natural monomorphisms, if $2i - \epsilon \geq (p-1)|x|$; the $\beta^\epsilon Q^i$ are natural monomorphisms, if $2i - \epsilon \geq |x|$.
b) A length two homology operation is given by $\beta^{\epsilon_1} e^{i_1} \beta^{\epsilon_2} e^{i_2}(x) = \beta^{\epsilon_1} e^{i_1}(\beta^{\epsilon_2} e^{i_2} x)$ and it acts non trivially if $2i_1 - \epsilon_1 \geq (p-1)(2i_2 - \epsilon_2)$;
and $\beta^{\epsilon_1} Q^{i_1} \beta^{\epsilon_2} Q^{i_2}(x) = \beta^{\epsilon_1} Q^{i_1}(\beta^{\epsilon_2} Q^{i_2}(x))$ and it acts non trivially if $2i_1 - \epsilon_1 \geq 2(p-1)i_2 - \epsilon_2$. Here $\beta^{\epsilon_1} Q^{i_1}(\beta^{\epsilon_2} Q^{i_2})$ acts as follows:
$H_*(X) \xrightarrow{\beta^{\epsilon_2} Q^{i_2}} H_*(\Sigma_p \int X) \xrightarrow{\beta^{\epsilon_1} Q^{i_1}} H_*(\Sigma_p \int \Sigma_p \int X)(\xrightarrow{i_*} H_*(\Sigma_{p^2} \int X)) \xrightarrow{(\theta_2)_*} H_*(X)$.
The inclusion i takes place if an Adem relation is allowed between $\beta^{\epsilon_1} Q^{i_1}$ and $\beta^{\epsilon_2} Q^{i_2}$.
c)$(\beta^{\epsilon_1} e^{i_1})(\beta^{\epsilon_2} e^{i_2}(\beta^{\epsilon_3} e^{i_3})) = (\beta^{\epsilon_1} e^{i_1}(\beta^{\epsilon_2} e^{i_2}))(\beta^{\epsilon_3} e^{i_3})$;
$(\beta^{\epsilon_1} Q^{i_1})(\beta^{\epsilon_2} Q^{i_2}(\beta^{\epsilon_3} Q^{i_3})) = (\beta^{\epsilon_1} Q^{i_1}(\beta^{\epsilon_2} Q^{i_2}))(\beta^{\epsilon_3} Q^{i_3})$.

Remark 1.15. A length n operation is of the form: $\beta^{\epsilon_1}Q^{i_1}\dots\beta^{\epsilon_n}Q^{i_n}$ and it acts non-trivially if $2i_t-\epsilon_t-\sum_{t+1}^{n}(2(p-1)i_s-\epsilon_s)\geq 0$, for $1\leq t\leq n-1$; (respectively, $\beta^{\epsilon_1}e^{i_1}\dots\beta^{\epsilon_n}e^{i_n}$, if $2i_t-\epsilon_t-(p-1)\sum_{t+1}^{n}(2i_s-\epsilon_s)\geq 0$, for $1\leq t\leq n-1$).

Proposition (1.14) and the remark above that Adem relations are allowed at certain positions characterize the structure of each family of operations which acts on a category of G_N-spaces for (G_n) fixed. Each family becomes a graded associative Hopf algebra and we examine its structure in the next section.

2. Extended Dyer-Lashof algebras and their duals

The algebra of homology operations as well as its dual structure is described here. Since the coalgebra structure plays an important role in the study of its dual which is isomorphic to a subalgebra of the ring $(E(x_1,\dots,x_n)\otimes P[y_1,\dots,y_n])^{P_n(N)}$, Adem relations should be explicitly evaluated. Here lies the advantage of using modular invariant theory to describe these algebras where Adem relations are overcome and the relation between homology operations and modular coinvariants is revealed. We proceed by analogy with May's computation of the dual of the Dyer-Lashof algebra. The Dyer-Lashof algebra R has been defined by allowing Adem relations everywhere and the extended Dyer-Lashof RN is defined by allowing Adem relations everywhere but on certain positions determined by a strictly increasing sequence $P=(\nu_0=0,\nu_1,\ \nu_2,\ \dots)$ of integers. If P is empty we allow Adem relations everywhere.

Let F be the free graded associative algebra on $\{e^i,\ i\geq 0\}$ and $\{\beta e^i,\ i>0\}$ over $\mathbf{Z}/p\mathbf{Z}$ with $|e^i|=2i$ and $|\beta e^i|=2i-1$. F becomes a coalgebra equiped with coproduct $\psi:\ F--->F\otimes F$ given by

$$\psi e^i=\sum e^{i-j}\otimes e^j \text{ and } \psi\beta e^i=\sum \beta e^{i-j}\otimes e^j\ +\ \sum e^{i-j}\otimes\beta e^j.$$

(2.1) Elements of F are of the form $e^I=\beta^{\epsilon_1}e^{i_1}\dots\beta^{\epsilon_n}e^{i_n}$ where $I=((\epsilon_1,i_1),\dots(\epsilon_n,i_n))$ with $\epsilon_j=0$ or 1 and i_j a non negative integer for $j=1,\ \dots\ ,n$. Let $l(I)$ denote the length of e^I and let the excess of e^I be denoted by $exc(e^I)=|e^{i_1}|-\epsilon_1-|e^{I'}|(p-1)$ where $I'=((\epsilon_2,i_2),\dots(\epsilon_n,i_n))$; and ∞, if $I=(0,\ \dots,\ 0)$.

F admits a Hopf algebra structure with unit $\eta:\mathbf{Z}/p\mathbf{Z}\longrightarrow F$ and augmentation $\epsilon:F\longrightarrow\mathbf{Z}/p\mathbf{Z}$ given by:

$$\epsilon(e^i)=\begin{cases}1, & \text{if } i=0\\ 0, & \text{otherwise.}\end{cases}$$

(2.2) We define $U=F/I_e$, where I_e is the two sided ideal generated by elements of negative excess. U is a Hopf algebra and if we let $U[n]$ denote the set of all elements of U with length n, then $U[n]$ is a coalgebra of finite type. We note here that the dual Steenrod algebra acts on U via Nishida relations, (see [9], the proof has been given by May in [19]).

(2.3) We extend the previous construction by restricting the degrees and imposing Adem relations. Let U' be the subalgebra of U generated by $\{e^{(p-1)i},\ i \geq 0\}$ and $\{\beta e^{(p-1)i},\ i > 0\}$. We denote these elements by Q^i and βQ^i respectively, and recall their degrees $|Q^i| = 2i(p-1)$ and $|\beta Q^i| = 2i(p-1)-1$. Let B be the quotient algebra of U' by the two sided ideal generated by elements of negative excess, where $exc(Q^I) = 2i_1 - \epsilon_1 - |Q^{I'}|$, with $I = ((\epsilon_1, i_1), \ldots (\epsilon_n, i_n))$ and $I' = ((\epsilon_2, i_2), \ldots (\epsilon_n, i_n))$; and ∞, if $I = (0,\ \ldots,\ 0)$. B is a Hopf algebra with the induced coproduct and $B[n]$ a coalgebra as before.

(2.4) Let $N = (0, n_1, n_2, \ldots)$ be a sequence defined before or $N = \emptyset$ and let $P = (\nu_0 = 0, \nu_1,\ \nu_2,\ \ldots)$, where $\nu_i = \sum_{j=1}^{i} n_j$. Let I_N be the two sided ideal of B generated by allowing Adem relations everywhere except at positions described by N counting from the left. We denote RN the quotient B/I_N and this quotient algebra is called **the extended Dyer-Lashof algebra.** If $N = \emptyset$, then $RN = R$ is the Dyer-Lashof algebra. We remark that B and R are special cases of RN. Finally, RN is a Hopf algebra and $RN[n]$ is again a coalgebra. Since $RN[n]$ and $U[n]$ are of finite type, they are isomorphic to their duals as vector spaces and these duals become algebras. We shall describe these duals giving an invariant theoretic description, namely: they are isomorphic to subalgebras of rings of invariants over the appropriate subgroup of $GL(n, \mathbf{Z})$. Respectively, if we let Adem relations everywhere except at positions described by N but counting from the right, the algebra we get is denoted by $R'N$ and is associated to the family described in 1.5-c) and acting on appropriate G'_N-spaces.

An element Q^I in $RN[n]$ is called admissible, if there are no Adem relations between its factors and primitive if $\psi Q^I = Q^I \otimes Q^0\ +\ Q^0 \otimes Q^I$. Here Q^0 means Q^0 $l(I)$ times. Knowing that the dual of a primitive is a generator, we shall find all the primitives and their relations to examine the algebraic structure of the dual algebras.

(2.5) The following algebra epimorphisms

$$F \longrightarrow U \longrightarrow B \longrightarrow RN \longrightarrow R$$

induce coalgebra epimorphisms and dually algebra monomorphisms (see diagram before remarks (2.24) below).

Note 2.6. The algebra U is an important tool to define homology operations; on the other hand, its algebraic structure is easier to be understood than of RN. We will examine RN in detail and discuss the U- case briefly.

Next we discuss the primitives of $U[n]$, $RN[n]$, and the primitive decomposition of an admissible element. We follow May [8].

(2.7) Let $I_{i,n} = (p^{i-2}(p-1), \ldots, (p-1), 1, 0, \ldots, 0)$, where there are $n-i$ zeros. Its degree is $|I_{i,n}| = 2p^{i-1}$ and $exc(I_{i,n}) = 0$. Here $1 \leq i \leq n$.

(2.8) Let $J_{i,n} = (p^{i-2}(p-1)/2, \ldots, (p-1)/2, (1,1), 0, \ldots, 0)$, where there are $n-i$ zeros and one Bockstein operation. Its degree is $|J_{i,n}| = p^{s-1}$ and $exc(J_{i,n}) = 0$. Here $1 \leq i \leq n$.

(2.9) Let $I_{\nu_j-i,\nu_j,n} = (p^{\nu_j-i-1}(p^i-1), \ldots, p^{n_j-i}(p^i-1), \ldots, (p^i-1), p^{i-1}, \ldots, p, 1, 0, \ldots, 0)$.
Here $1 \le j \le \ell$, $1 \le i \le n_j$, $l(Q^{I_{\nu_j-i,\nu_j,n}}) = n$, i is the number of p-th powers, and the first $n - \nu_j$ entries are zeros. The degree $|Q^{I_{\nu_j-i,\nu_j,n}}| = 2p^{\nu_j-i}(p^i-1)$ and the $exc(Q^{I_{\nu_j-i,\nu_j,n}}) = 0$, if $i < \nu_j$, and 1 if $j = 1$ and $i = \nu_1$.
(2.10) Let $J_{\nu_j-i,\nu_j,n} = (p^{\nu_j-i-1}(p^i-1), \ldots, p^{n_j-i}(p^i-1), \ldots, (p^i-1), (1, p^{i-1}), \ldots, p, 1, 0, \ldots, 0)$.
Here $1 \le j \le \ell$, $1 \le i \le n_j$. The only difference between $I_{\nu_j-i,\nu_j,n}$ and $J_{\nu_j-i,\nu_j,n}$ is the appearance of one Bockstein operation. The degree $|Q^{J_{\nu_j-i,\nu_j,n}}| = 2p^{\nu_j-i}(p^i-1) - 1$ and the $exc(Q^{J_{\nu_j-i,\nu_j,n}}) = 1$.
(2.11) Let $K_{\nu_j-s,\nu_j-i,\nu_j,n} = (p^{\nu_j-i-1}(p^i-1) - p^{\nu_j-s-1}, \ldots, (1, p^{s-i-1}(p^i-1)), \ldots, p^i-1, (1, p^{i-1}), \ldots, p, 1, 0, \ldots, 0)$.
Here $1 \le j \le \ell$, $1 \le i \le n_j$, and $i < s \le \nu_j$. There are two Bockstein operations in this element: at the i-th position after the zeros and i+s-th position after the zeros. The degree $|Q^{K_{\nu_j-s,\nu_j-i,\nu_j,n}}| = 2(p^{\nu_j-i}(p^i-1) - p^{\nu_j-s})$ and the $exc(Q^{K_{\nu_j-s,\nu_j-i,\nu_j,n}}) = 0$.

We shall show that these elements are primitives and any primitive in $RN[n]$ is a linear combination of these elements.

Lemma 2.12. *The set* $P[n] = \{ Q^{I_{\nu_j-i,\nu_j,n}}, Q^{J_{\nu_j-i,\nu_j,n}}, Q^{K_{\nu_j-s,\nu_j-i,\nu_j,n}} | 1 \le j \le l,\ 1 \le i \le n_j, \text{ and } i < s \le \nu_j \}$ *consists of primitive elements and any primitive monomial belongs in this set.*

Proof. First we consider primitive monomials with no Bockstein operations. The lemma above has been proved by May in the case $n = \nu_1$, we proceed to the case $n > \nu_1$. We note that the first $n - \nu_j$ entries from the right may all be zero and the ν_j + 1-st should be 1 (there is no Adem relation between the ν_j and the ν_j + 1-st). The number of p-th powers can be at most n_j; otherwise we can write $p^t = p^t - p^{t-n_j-1} + p^{t-n_j-1}$ which implies that the sequence I above can be written as a sum of two admissible sequences. This implies that for $t > i$, there is only one choice: $p^{t-i-1}(p-1)$.
For the case with a number of Bockstein operations, we are left only to show that one of them must be inside the last block (the n_j-th entries after the zeros) using May's lemma [3.1] in [8]. Let $I = (I'', (\beta, k_i), I')$ such that no βs in I'. Then I' is primitive and of the form $I' = I_{i-1-s,i-1,n-\nu_j+i-1}$, where $i > n_j$. If the last entry in I' is $p^{i-2-s}(p^s-1)$, then $k_i = p^{i-1-s}(p^s-1)+1$, otherwise the excess is negative. But then there is an Adem relation between the elements above: if $i = n_j + 1$, we have a non-primitive element i.e. ψQ^I contains
$Q^{I-(p^{\nu_j-i-2}(p-1),\ldots,(p-1),(1,1),0,\ldots,0)} \otimes Q^{(p^{\nu_j-i-2}(p-1),\ldots,(p-1),(1,1),0,\ldots,0)}$ This forces $k_{i-1} = p^{i-2}$ and $k_i = p^{i-1}$. Thus $i \le n_j$ and $k_t = p^{t-1}$, for $t \le i$, and the last implies that I is of the form $J_{\nu_j-i,\nu_j,n}$ or $K_{\nu_j-s,\nu_j-i,\nu_j,n}$.

For the $U[n]$-case we have the analogue.

Lemma 2.13. *The set*

$$P(L)[n] = \{\ e^{I_{j,n}},\ e^{J_{s,n}},\ \mid 1 \le j \text{ and } s \le n\}$$

consists of primitive elements and any primitive monomial belongs in this set.

Next we shall show that any admissible sequence, I, can be written uniquely as a sum of primitives. We follow May's notation.

(2.14) To any sequence I we associate a sequence $\epsilon^{I(t)} = \{e_1, \dots, e_j\}$ with $1 \le e_1 < \cdots < e_j \le \nu_t \le n$ and $1 \le t \le \ell$ such that its elements correspond to the positions where Bocksteins appear in I, and to any such sequence $\epsilon^{I(t)}$ we associate a set $I_{\epsilon^{I(t)}}[n]$ which contains all admissible sequences such that their Bocksteins appear at positions described by $\epsilon^{I(t)}$.

$$I_{\epsilon^{I(t)}}[n] = \{\ I \mid I \text{admissible}, exc(I) \ge 0,\ l(I) = n, \text{and } \epsilon_{\nu_t - a} = 1 \Leftrightarrow a \in \epsilon^{I(t)}\ \}.$$

If $\epsilon^{I(t)}$ is empty, $I_{\epsilon^{I(t)}}[n] = I[n]$. For each e_{2m} in $\epsilon^{I(t)}$ we find $t_{(m)}$ such that $\nu_{t_{(m)}-1} < e_{2m} \le \nu_{t_{(m)}}$.
If j is even we define:

$$L_{\epsilon^{I(t)},\nu_{t_{(j/2)}},n} = K_{e_1-1,e_2-1,\nu_{t_{(1)}},n} + K_{e_3-1,e_4-1,\nu_{t_{(2)}},n} + \cdots + K_{e_{j-1}-1,e_j-1,\nu_{t_{(j/2)}},n}.$$

If j is odd we define:

$$L_{\epsilon^{I(t)},\nu_{t_{((j+1)/2)}},n} = K_{e_1-1,e_2-1,\nu_{t_{(1)}},n} + \cdots + K_{e_{j-2}-1,e_{j-1}-1,\nu_{t_{((j-1)/2)}},n} + J_{e_j-1,\nu_{t_{((j+1)/2)}},n}.$$

Lemma 2.15. *Let $I \in I[n]$. Then I can be written uniquely in the form* $\sum_{1\le j\le \ell,\ 1\le i\le n_j} m_{j,i} I_{\nu_j - i,\nu_j,n}$.

Proof. This lemma has been proven by May in the case $N = \emptyset$. For completeness we quote his proof:
Let N be the non-negative integers and $f{:}\mathrm{N}^n \longrightarrow I[n]$ given by $f(m_1, \dots, m_n) = \sum m_i I_{n-i,n,n}$. Then f is a bijection with inverse $f^{-1}(s_1, \dots, s_n) = (m_1, \dots, m_n)$, where $m_q = ps_q - s_{q-1}$ if $q > 1$ and $m_1 = s_n - \sum_2^k m_q$. For the general case we induct on the number of blocks.
Let $I = (i_1, \dots, i_{\nu_{\ell-1}}, i_{\nu_{\ell-1}+1}, \dots, i_{\nu_\ell})$ and $I_1 = (i_{\nu_{\ell-1}+1}, \dots, i_{\nu_\ell})$, then I_1 can be written uniquely as $\sum_1^{n_\ell} m_i I_{n_\ell - i, n_\ell, n_\ell}$, by induction. We extend $I_{n_\ell-i,n_\ell,n_\ell}$ to $I_{\nu_\ell - i,\nu_\ell,n}$ such that the last n_ℓ elements of the last sequence are equal to elements in $I_{n_\ell-i,n_\ell,n_\ell}$ respectively. We note that the excess of the last m elements of $I_{\nu_\ell-i,\nu_\ell,n}$ is zero for any $m > n_\ell$ and any i. Let $I' = (i'_1, \dots, i'_{\nu_{\ell-1}}, 0, \dots, 0) = I - \sum_1^{n_\ell} m_i I_{\nu_\ell-i,\nu_\ell,n}$, then I' can be written uniquely in the required form by induction as soon as I' is admissible.
Claim: I' is admissible with non-negative excess.
For any two consecutive elements of $I_{\nu_\ell-i,\nu_\ell,n}$ we have $s_{i,q-1} = ps_{i,q}$ for $q \le \nu_{\ell-1}$. And since $\ell > 1$, $\mathrm{exc}(I_{\nu_\ell-i,\nu_\ell,n}) = 0$, for any i. $2i_1 - 2\sum i_q(p-1) \ge exc(\sum_1^{n_\ell} m_i I_{\nu_\ell-i,\nu_\ell,n}) = 0$ implies $exc(I') \ge 0$, and for any two consequtive

elements of I' we have $i'_q = i_q - \sum_1^{n_t} s_{i,q}$ where $i_{q-1} \leq pi_q$ and $s_{i,q-1} = ps_{i,q}$ which implies $i'_{q-1} \leq pi'_q$. This completes the proof.

We proceed to the case I contains a number of Bocksteins.

Lemma 2.16. *For each non empty $\epsilon^{I(t)}$, let $f_{\epsilon^{I(t)}} : I[n] \longrightarrow I_{\epsilon^{I(t)}}[n]$ given by $f_{\epsilon^{I(t)}}(I) = I + L_{\epsilon^{I(t)},\nu_t,n}$, then $f_{\epsilon^{I(t)}}$ is a bijection.*

Proof. $f_{\epsilon^{I(t)}}$ is well defined since the sum of two admissible sequences is again admissible and $f_{\epsilon^{I(t)}}(I) \in I_{\epsilon^{I(t)}}[n]$. We shall show that $f^{-1}_{\epsilon^{I(t)}}(J) = J - L_{\epsilon^{I(t)},\nu_t,n}$ is admissible and has no negative excess. Let $J = ((\epsilon_1, i_1), \ldots, (\epsilon_n, i_n)) \in I_{\epsilon^{I(t)}}[n]$. If the number of Bocksteins in J is odd, then $exc(J) \geq 1$. Now for any $q \leq n$, $i_{q-1} \leq pi_q - \epsilon_q$, since J is admissible. We recall that $L_{\epsilon^{I(t)},\nu_t,n} = ((\delta_1, r_1), \ldots, (\delta_{\nu_t}, r_{\nu_t}), 0, \ldots, 0)$, where $r_{q-1} = pr_q - \delta_q$, $\delta_q = \epsilon_q$, and $exc(K_{\nu_j-s,\nu_j-i,\nu_j,n}) = 0$. If the number is even, then $exc(L_{\epsilon^{I(t)},\nu_t,n}) = 0$ implies $exc(J - L_{\epsilon^{I(t)},\nu_t,n}) \geq 0$. Moreover, $i_{q-1} \leq pi_q - \epsilon_q$ and $r_{q-1} = pr_q - \delta_q$ imply $i_{q-1} - r_{q-1} \leq p(i_q - r_q)$, so $J - L_{\epsilon^{I(t)},\nu_t,n}$ is admissible in either case and this completes the proof.

Proposition 2.17. *Let $e^I \in U[n]$ and $\beta(I) = (i_1, \ldots, i_k)$ be the position where Bockstein operations appear in I ($1 \leq k \leq n$ and $\beta(I)$can be $\emptyset$). Then I can be written uniquely in the form:*

$$I = \sum_{j=1}^{n} m_j I_{j,n} \;+\; \sum_{t \in \beta(I)} J_{i_t,n}.$$

We are ready to consider $(RN[n])^*$ and discuss its algebra generators among with their relations, its relation with modular invariants, the Steenrod algebra action and its coalgebra structure.

(2.18) Define the generators as follows:

$$\begin{aligned}
&\xi_{0,n} = ((Q^0)^n)^*, && 0 \leq n.\\
&\xi_{\nu_j-i,\nu_j,n} = (Q^{I_{\nu_j-i,\nu_j,n}})^*, && 1 \leq j \leq \ell,\ 1 \leq i \leq n_j.\\
&\tau_{\nu_j-i,\nu_j,n} = (Q^{J_{\nu_j-i,\nu_j,n}})^*, && 1 \leq j \leq \ell,\ 1 \leq i \leq n_j.\\
&\sigma_{\nu_j-s,\nu_j-i,\nu_j,n} = (Q^{K_{\nu_j-s,\nu_j-i,\nu_j,n}})^*, && 1 \leq j \leq \ell,\ 1 \leq i \leq n_j, \text{ and } i < s \leq \nu_j.
\end{aligned}$$

We have the following generalized version of May's theorem 3.7 [8].

Theorem 2.19. *Let $PM[n]$ be the free associative commutative algebra generated by $\{\xi_{\nu_j-i,\nu_j,n},\ \tau_{\nu_j-i,\nu_j,n}$, and $\sigma_{\nu_j-s,\nu_j-i,\nu_j,n}\ /\ 1 \leq j \leq \ell,\ 1 \leq i \leq n_j$, and $i < s \leq \nu_j\ \}$ modulo the following relations:*

a) $\tau_{\nu_j-i,\nu_j,n}\,\tau_{\nu_j-i,\nu_j,n} = 0$, $1 \leq j \leq \ell$, $1 \leq i \leq n_j$.

b) $\tau_{\nu_{j_s}-s,\nu_{j_s},n}\,\tau_{\nu_j-i,\nu_j,n} = \sigma_{\nu_{j_s}-s,\nu_j-i,\nu_j,n} \prod_{t=1}^{j_s} \xi_{\nu_{t-1},\nu_t,n}$. *Here* $1 \leq s \leq n_{j_s}$, $1 \leq j \leq \ell$, $1 \leq i \leq n_j$, *and* $j_s \leq j$.

c) $\tau_{\nu_{j_s}-s,\nu_{j_s},n}\tau_{\nu_{j_1}-i,\nu_{j_1},n}\tau_{\nu_{j_2}-k,\nu_{j_2},n} = \sigma_{\nu_{j_s}-s,\nu_{j_1}-i,\nu_{j_1},n}\tau_{\nu_{j_2}-k,\nu_{j_2},n}\prod_{t=1}^{j_s}\xi_{\nu_{t-1},\nu_t,n}.$

Here $1 \le s < n_{j_s}$, $1 \le j_1$, $j_2 \le \ell$, $1 \le i \le n_{j_1}$, $j_s \le j_1 \le j_2$, *and* $1 \le k \le n_{j_2}$. *Then* $PM[n] \equiv RN[n]^*$ *as algebras.*

Theorem 2.20. $U[n]^*$ *is the free associative commutative algebra generated by*

$$\{ (e^{I_{i,n}})^*, (e^{J_{j,n}})^*, \mid 0 \le i \le n \text{ and } 1 \le j \le n\}$$

modulo the single relation $((e^{J_{j,n}})^*)^2 = 0$. *That is* $U[n]^*$ *is isomorphic to a polynomial tensor an exterior algebra over* $\mathbf{Z}/p\mathbf{Z}$.

We procced to the relation between $RN[n]^*$ and $(E(x_1, \dots, x_n) \otimes P[y_1, \dots, y_n])^{P_n(N)}$. We recall that $P_n(N)$ stands for the first length n part of the infinite sequence N. We advise the reader to read theorem (4.25, 4.26, 4.27), proposition (4.28), and corollary (4.28) from section 4 before the rest of this section.

(2.21) Let $S^{P_n(N)}$ be the subalgebra of $(E(x_1, \dots, x_n) \otimes P[y_1, \dots, y_n])^{P_n(N)}$ generated by:

$$\{(L_{\nu_j}/L_{\nu_{j-1}})^{p-1}, Q_{\nu_j,\nu_j-i} - Q^{p^{n_j}}_{\nu_{j-1},i-n_j}, M_{\nu_j;\nu_j-i}(L_{\nu_j})^{p-2}, \text{and}$$
$$M_{\nu_{j_s};\nu_{j_s}-s}(L_{\nu_{j_s}})^{p-2}M_{\nu_j;\nu_j-i}(L_{\nu_j})^{p-2}/L^{p-1}_{\nu_{j_s}} \mid$$
$$\mid 1 \le j \le \ell,\ 1 \le i \le n_j - 1,\ j_s \le j, \text{and } 1 \le s \le n_{j_s}\}.$$

Here we understand $Q_{\nu_{j-1},i-n_j}$ to be zero, if $i < n_j$; modulo the relations:
a) $[M_{\nu_j;s}(L_{\nu_j})^{p-2}]^2 = 0$;
b) $M_{\nu_{j_s};\nu_{j_s}-s}(L_{\nu_{j_s}})^{p-2}M_{\nu_j;\nu_j-i}(L_{\nu_j})^{p-2} = (L_{\nu_{j_s}})^{p-1}M_{\nu_{j_s};\nu_{j_s}-s}L^{p-2}_{\nu_{j_s}}/L^{p-1}_{\nu_{j_s}}$ $M_{\nu_j;\nu_j-i}(L_{\nu_j})^{p-2}$. Here $1 \le j \le \ell$ and $\nu_{j-1} \le s < \nu_j$.

Now the following corollary is deduced.

Theorem 2.22. $(RN[n])^* \cong S^{P_n(N)}$ *as algebra over the Steenrod algebra and the isomorphism* Φ *is given by*
$\Phi(\xi_{\nu_j-i,\nu_j,n}) = Q_{\nu_j;\nu_j-i} - (Q_{\nu_{j-1},i-n_j})^{p^{n_j}}$, *if* $i \ge n_j$;
$\Phi(\tau_{\nu_j-i,\nu_j,n}) = M_{\nu_j;\nu_j-i}(L_{\nu_j})^{p-2}$;
$\Phi(\sigma_{\nu_{j_s}-s,\nu_j-i,\nu_j,n}) = M_{\nu_{j_s};\nu_{j_s}-s}(L_{\nu_{j_s}})^{p-2}M_{\nu_j;\nu_j-i}(L_{\nu_j})^{p-2}/(L_{\nu_{j_s}})^{p-1}$.
Here $1 \le j \le \ell$, $1 \le i \le n_j$, $j_s \le j$, and $1 \le s \le n_{j_s}$.

As it is expected the $U[n]$-case is simpler.

Theorem 2.23. $(U[n])^* \equiv S^{U_n}$ *as Steenrod algebras and the isomorphism* Φ *is given by*
$\Phi((e^{I_{j,n}})^*) = V_j$;
$\Phi((e^{J_{s,n}})^*) = M_{s;s-1}(L_{s-1})^{(p-3)/2}$.
Here $0 \le j \le n$ and $1 \le s \le n$.

We have arrived at the following picture where the downward arrows are coalgebra epimorphisms over the Steenrod algebra induced by the Adem relations, the upward arrows are monomorphisms over the Steenrod algebra (actually inclusions), and the first column horizontal arrows are vector space isomorphisms, while the second column arrows are isomorphisms over the Steenrod algebra.

$$
\begin{array}{ccccccc}
F[n] & & & & & & \\
\downarrow & & & & & & \\
U[n] & \xrightarrow{\equiv} & U[n]^* & \xleftarrow{\Phi} & S^{U_n} & \le (E(x_1,\dots,x_n)\otimes P[y_1,\dots,y_n])^{U_n} \\
\downarrow & & \uparrow & & \uparrow & \uparrow \\
B[n] & \xrightarrow{\equiv} & B[n]^* & \xleftarrow{\Phi} & S^{B_n} & \le (E(x_1,\dots,x_n)\otimes P[y_1,\dots,y_n])^{B_n} \\
\downarrow & & \uparrow & & \uparrow & \uparrow \\
RN[n] & \xrightarrow{\equiv} & RN[n]^* & \xleftarrow{\Phi} & S^{P_n(N)} & \le (E(x_1,\dots,x_n)\otimes P[y_1,\dots,y_n])^{P_n(N)} \\
\downarrow & & \uparrow & & \uparrow & \uparrow \\
R[n] & \xrightarrow{\equiv} & R[n]^* & \xleftarrow{\Phi} & S^{GL_n} & \le (E(x_1,\dots,x_n)\otimes P[y_1,\dots,y_n])^{GL_n}
\end{array}
$$

REMARK (2.24). We have shown that the n-th extended Dyer-Lashof coalgebra $RN[n]$ associated to $P_n = (0, n_1, n_1 + n_2, \dots, n = \sum_{i=i}^{k} n_i)$ is isomorphic to $(S^{P_n(N)})^*$ as opposite Steenrod coalgebras via Nishida relations. Since $S^{P_n(N)} \equiv S^{\eta_n P'_n(N)^t \eta_n}$ as Steenrod algebras, $RN[n] \equiv \left(S^{\eta_n P'_n(N)^t \eta_n}\right)^*$ as opposite Steenrod coalgebras. where $P'_n = (0, n_k, n_k + n_{k-1}, \dots, n = \sum_{i=i}^{k} n_i)$. Therefore, if we are given an increasing sequence of positive integers N as before, we can construct the extended Dyer-Lashof algebra, denoted $R'N$, by letting Adem relations everywhere but on the positions described by N but in the reverse order and this algebra of homology operations acts on a G_N-spaces associated with the family of groups described in (1.5)-c). In simple words, $RN[n]$ is associated to $\Sigma_{p^{n_1}} \int \cdots \int \Sigma_{p^{n_k}}$ and $R'N[n]$ to $\Sigma_{p^{n_k}} \int \cdots \int \Sigma_{p^{n_1}}$.

Since Q^0 does not act as an identity, the degree zero homology operations denoted by $Q^{I_{n,n}}$ do not form a finite dimensional vector space and in general RN_+ is not of finite type (any number of zeros is allowed). RN^* is not a Hopf algebra. As an algebra $RN^* = \prod_{n\ge 0} RN[n]^*$ and for each n there are n polynomial generators $\{\xi_{i,\nu_j,n}\}$ or $\{Q_{\nu_j,i}\}$ in degrees $2(p^{\nu_j} - p^i)$, n exterior generators $\{\tau_{i,\nu_j,n}\}$ or $\{M_{\nu_j;i}\}$ in degrees $2(p^{\nu_j} - p^i - 1) + 1$, and also $n(n-1)/2$ exterior generators $\{\sigma_{s,i,\nu_j,n}\}$ or $\{M_{\nu_j;s,i}\}$ in degrees $2(p^{\nu_j} - p^i - p^s)$ respectively, modulo the single relation b) given in theorem (4.25). In the invariant theoretic description of the generators we are not allowed to use the known relations between generators of different height, where the height of elements of $RN[n]^*$ is n. For example, let $p = 2$ and $N = (0, 1, 1, \dots)$, then in the isomorphic image of B^* the generator V_1^{p-1} appears infinitely many times with a different height.

The Steenrod algebra action on $(E(x_1, \dots, x_n) \otimes P[y_1, \dots, y_n])^{P_n(N)}$ has been calculated in [14], hence the same action on $RN[n]^*$ is known and it can be used to deduce the dual Steenrod action on $RN[n]$.

Since $S^{P_n(N)}$ is a subalgebra of S^{B_n}, we shall show that the monomorphisms induced from the epimorphisms which impose Adem relations are inclusions.

We will use this result to calculate the coproduct in the dual algebras. The idea is to find all $Q^I \in B[n]$ such that after applying Adem relations their sums contain a certain primitive of $R[n]$. Let ψ also denote the coproduct in RN^* ($\psi: RN^* \longrightarrow RN^* \otimes RN^*$) which is dual to the given product in RN. We shall find the coproduct on generators of RN^* and this is equivalent to find all elements of RN which contain a certain primitive element. This case reduces to consider the coproduct in R^* since a generator of RN^* is an algebraic combination of R^*-generators. The coproduct in R^* has been given by May in [8], here we use an invariant theoretic proof of his formulas.

REMARK (2.28). It is known that the induced inclusion $i: \Sigma_p \int \cdots \int \Sigma_p \longrightarrow \Sigma_{p^n}$ induces Adem relations in homology [19]. Since the same inclusion is used in cohomology to give an invariant theoretic description of $H^*(\Sigma_{p^n})$ and $H^*(\Sigma_p \int \cdots \int \Sigma_p)$, as the following diagram suggests, the dual of every monomial of $B[n]$ which contains the primitive $Q^{I_{n-i,n,n}}$ in its sum after applying Adem relations should be a summand of the dual $(Q^{I_{n-i,n,n}})^*$ in its B_n-decomposition.

$$\begin{array}{ccc} H^*(\Sigma_p \int \cdots \int \Sigma_p) & \longrightarrow & H^*(E^n)^{B_n} \\ \uparrow & & \uparrow \\ H^*(\Sigma_{p^n}) & \longrightarrow & H^*(E^n)^{GL_n} \end{array}$$

Using our isomorphism between $R[n]^*$ and S^{GL_n}, we can find all monomials in $B[n]$ which contain $Q^{I_{n-i,n,n}}$ after applying Adem relations. Now suppose that we do not allow Adem relations only in one position. The dual of such a monomial of $B[n]$ must be of the form $((V_{t+1} \cdots V_{t+k})^{(p-1)})^{p^t} (V_{k+i+1} \cdots V_n)^{(p-1)}$ which is a summand in $Q_{n,i}$, for $1 \leq k \leq n-i$ and $0 \leq t \leq i$. Here we used lemma (4.8) c). Hence, $Q^{I_{i,n,n}}$ can be written in terms of a product of two admissible elements as follows:

$$Q^{p^t I_{0,k,k} + (p^{n-k} - p^t) I_{k-1,k,k}} Q^{I_{i,n-k,n-k}} = Q^{I_{i,n,n}}.$$

Using the corresponding decomposition for $M_{n;i} L_n^{(p-2)}$ in $(E(x_1, \ldots, x_n) \otimes P[y_1, \ldots, y_n])^{B_n}$, we derive the corresponding formula for $Q^{J_{i,n,n}}$ (theorem (4.24)-b): Let us pick any summand $M_{r;r-1} L_r^{(p-2)} (L_n/L_r)^{p-1} Q_{r-1,i}$ in $M_{n;i} L_n^{(p-2)}$. The monomial that contributes an Adem relation to the left of a Bockstein in the dual side is $Q_{r-1,i}$ between $r-1$ and r position from left and the farthest that it can be moved is the t-1-i-th position from the left. Since in this case, the Bockstein operation should be contained in the left factor, there are $r - i - 1$ cases to consider, namely Adem relations described above. By a straightforward bookkeeping we get the formula in the lemma bellow. Let us note that there are no Adem relations to the right of the Bockstein. The next lemma follows.

Lemma 2.29. *Let $I = I_{i,n,n}$ or $J_{i,n,n}$, then all possible choices for admissible sequences J and K such that $Q^K Q^J$ contains Q^I in its summand after applying*

Adem relations are given by the following formulas.

a) $Q^{p^{k-1-i}(p-1)I_{i-1,i,i}}Q^{I_{k-1-i,k-i,n-i}} = Q^{I_{k-1,k,n}}$, for $1 \leq i < k \leq n$.

b) $Q^{I_{k-1,k,n-i}}Q^{(0,\ldots,0)} = Q^{I_{k-1,k,n}}$, for $1 \leq i \leq n-k$.

c) $Q^{p^{k-1-i}(p-1)I_{i-1,i,i}}Q^{J_{k-1-i,k-i,n-i}} = Q^{J_{k-1,k,n}}$, for $1 \leq i < k \leq n$.

d) $Q^{J_{k-1,k,n-i}}Q^{(0,\ldots,0)} = Q^{J_{k-1,k,n}}$, for $1 \leq i \leq n-k$.

e) $Q^{(p^{n-k}-p^j)I_{0,k,k}+p^j I_{i-j,k,k}}Q^{I_{j,n-k,n-k}} = Q^{I_{i,n,n}}$. for $0 \leq j \leq i,\ j \leq n-k$.

f) $Q^{(p^{n-k}-p^j)I_{0,k,k}+p^j I_{i-j,k,k}}Q^{J_{j,n-k,n-k}} = Q^{J_{i,n,n}}$.for $0 \leq j \leq i,\ j \leq n-k$.

g) $Q^{(p^{n-k}-1)I_{0,k,k}+J_{i,k,k}}Q^{I_{0,n-k,n-k}} = Q^{J_{i,n,n}}$, for $n-i < k \leq n$.

Note 2.30. If we replace Q^i and βQ^i by e^i and βe^i respectively in a) and b) above, then we get the analogous formulas for the $U[n]$-primitives.

Let us also note here that the analogue formula for $Q^{K_{s,i,n,n}}$ turns out to be complicated and this is the same for the corresponding formula with respect to invariants. As May pointed out in [8] page 35, we can find the coproduct for $M_{n;s,i}L_n^{(p-2)}$ using the relations between the generators. Let us also note that the invariant description of RN^* is simpler but not simple enough to reduce every case.

Before we express the next theorem which has been proved by May [8], we make a note on the notation. Let ${}_nV_k^{(p-1)}$ and ${}_nM_{k;k-1}L_k^{(p-2)}$ denote the generators $V_k^{(p-1)}$ and $M_{k;k-1}L_k^{(p-2)}$ respectively in $(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n])^{B_n}$ and 1_m the identity element in $B[n]^*$ and $R[n]^*$ dual to $Q^{(0,\ldots,0)}$.

Theorem 2.31. *The coproduct on generators of B^* and R^* is given by the following formulas (here we use the isomorphic image of these algebras).*

i) $\psi({}_nV_k^{(p-1)}) = \sum_{1\leq i\leq k-1} ({}_iV_i^{(p-1)})^{p^{k-1-i}(p-1)} \otimes ({}_{n-i}V_{k-i}^{(p-1)}) +$
$+ \sum_{0\leq i\leq n-k} ({}_{k+i}V_k^{(p-1)}) \otimes 1_{n-k-i}.$

ii) $\psi({}_nM_{k;k-1}L_k^{(p-2)}) = \sum_{1\leq i\leq k-1} ({}_iV_i^{(p-1)})^{p^{k-1-i}(p-1)} \otimes {}_{n-i}(M_{k-i;k-i-1}L_{k-i}^{(p-2)})$
$+ \sum_{0\leq i\leq n-k} ({}_{k+i}M_{k;k-1}L_k^{(p-2)}) \otimes 1_{n-k-i}.$

iii) $\psi(Q_{n,i}) = \sum_{(i,k)} Q_{k,0}^{p^{n-k}-p^j} Q_{k,i-l}^{p^j} \otimes Q_{n-k,j}.$

iv) $\psi(M_{n;i}L_n^{(p-2)}) = \sum_{(i,k)} Q_{k,0}^{p^{n-k}-p^j} Q_{k,i-j}^{p^j} \otimes M_{n-k;j}L_{n-k}^{(p-2)} +$
$+ \sum_{(k)} Q_{k,0}^{p^{n-k}-1} M_{k;i}L_k^{(p-2)} \otimes Q_{n-k,0}.$

v) $\psi(M_{n;s,k}L_n^{(p-2)}) =$
$= \sum_{(i,j,f)} Q_{i,0}^{p^{n-i}-p^f-p^j}(Q_{i,i+j-k}^{p^j}Q_{i,i+f-s}^{p^f} - Q_{i,i+f-k}^{p^f}Q_{i,i+j-s}^{p^j})\otimes M_{n-i;f,j}L_{n-i}^{(p-2)} -$

$$\sum_{(i,j)} Q_{i,0}^{p^{n-i}-p^j-1}(Q_{i,i+j-k}^{p^j} M_{i;i-s} L_i^{(p-2)} - Q_{i,i+j-s}^{p^j} M_{i;i-k} L_i^{(p-2)}) \otimes M_{n-i;j} L_{n-i}^{(p-2)}$$
$$+ \sum_{(i)} Q_{i,0}^{p^{n-i}-1} M_{i;i-s,i-k} L_i^{(p-2)} \otimes Q_{n-i,0}.$$

3. The homology of $G_N X$

Theorem 3.1. *Let X be connected in $\Im$ and $G_n = \Sigma_{p^{n_1}} \int \cdots \int \Sigma_{p^{n_1}}$ associated to $N = (0, n_1, n_2, \dots)$ an increasing sequence of positive integers or $G_n = \Sigma_{p^n}$. Then $\tilde{H}_*(G_N X) \equiv \sum \tilde{H}_*(F_k/F_{k-1}(X))$ is isomorphic to a free non associative (associative, if $G_n = \Sigma_{p^n}$ for all n) commutative graded algebra over $\mathbf{Z}/p\mathbf{Z}$ generated by the free RN-module basis$(H_*(X))$ modulo the relations: $Q^s x = x^p$, if $2s = |x|$ and $Q^I x = 0$, if $e(I) + b(I) < |x|$. Here $x \in$ basis$(H_*(X))$. Moreover $H_*(G_N X)$ is a coalgebra, where the coproduct is given by:*

$$\psi Q^I x = \sum_{K+J=I} \sum Q^K x' \otimes Q^J x", \quad \text{with } \psi x = \sum x' \otimes x".$$

Here basis$(H_*(X))$ is a fixed homogeneous basis of $H_*(X)$ over $\mathbf{Z}/p\mathbf{Z}$.

Proof. First, we shall show that $\tilde{H}_*(G_N X)$ and the algebra described in theorem above are isomorphic as comodules, then the theorem will follow from the fact that the product θ in $G_N X$ induces the tensor product in homology which was used to calculate the homology of $G_n \int X$ and $G_\infty \int X$ in [15]. We recall the filtration $F_k(X)$: the image of $\sqcup_{i=0}^{k}(EG_n \times X^{p^n}) \sqcup \{*, *\}$ in $G_N X$ and hence $\tilde{H}_*(G_N X) \equiv \lim \tilde{H}_*(F_k(X))$. It is known that $F_k/F_{k-1}(X) \equiv EG_k \times X^{[p^k]}/\approx$ $\equiv (EG_k)_+ \wedge X^{[p^k]}$, where $X^{[p^k]}$ is the smash product and the relation is given by $(e; *) \approx (e'; *)$ for any e, $e' \in EG_k$, and $(ge; x) \approx (e; gx)$ for $g \in G_k$ and $x \in X^{[p^k]}$. X_+ denotes $X \sqcup \{*\}$. If we let $Y = X \sqcup \{*\}$ and consider $\{*\}$ as the base point of Y, then $F_k/F_{k-1}(Y) = G_k \int X \sqcup \{*\}$ and the obvious map p between Y and X induces an epimorphism in mod-p reduced homology:

$$p_* : \tilde{H}_*(F_k/F_{k-1}(Y)) \longrightarrow \tilde{H}_*(F_k/F_{k-1}(X)).$$

Using the known comodule decomposition of $\tilde{H}_*(F_k/F_{k-1}(Y))$ (see [15] discussion after theorem (4.2)) and the map p we conclude that:

$$\tilde{H}_*(F_k/F_{k-1}(X)) \equiv \tilde{H}_*(G_k \int Y)/M_k.$$

Here M_k is the submodule of $\tilde{H}_*(G_k \int Y)$ generated by all elements of the form $Q^I x_0$ and $\otimes_{i=1}^{p^l} Q^{J_i} x_i$ where x_0 is the zero class of X, $l(I) = p^k$, $\sum l(J_i) = p^k$, and $x_j = x_0$ for some $j \in \{1, \dots, p^l\}$. Our assertion follows by the natural decomposition:

$$\tilde{H}_*(F_k(X)) \equiv \tilde{H}_*(F_{k-1}(X)) \oplus \tilde{H}_*(F_k/F_{k-1}(X)).$$

The relations are due to the definition of the Q^I's.

REMARK (3.2). The stable homotopy groups of spheres are the homotopy groups of $Q(S^0)$. On the other hand, its homotopy type is the same as to the realization of the simplicial monoid $\sqcup_{n\geq 0} B\Sigma_n$ by adjoining inverses. Moreover, if we let X be a group, $\sqcup_{n\geq 0} B\Sigma_n \int X$ is a simplicial monoid and its group completion has the homotopy type of $Q(BX_+)$. We are interested in studing the analogue of our $G_N X$, for X a space.

4. Modular invariant theory

In this section, we discuss the rings of invariants of certain subgroups of $GL_n(\mathbf{Z}/p\mathbf{Z})$. Although, it can be thought of as an independent section, we are mostly interested in the cases $G = U_n, B_n$, or $P_n(N)$ because of their relation with homology operations. The main result is given in theorem (4.25).

(4.1) Here $P_n(N)$ denotes the parabolic subgroup of $GL_n(\mathbf{Z}/p\mathbf{Z})$ associated to $N = (0, n_1, n_2, \ldots)$ with $\sum n_i = n$ and $P'_n(N)$ the parabolic associated with the sequence N but with the order reversed as follows:

$$P_n(N) = \begin{pmatrix} [n_1 \times n_1] & & & * \\ & [n_2 \times n_2] & & \\ & & \ddots & \\ 0 & & & [n_k \times n_k] \end{pmatrix} \text{ and } \eta_n = \begin{pmatrix} 0, \ldots, 0, 1 \\ \vdots \\ 1, 0, \ldots, 0 \end{pmatrix}.$$

(4.2) We have k blocks along the main diagonal, anything above and 0 below, where any block $[n_i \times n_i]$ is an element of $GL_{n_i}(\mathbf{Z}/p\mathbf{Z})$. Note that if $k = 1$, then $P_n(N) = GL_n(\mathbf{Z}/p\mathbf{Z})$ and if $n_i = i$, then $P_n(N)$ is denoted by B_n, a Borel subgroup of $GL_n(\mathbf{Z}/p\mathbf{Z})$. Finally, let U_n be the subgroup of $GL_n(\mathbf{Z}/p\mathbf{Z})$ consisting of matrices with 1's along the main diagonal, anything above, and zero below. It is well known that U_n is a p-Sylow subgroup of $GL_n(\mathbf{Z}/p\mathbf{Z})$. We note that $\eta_n P_n(N) \eta_n = P'_n(N)^t$.

The notation we use follows Hyunh [11]. First, we recall some details here for the convenience of the reader and quote some lemmas and theorems from [11] and [4]. We recall that the object usually referred to as Dickson's algebra is $(P[y_1, \ldots, y_n])^{GL_n(\mathbf{Z}/p\mathbf{Z})}$.

The $GL_n(\mathbf{Z}/p\mathbf{Z})$ action on $E(x_1, \ldots, x_n) \otimes P[y_1, \ldots, y_n] \cong H^*(E_1 \times \cdots \times E_n, \mathbf{Z}/p\mathbf{Z})$ is induced by the contragradient representation on $H^1(E_1 \times \cdots \times E_n, \mathbf{Z}/p\mathbf{Z})$, where this action is extended to the y_i's, via the Bockstein monomorphism $\beta x_i = y_i$.

$$(g_{ij})x_s = \sum_{i=1}^{n} g_{is} x_i, \ (g_{ij})y_s = \sum_{i=1}^{n} g_{is} y_i, \text{for } 1 \leq s \leq n.$$

The idea is to define some elements of $(E(x_1, \ldots, x_n) \otimes P[y_1, \ldots, y_n])^{GL_n}$ and show that they form a generating set for this algebra.

(4.3) Let L_n, $L_{n,i}$, and $M_{n;s_1,\ldots,s_m}$ denote respectively the following graded determinants (in the sense of Hyunh [10] page 321):

$$L_n = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1^p & \cdots & y_n^p \\ \vdots & & \\ y_1^{p^{n-1}} & \cdots & y_n^{p^{n-1}} \end{vmatrix}, \quad L_{n,i} = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1^p & \cdots & y_n^p \\ \vdots & & \\ y_1^{p^n} & \cdots & y_n^{p^n} \end{vmatrix},$$

$$M_{n;s_1,\dots,s_m} = \frac{1}{m!} \begin{vmatrix} x_1 & \cdots & x_n \\ \vdots & & \\ x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \\ \vdots & & \\ y_1^{p^{n-1}} & \cdots & y_n^{p^{n-1}} \end{vmatrix}.$$

Here the i-th power is omitted from the second determinant, $0 \leq i \leq n-1$. There are m rows of x_i's and the s_i-th's powers are omitted, where $0 \leq s_1 < \cdots < s_m \leq n-1$, in the last determinant. (4.4) Also let

$$Q_{n,i} = \frac{L_{n,i}}{L_n}.$$

Note 4.5. $L_n = \prod_{i=1}^n V_i$, where $V_i = \prod_{a_j \in \mathbf{Z}/p\mathbf{Z}} (a_1 y_1 + \cdots + a_{i-1} y_{i-1} + y_i)$ and $L_{n,0} = L_n^p$. The degrees of the previous invariants (of U_n, $SL_n(\mathbf{Z}/p\mathbf{Z})$, and $GL_n(\mathbf{Z}/p\mathbf{Z})$) are given by:
$|V_i| = 2p^{i-1}$ $[2^{i-1}$, if $p=2]$. $|L_n| = 2(1 + \cdots + p^{n-1})$ $[2^n - 1$, if $p=2]$, $|Q_{n,i}| = 2(p^n - p^i)$ $[2^n - 2^i$, if $p = 2]$, and $|M_{n;s_1,\dots,s_m}| = m + 2((1 + \cdots + p^{n-1}) - (p^{s_1} + \cdots + p^{s_m}))$.

We recall here the following theorems concerning the ring of invariants of $P_n(N)$ over a polynomial algebra.

Theorem 4.6. *(Campbell) The invariants* $\{L_{\nu_j}/L_{\nu_{j-1}}, Q_{\nu_j,i} - (Q_{\nu_{j-1},i-n_j})^{p^{n_j}} \mid$ $\mid 1 \leq j \leq k \ \nu_{j-1} < i \leq \nu_j - 1\}$ *of* $SP_n(N)$ *are algebraically independent and form a polynomial basis for the invariants of* $SP_n(N)$ *. Moreover,*

$$\begin{aligned}(P[y_1,\dots,y_n])^{P_n(N)} = & P\big(L_{\nu_j}/L_{\nu_{j-1}},\ Q_{\nu_j,i} - (Q_{\nu_{j-1},i-n_j})^{p^{n_j}} \\ & \mid 1 \leq j \leq k \ \nu_{j-1} < i \leq \nu_j - 1\}\big).\end{aligned}$$

Here we understand $Q_{\nu_{j-1},i-n_j}$ *to be* 0, *if* $i < n_j$.

The following lemma gives the relation among the generators of the previous invariants.

Lemma 4.7. *a)* $L_n = (-1)^{n-1} L_{n-1} (\sum_0^{n-1} (-1)^s Q_{n-1,s} y_n^{p^s})$.
b) $Q_{n,s} = Q_{n-1,s-1} V_n^{p-1} + Q_{n-1,s}^p$.
c) $Q_{n,i} = \sum_{1 \leq j_1 < \cdots < j_{n-i} \leq n} \prod_1^{n-i} (V_{j_s}^{p-1})^{p^{i+s-j_s}}$.

(4.8) Let $I = (i_1, \ldots, i_m)$ be a totally ordered sequence and $I' = (i_{m+1}, \ldots, i_n)$ its ordered complement in $\{1, \ldots, n\}$. We denote its length by $l(I) = m$ and an order between sequences of the same length is defined: $I < J$, if there is a t, $1 \leq t \leq m$, such that $i_t < j_t$ and $i_s = j_s$, for $t+1 \leq s \leq m$. Let also $x_I := x_{i_1} \ldots x_{i_m}$ and

$$[s_1, \ldots, s_{n-m}]_{I'} = \begin{vmatrix} y_{i_{m+1}}^{p^{s_1}} & \ldots & y_{i_n}^{p^{s_1}} \\ \vdots & \ldots & \\ y_{i_{m+1}}^{p^{s_{n-m}}} & \ldots & y_{i_n}^{p^{s_{n-m}}} \end{vmatrix}.$$

Lemma 4.9. *(Huynh) Let $0 \leq s_1 < \cdots < s_m \leq n-1$.*
a) $M_{n;s_1,\ldots,s_m} L_n^{m-1} = (-1)^{m(m-1)/2} M_{n;s_1} \ldots M_{n;s_m}$.
b) $(M_{n;s})^2 = 0$.
c) $M_{n;0,1,\ldots,n-1} L_n^{n-1} = (-1)^{n(n-1)/2} M_{n;0} \ldots M_{n;n-1} = x_1 x_2 \ldots x_n$.
d) $M_{n-1;s_1,\ldots,s_m} V_n = M_{n;s_1,\ldots,s_m} - \sum_1^m (-1)^{m+i} M_{n;s_1,\ldots,\widehat{s_i},\ldots,n-1} Q_{n-1,s_i}$.
e) $M_{k;s_1,\ldots,s_m} V_{k+1} \ldots V_n = M_{n;s_1,\ldots,s_m} + \sum_{(t_1,\ldots,t_m) > (s_m - m+1,\ldots,s_m)} M_{n;t_1,\ldots,t_m} f_{t_1,\ldots,t_m}$

Here $f_{t_1,\ldots,t_m} \in (P[y_1 \ldots, y_n])^{U_n}$.
f) $M_{n;s_1,\ldots,s_m} L_n^{m-1} = (-1)^{m(m-1)/2} \prod_1^m \left(\sum_{r=s_j+1}^n M_{r;r-1} V_{r+1} \ldots V_n Q_{r-1,r-1-s_j} \right)$
g) $M_{n-1;s_1,\ldots,s_{m-1}} x_n = M_{n;s_1,\ldots,s_{m-1},n-1} -$
$- \sum_{|I|=m,\ n \notin I} sgn\sigma_I x_I [0, \ldots, \widehat{s}_i, \ldots, \widehat{s}_{m-1}, \ldots, n-2]'_I$.

Note 4.10. i) Let f be a $P_n(N)$-invariant given by $f = \sum_{I \leq J} x_I f_I$ and let J take m_j values from $\{\nu_{j-1}+1, \ldots, \nu_j\}$, for $1 \leq j \leq t$, $l(I) = m$ and $\sum m_j = m$. Then J is of the form:
$(\nu_1 - m_1 + 1, \ldots, \nu_1, \nu_2 - m_2 + 1, \ldots, \nu_2, \ldots, \nu_t - m_t + 1, \ldots \nu_t)$. We also note that not all indices less than J might appear in the sum of any element in $P_n(N)$. Since each block is an element of GL_{n_i}, I can take at most m_t values from the t-th block, $min(m_t + m_{t-1}, n_{t-1})$ values from the t-1-st block, and so on. In general, it can take $min(m_t + \cdots + m_i, n_i)$ values from the i-th block.
ii)Let f be an $SP_n(N)$-invariant given by $f = \sum_{I<J} x_I f_I$, where $J = (j_1, \ldots, j_m)$, $j_{m-t} = \nu_{k-t}$, $0 \leq t \leq n_k - 1$, and $j_{m-n_k} = \nu_\ell$. If we let $F = \sum_{I' \leq J'} x_{I'} f_{I'}$, where $J' \sqcup (\nu_{k-1}+1, \ldots, \nu_k) = J$, and $f = (\sum_{I' \leq J'} x_{I'} f_{I'}) x_{(\nu_{k-1}+1, \ldots \nu_k)}$ $+ \sum_{I<J} x_I f_I$, then F is also an $SP_n(N)$-invariant.

Lemma 4.11. *(Huynh) Let f be a U_n-invariant given by $f = \sum_{I<J} x_I f_I$. Then f_J is a U_n-invariant having the factor $V_{J'} := \prod_{i \in J'} V_i$, where J' is the complement of J in $\{1, 2, \ldots, j_m\}$.*

Lemma 4.12. *Let f be an $SP_n(N)$-invariant given by $f = \sum_{I \leq J} x_I f_I$. Let I be an index set involved in the above sum. If there exists an element g of $P_n(N)$ such that $gx_I = sign(g) x_J$ and fixes $x_{(I \cup J)'}$, then f_I contains (y_J) as a factor.*

Proof. There always exists a σ, product of transpositions, in Σ_n such that its associated element $g(\sigma)$, in GL_n satisfies the required properties. If $g(\sigma) \in P_n(N) - SP_n(N)$ then by changing the sign of its elements in the first column and applying it on the f, noting also that f is a U_n-invariant, we have the required property. Here $(g'(\sigma))^{-1} = g(\sigma)$, $g'(\sigma)f = f \Rightarrow g'(\sigma)x_I f_I = x_J g'(\sigma) f_I = x_J y_{J'} f'_J \Rightarrow f_I = (g'(\sigma))^{-1} y_{J'} f'_J \Rightarrow f_I = y_{I'}(g'(\sigma))^{-1} f'_J$.

Note. The existance of g in the lemma above depends on the form of I, i.e. I and J must contain the same number of elements from the same block.

Note 4.13. i) Let f be an $SP_n(N)$-invariant given by $f = \sum_{I \leq J} x_I f_I$ such that $j_m = \nu_k$, $j_{m-q} = \nu_{k-\ell}$, and there is no $\nu_{k-\ell+i}$ between j_m and j_{m-q}, for $1 \leq i < \ell$. Then f_{I_1} contains V_{ν_k} as a factor where $I_1 = (j_1, \ldots, j_{m-q}, \nu_k - d, \ldots, \nu_k - 1)$.
ii) Let $I = (i_1, \ldots, i_k)$ with $i_k = m < n$ and
$F = \sum_{0 \leq s_1 < \cdots < s_k = m-1} [0, 1, \ldots, \widehat{s}_1, \ldots, \widehat{s}_k]_I f_{(s_1, \ldots, s_k)}$ such that F contains the factor L_n/L_m, and each $f_{(s_1, \ldots, s_k)}$ is an invariant of U_n, then each $f_{(s_1, \ldots, s_k)}$ contains the factor L_n/L_m.
iii) Given $f = \sum_{I \leq J} x_I f_I$ an $SP_n(N)$-invariant, we shall express it in terms of the invariant generators. First we show that the exterior factors and their invariant polynomial coefficients can be expressed in terms of invariant matrices $M_{n;s_1,\ldots,s_m}$ and $SP_n(N)$-invariant polynomials. The next lemma proves the last part of the last sentence.

Lemma 4.14. *Let f be zero, where f is given by* $\sum_{0 \leq s_1 < \cdots < s_m \leq \nu_1 - 1} M_{\nu_1; s_1, \ldots, s_m} f_{(\nu_1; s_1, \ldots, s_m)} + \sum_{j=2}^{k} \sum_{\ell=1}^{n_j} \sum_{0 \leq s_1 < \cdots < s_m = \nu_j - \ell} M_{\nu_j; s_1, \ldots, s_m} f_{(\nu_j; s_1, \ldots, s_m)}$.
Then all the $f_{(\nu_j; s_1, \ldots, s_m)}$ are zero. Moreover, if f is an $SP_n(N)$-invariant, then all the $f_{(\nu_j; s_1, \ldots, s_m)}$ are also $SP_n(N)$ -invariants.

Proof. The lemma is proved by induction on j, lemma (4.9) is extensively used. We formulate the idea for the first steps.
Let $F_i = \sum_{\ell=1}^{n_i} \sum_{0 \leq s_1 < \cdots < s_m = \nu_i - \ell} M_{\nu_i; s_1, \ldots, s_m} f_{(\nu_i; s_1, \ldots, s_m)}$ and then $f = \sum F_i$. We show that $f_{(\nu_i; s_1, \ldots, s_m)} = 0$ for all indeses in the sum.

$$F_i(L_{\nu_k}/L_{\nu_i}) = \sum_{\ell=1}^{n_j} \sum_{0 \leq s_1 < \cdots < s_m = \nu_k - \ell} M_{\nu_k; s_1, \ldots, s_m} f_{(\nu_i; s_1, \ldots, s_m)} + \\ + \sum_{(t_1, \ldots, t_m) > (\nu_i - m + 1, \ldots, \nu_i - 1)} M_{\nu_k; t_1, \ldots, t_m} h_{(t_1, \ldots, t_m)}.$$

Here the $h_{(t_1, \ldots, t_m)}$ are algebraic combinations of $f_{(\nu_i; s_1, \ldots, s_m)}$ and elements from $P[y_1, \ldots, y_{\nu_n}]^{U_n}$.
First step:
$f(L_{\nu_k}/L_{\nu_1}) = M_{\nu_k; 0, \ldots, m-1} f_{\nu_1; 0, \ldots, m-1} + \sum_{(t_1, \ldots, t_m), t_m \geq m} M_{\nu_k; t_1, \ldots, t_m} h_{(t_1, \ldots, t_m)}$

$$f(L_{\nu_k}/L_{\nu_1})M_{\nu_k;m,m+1,\dots,\nu_k-1} = M_{\nu_k;0,\dots,m-1}M_{\nu_k;m,m+1,\dots,\nu_k-1}f_{\nu_1;0,\dots,m-1} + \\ + \sum_{(t_1,\dots,t_m),t_m\geq m} M_{\nu_k;t_1,\dots,t_m}M_{\nu_k;m,m+1,\dots,\nu_k-1}h_{(t_1,\dots,t_m)}.$$

Applying (4.9) again, we see that $f_{\nu_1;0,\dots,m-1} = 0$ and this is the smallest index set. Inducting on ν_1 and m we show that $f_{(\nu_1;s_1,\dots,s_m)} = 0$, and hence $F_1 = 0$.

Lemma 4.15. *Let $f = H + Fx_{\nu_k}$ be an $SP_n(N)$-invariant, where $f \in E(x_1,\dots,x_{\nu_k})\otimes P[y_1,\dots,y_{\nu_k}]$, while $H, F \in E(x_1,\dots,x_{\nu_k-1})\otimes P[y_1,\dots,y_{\nu_k}]$ and F given by:*

$$\sum_{0\leq s_1<\cdots<s_{m-1}\leq\nu_1-1} M_{\nu_1;s_1,\dots,s_{m-1}}f_{(\nu_1;s_1,\dots,s_{m-1})} + \\ + \sum_{j=2}^{k-e}\sum_{\ell=1}^{n_j}\sum_{0\leq s_1<\cdots<s_{m-1}=\nu_j-\ell} M_{\nu_j;s_1,\dots,s_{m-1}}f_{(\nu_j;s_1,\dots,s_{m-1})}.$$

Then H and F are $SP_{n-1}(N)$-invariants and $x_{\nu_k}F$ can be written in the form:

$$x_{\nu_k}F = \sum_{0\leq s_1<\cdots<s_{m-1}=\nu_k-\ell} M_{\nu_k;s_1,\dots,s_{m-1}}f_{(\nu_k;s_1,\dots,s_{m-1})} - \sum_{|I|=m,\ I\leq J} sgn\sigma_I x_I \\ \sum_{0\leq s_1<\cdots<s_{m-1}\leq\nu_k-2} [0.\dots,\widehat{s}_i,\dots,\widehat{s}_{m-1},\dots\nu_k-2]_I f'_{(\nu_k;s_1,\dots,s_{m-1})}.$$

Here $J = (\nu_k - m,\dots,\nu_k-1)$ and $f'_{(\nu_k;s_1,\dots,s_{m-1})}$ are both $SP_{n-1}(N)$ and U_n-invariants.

Sketch of proof. We outline the proof which is technical but straitforward. We prove the lemma using induction on j, the number of blocks. We start with the biggest index set $I^* = (\nu_{k-e} - m + 2,\dots,\nu_{k-e})$ involved in F and we note that its polynomial coefficient $\sum_{\ell=1}^{n_{k-e}} \sum_{0\leq s_1<\cdots<s_{m-1}\leq\nu_{k-e-\ell}} [0,\dots,\widehat{s}_{m-1},\dots,\nu_{k-e}-1]_I f_{(\nu_{k-e};s_1,\dots,s_{m-1})}$ is divided by $L_{\nu_k-1}/L_{\nu_{k-e}}$, (lemma 4.11). Hence each $f_{(\nu_{k-e};s_1,\dots,s_{m-1})}$ has the same property, (Note 4.13 ii)). This is crusial, since using lemma 4.9 e), we can rewrite

$$M_{\nu_{k-e};s_1,\dots,s_{m-1}}f_{(\nu_{k-e};s_1,\dots,s_{m-1})} = M_{\nu_k-1;s_1,\dots,s_{m-1}}f'_{(\nu_{k-e};s_1,\dots,s_{m-1})} + \\ \sum_{(t_1,\dots,t_{m-1})>(\nu_{k-e-1}-m+2,\dots,\nu_{k-e-1})} M_{\nu_k-1;t_1,\dots,t_{m-1}}f_{(t_1,\dots,t_{m-1})}".$$

The last step is to rewrite $M_{\nu_k-1;s_1,\dots,s_{m-1}}x_{\nu_k}$ or $M_{\nu_k-1;t_1,\dots,t_{m-1}}x_{\nu_k}$ in the required form using lemma 4.9 g). We use the technique above in the following example.

EXAMPLE: Let $n = (2,2)$ and f given by the following sum be an $SP_4(N)$ -invariant.

$f = x_{(1,2)}f_{(1,2)} + x_{(1,3)}f_{(1,3)} + x_{(1,4)}f_{(1,4)} + x_{(2,3)}f_{(2,3)} + x_{(2,4)}f_{(2,4)}$

$f = x_{(1,2)}f_{(1,2)} + x_{(1,3)}f_{(1,3)} + x_{(2,3)}f_{(2,3)} + (x_1 f_{(1,4)} + x_2 f_{(2,4)})x_4$.
$f = H + Fx_4$, $H = x_{(1,2)}f_{(1.2)} + x_{(1,3)}f_{(1,3)} + x_{(2,3)}f_{(2,3)}$, and $F = (x_1 f_{(1,4)} + x_2 f_{(2,4)}) = M_{2;0}f_{(2;0)} + M_{2;1}f_{(2;0)}$. H and F are $SP_3(2,1)$-invariants and F is a U_n-invariant. By lemma 4.11 both $f_{(1,4)}$ and $f_{(2,4)}$ contain the factor V_3 and by lemma 4.12 the same holds for $f_{(2;0)}$ and $F_{(2;1)}$. Now we apply lemma 4.9 d):
$M_{2;0}V_3 f'_{(2;0)} = M_{3;0}f'_{(2;0)} - M_{3;2}Q_{2,0}f'_{(2;0)}$ and
$M_{2;0}V_3 f'_{(2;0)} = M_{3;0}f'_{(2;0)} - M_{3;2}Q_{2,0}f'_{(2;0)}$ and Lemma 4.9 g) is applied in the following: $M_{3;i}x_4 = M_{4;i,3} - \sum_{I<(3,4)} sgn(I)x_I[0,\widehat{i},2]_I$, for $i = 0,1,2$.
Lastly, $Fx_4 = M_{4;0,3}f'_{(2;0)} + M_{4;1,3}f'_{(2;1)} - M_{4;2,3}(Q_{2,0}f'_{(2;0)} + Q_{2,1}f'_{(2;1)}) - \sum_{l(I)=2} sgn(I)x_I[1,2]_I f'_{(2;0)} - \sum_{l(I)=2} sgn(I)x_I[0,2]_I f'_{(2;1)}$
$+ \sum_{l(I)=2} sgn(I)x_I[0,1]_I(Q_{2,0}f'_{(2;0)} + Q_{2,1}f'_{(2;1)})$.

Note 4.16. The lemma above remains true, if ν_{k-e} is replaced by $\nu_k - 1$, since $M_{\nu_k-1;s_1,\dots,s_m}$ is still an $SP_{n-1}(N)$-invariant.

Lemma 4.17. *Let f be an $SP_n(N)$-invariant given by $f = \sum_{I\leq I^*} x_I f_I$, where $I^* = (i_1^*,\dots,i_m^*)$, $i_{d_1+\dots+d_t} = \nu_t$, $1 \le t \le k$, $m = d_1 + \dots + d_{k^*}$, and $d_{k^*} < n_{k^*}$, if $n_{k^*} > 1$. Then f can be written in the form:*

$$f = \sum_{j=j_m}^{k^*} \sum_{\ell=1}^{n_j} \sum_{0\le s_1<\dots<s_m=\nu_j-\ell} M_{\nu_j;s_1,\dots,s_m} f_{(\nu_j;s_1,\dots,s_m)}.$$

Here $\nu_{j_m} - 1 < m \le \nu_{j_m}$ and $j_m \ge 2$. Moreover all $f_{(\nu_j;s_1,\dots,s_m)}$ are $SP_n(N)$-invariants.

Proof. Since the proof is lengthy and technical, let us discuss the main points and work out a complete example. The idea is to use induction on the number of x_i's and the length of I^*. Since $i_m^* = \nu_{k^*}$, induction on I^* is equivalent inducting on the size of the k- block. First we prove the case $m = 1$ and then $m > 1$ using three steps:
i) We prove the lemma for $i_m^* = n_1$;
ii) We suppose the lemma is true for $N = (n_1,\dots,n_{k-1})$ and prove it for the case $n_k = 1$;
iii) We suppose the lemma is true for $n_k = u$ and prove it for the case $n_k = u+1$. To be reduced to the case where induction holds, we factor out $x_{i_m^*}$, ($f = \sum_{I<I^*, i_m^*\notin I} x_I f_I + (\sum_{J\leq J^*} x_J f_J)x_{i_m^*}$), and we note that f is an $SP_{\nu_\ell-1}(N)$ and U_n-invariant, where $P_{\nu_\ell-1} = (\nu_1,\dots,\nu_\ell-1)$ and $i_m^* = \nu_\ell$. Then we start with the sum over J noting that the length of J, $l(J)$, is $m-1$ and induction hypothesis can be applied. Since this is a U_n-invariant, using lemma (4.9), we bring the new form into the required one plus a sum where the exterior factors do not contain $x_{i_m^*}$. This is because induction hypothesis provides generators with exterior factors of length $m-1$ and invariants of $SP_{\nu_\ell-1}(N)$ instead of $SP_n(N)$. The last step is to write those invariants as $SP_n(N)$-invariants. This

is again achieved using mostly lemma (4.12).

EXAMPLE: Let $n = (2,2)$ and f given by the following sum be an $SP_4(N)$ -invariant as in lemma 4.15.

$f = x_{(1,2)}f_{(1,2)} + x_{(1,3)}f_{(1,3)} + x_{(1,4)}f_{(1,4)} + x_{(2,3)}f_{(2,3)} + x_{(2,4)}f_{(2,4)}$
$= x_{(1,2)}f_{(1,2)} + x_{(1,3)}f_{(1,3)} + x_{(2,3)}f_{(2,3)} + (x_1 f_{(1,4)} + x_2 f_{(2,4)})x_4$.

Here $I^* = (2,4)$ and $x_1 f_{(1,4)} + x_2 f_{(2,4)}$ is an $SP_4(N)$ and U_4-invariant. Hence $x_1 f_{(1,4)} + x_2 f_{(2,4)} = M_{2;0} f_{(2;0)} + M_{2;1} f_{(2;0)}$ by induction. From example in lemma 4.15 we have the following $SP_4(N)$ and U_4-invariant.

$f - M_{4;0,3}f'_{(2;0)} - M_{4;1,3}f'_{(2;1)} + M_{4;2,3}(Q_{2,0}f'_{(2;0)} + Q_{2,1}f'_{(2;1)}) = \sum_{I<(2,4),\ \{4\}\notin I} x_I h$

and this sum is an $SP_3(2,1)$ and U_4-invariant.

By induction $\sum_{I<(2,4),\ \{4\}\notin I} x_I h = M_{2;0,1}h'_{(2;0)} - M_{3;0,2}h'_{(3,0,2)} + M_{3;1,2}h'_{(3,1,2)}$.

Now $x_{(2,3)}h_{(2,3)}$ is a summand in the sum above. Since $f_{(2,3)}$ contains y_4 as a factor and the polynomial coefficient of $x_{(2,3)}$ in $M_{4;0,3}f'_{(2;0)} - M_{4;1,3}f'_{(2;1)} + M_{4;2,3}(Q_{2,0}f'_{(2;0)} + Q_{2,1}f'_{(2;1)})$ also contains y_4 as a factor, $h_{(2,3)}$ contains y_4 as a factor as well. Finally $h'_{(3.1,2)}$ and $h'_{(3,0,2)}$ contain y_4 as a factor and hence $M_{3;0,2}V_4 h''_{(3,0,2)} = M_{4;0,2}h''_{(3\,0,2)} + M_{4;2,3}h''_{(3,0,2)}Q_{3,2} - M_{4;0,3}h''_{(3,0,2)}Q_{3,2}$.

The required expression for f follows using the expression above and the appropriate one for $M_{3;1,2}h'_{(3.1\,2)}$.

REMARK(4.18). If in the lemma above $I^* = (i_1^*, \ldots, i_m^* = \nu_{k_*})$ and $i^*_{m-n_{k_*}+1} = \nu_{k_*-1} + 1$, then we prove the lemma following almost the same technique. First we decompose f as follows:

$$\sum_{I<I^*,\{\nu_{k_*-1}+1,\ldots,\nu_{k_*}\}\notin I} x_I f_I + \Big(\sum_{J\leq J^*} x_J f_J\Big) x_{\nu_{k_*-1}+1}\cdots x_{\nu_{k_*}}.$$

Here $J^* \cup \{\nu_{k_*-1}+1, \ldots, \nu_{k_*}\} = I^*$. Now $\sum_{J\leq J^*} x_J f_J$ is an $SP_n(N)$-invariant by lemma (4.13) and by lemma (4.17) we can write this sum in the following form:

$$a)\ \sum_{J\leq J^*} x_J f_J = \sum_{0\leq s_1<\cdots<s_{m-n_{k_*}}\leq \nu_1-1} M_{\nu_1;s_1,\ldots,s_{m-n_{k_*}}} f_{(\nu_1;s_1,\ldots,s_{m-n_{k_*}})} +$$

$$+ \sum_{j=2}^{e}\sum_{t=1}^{n_j} \sum_{0\leq s_1<\cdots<s_{m-n_{k_*}}=\nu_j-1} M_{\nu_j;s_1,\ldots,s_{m-n_{k_*}}} f_{(\nu_j;s_1,\ldots,s_{m-n_{k_*}})}.$$

Here the $f_{(\nu_j;s_1,\ldots,s_{m-n_{k_*}})}$ are $SP_n(N)$-invariants. We note that if J^* has a similar form as I^*, namely: $J^* = (i_1^*, \ldots, i_{m-n_{k_*}})$ such that $i^*_{m-n_{k_*}} = \nu_t$ and $i^*_{m-n_{k_*}-n_t+1} = \nu_{t-1} + 1$, we proceed to the longest subset of I^* where its last part does not cover the whole block. At this point we observe that $L_{\nu_{k_*}-1}/L_{\nu_*}$ divides $f_{(\nu_*;s_1,\ldots,s_{m-n_{k_*}})}$ and this is the key point to express the invariants above in the right form using the following formulas:

$$(4.19)\ M_{n-e;s_1,\ldots,s_k} x_{n-e+1}\cdots x_n = M_{n;s_1,\ldots,s_k,n-e,n-e+1,\ldots,n-1} - \sum_{l(I)=k+e,\ \{n-e-k,\ldots,n-e\}\notin I} sgn\sigma_I x_I[0,1,\ldots,\widehat{s}_1,\ldots\widehat{s}_m,\ldots,n-e-1]_I$$

(For proof we use the Laplace expansion.)
(4.20) $L_{n-e}x_{n-e+1}\dots x_n = M_{n;n-e,n-e+1,\dots,n-1} - \sum_{l(I)=e,\ I<\{n-e+1,\dots,n\}} sgn\sigma_I x_I$
$[0,1,\dots,n-e]_I$.
The last step is to induct on $j = 1,\dots,e-1$ and mimic the proof of lemma (4.15).

Theorem 4.21. *The algebra $(E(x_1,\dots,x_n)\otimes P[y_1,\dots,y_n])^{SP_n(N)}$ is a tensor product between the polynomial algebra $P((L_{\nu_j}/L_{\nu_{j-1}}),\ Q_{\nu_j,i}-(Q_{\nu_{j-1},i-n_j})^{p^{n_j}} \mid 1\le j\le k\ \ \nu_{j-1}<i\le \nu_j-1)$ and the $\mathbf{Z}/p\mathbf{Z}$ -module $SM[n]$, where $SM[n]$ is spanned by the set of elements consisting of the following monomials:*

$$M_{\nu_j;s_1,\dots,s_m};\ 1\le j\le k\ \ 1\le m\le \nu_j,\ 1\le \ell\le \nu_j, \text{and } 0\le s_1<\dots<s_m=\nu_j-\ell.$$

Here we understand $Q_{\nu_{j-1},i-n_j}$ to be 0, if $i<n_j$. Its algebra structure is determined by the following relations:
a) $(M_{\nu_j;s})^2=0$, for $1\le j\le k,\ \ \nu_{j-1}<s\le\nu_j-1$.

$$b)\ M_{\nu_j;s_1,\dots,s_m}(L_{\nu_j})^{m-1} = (-1)^{m(m-1)/2}\prod_{q=1}^{m}(M_{\nu_j-k_{(q)};s_q}(L_{\nu_j}/L_{\nu_j-k_{(q)}}) + \sum_{r=0}^{k_{(q)}-1}\sum_{\lambda=1}^{n_{j-r}} M_{\nu_{j-r};\nu_{j-r}-\lambda}(L_{\nu_j}/L_{\nu_{j-r}})\sum_{\mu=1}^{n_{j-r}-\lambda+1}(-1)^{\mu-1}\prod_{\alpha_0=s_q,\ \nu_{j-r-1}\le\alpha_i<\dots<\alpha_\mu=\nu_{j-r}-\lambda} Q_{\alpha_i,\alpha_{i-1}}).$$

Here $\nu_{j-1}<s_m\le\nu_j-1,\ \nu_{j_{(q)}-1}<s_q\le\nu_{j_{(q)}}-1$, and $k_{(q)}=j-j_{(q)}$.

Proof. The theorem follows from lemma (4.9)-b) and f).

Corollary 4.22. *(Huynh) The algebra $(E(x_1,\dots,x_n)\otimes P[y_1,\dots,y_n])^{U_n}$ is a tensor product between the polynomial algebra $P(V_i \mid 1\le i\le n)$ and the $\mathbf{Z}/p\mathbf{Z}$ -module $UM[n]$, where $UM[n]$ is spanned by the set of elements consisting of the following monomials:*

$$M_{s;s_1,\dots,s_m};\ 1\le m\le n,\ m\le s\le n, \text{ and } 0\le s_1<\dots<s_m=s-1.$$

Its algebra structure is determined by the following relations:
a) $(M_{\nu_j;s})^2=0$, for $1\le j\le k,\ \ \nu_{j-1}<s\le\nu_j-1$.
b) $M_{s;s_1,\dots,s_m}(V_1\dots V_s)^{m-1} = (-1)^{m(m-1)/2}\prod_{q=1}^{m}$
$(\sum_{r=s_q+1}^{s} M_{r;r-1}V_{r+1}\dots V_sQ_{r-1,s_q})$.
For $1\le m\le n,\ m\le s\le n$, and $0\le s_1<\dots<s_m=s-1$.

Now we proceed to the $P_n(N)$-invariants.

Lemma 4.23. *Let f be a B_n-invariant given by $f = \sum_{I \leq I^*} x_I f_I$. Then $I^* = (i_1^*, \ldots, i_m^*)$, then every f_I has the factor $L_{i_m^*}^{p-2}$, where $I = (i_1, \ldots, i_m)$.*

Theorem 4.24. *The algebra $(E(x_1, \ldots, x_n) \otimes P[y_1, \ldots, y_n])^{B_n}$ is a tensor product between the polynomial algebra $P((V_i)^{p-1} \mid 1 \leq i \leq n)$ and the $\mathbb{Z}/p\mathbb{Z}$ -module $BM[n]$, where $BM[n]$ is spanned by the set of elements consisting of the following monomials:*

$$M_{s;s_1,\ldots,s_m} L_s^{p-2};\ 1 \leq m \leq n,\ m \leq s \leq n, \text{ and } 0 \leq s_1 < \cdots < s_m = s-1.$$

Its algebra structure is determined by the following relations:
a) $(M_{\nu_j;s} L_s^{p-2})^2 = 0$, *for* $1 \leq j \leq k,\ \ \nu_{j-1} < s \leq \nu_j - 1$.
b) $M_{s;s_1,\ldots,s_m} L_s^{p-2} (L_s^{p-1})^{m-1} = (-1)^{m(m-1)/2} \prod_{q=1}^m$
$\left(\sum_{r=s_q+1}^{s} M_{r;r-1} L_r^{p-2} (V_{r+1})^{p-1} \ldots (V_s)^{p-1} Q_{r-1,s_q} \right)$
Here $1 \leq m \leq n,\ m \leq s \leq n$, *and* $0 \leq s_1 < \cdots < s_m = s-1$.

Theorem 4.25. *The algebra $(E(x_1, \ldots, x_n) \otimes P[y_1, \ldots, y_n])^{P_n(N)}$ is a tensor product between the polynomial algebra $P((L_{\nu_j}/L_{\nu_{j-1}})^{p-1}, Q_{\nu_j,i} - (Q_{\nu_{j-1},i-n_j})^{p^{n_j}} \mid 1 \leq j \leq k\ \ \nu_{j-1} < i \leq \nu_j - 1)$ and the $\mathbb{Z}/p\mathbb{Z}$ -module $PM[n]$, where $PM[n]$ is spanned by the set of elements consisting of the following monomials:*

$$M_{\nu_j;s_1,\ldots,s_m} L_{\nu_j}^{p-2}; 1 \leq j \leq k, 1 \leq m \leq \nu_j, 1 \leq \ell \leq \nu_j, 0 \leq s_1 < \cdots < s_m = \nu_j - \ell.$$

Here we understand $Q_{\nu_{j-1},i-n_j}$ to be 0, if $i < n_j$. Its algebra structure is determined by the following relations:
a) $(M_{\nu_j;s} L_{\nu_j}^{p-2})^2 = 0$, *for* $1 \leq j \leq k,\ \ \nu_{j-1} < s \leq \nu_j - 1$.

$$b) M_{\nu_j;s_1,\ldots,s_m} L_{\nu_j}^{p-2} (L_{\nu_j}^{p-1})^{m-1} = (-1)^{m(m-1)/2} \prod_{q=1}^{m} (M_{\nu_j - k_{(q)};s_q} L_{\nu_j - k_{(q)}}^{p-2}$$
$$(L_{\nu_j}/L_{\nu_j - k_{(q)}})^{p-1} + \sum_{r=0}^{k_{(q)}-1} \sum_{\lambda=1}^{n_{j-r}} M_{\nu_{j-r};\nu_{j-r}-\lambda} L_{\nu_{j-r}}^{p-2} (L_{\nu_j}/L_{\nu_{j-r}})^{p-1}$$
$$\sum_{\mu=1}^{n_{j-r}-\lambda+1} (-1)^{\mu-1} \prod_{\alpha_0 = s_q,\ \nu_{j-r-1} \leq \alpha_i < \cdots < \alpha_\mu = \nu_{j-r} - \lambda} Q_{\alpha_i,\alpha_{i-1}}).$$

For $\nu_{j-1} < s_m \leq \nu_j - 1,\ \ \nu_{j_{(q)}-1} < s_q \leq \nu_{j_{(q)}} - 1$, *and* $k_{(q)} = j - j_{(q)}$.

Note. The last summation in relation b) above can be expressed as a polynomial with respect to generators of $(P[y_1, \ldots, y_n])^{P_n(N)}$ by lemma (4.15).

The rings of invariants of lower parabolic subgroups denoted by $P_n(N)^t$ can be deduced from the theorem above after noting that $P_n(N)^t = \eta_n P_n'(N) \eta_n$, where $P_n' = (w_1 = n_k, \ldots, w_k = n_k + \cdots + n_1)$.

Theorem 4.26. *The algebra* $(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n])^{P_n(N)^t}$ *is a tensor product between the polynomial algebra* $P(\eta_n(L_{w_j}/L_{w_{j-1}})^{p-1},\ \eta_n Q_{w_j,i} - (\eta_n Q_{w_{j-1},i-n_{k+1-j}})^{p^{n_{k+1-j}}} \mid 1\le j\le k\ \ w_j=\sum^{j_{t=1}} n_{k+1-t}\ w_{j-1}<i\le w_j-1)$ *and the* $\mathbf{Z}/p\mathbf{Z}$ *-module* $P^tM[n]$, *where* $P^tM[n]$ *is spanned by the set of elements consisting of the following monomials:* $\eta_n(M_{\nu_j;s_1,\ldots,s_m}L_{\nu_j}^{p-2})$; $1\le j\le k$, $1\le m\le w_j$, $1\le \ell\le n_{k+1-j}$, and $0\le s_1<\cdots<s_m=w_j-\ell$. *Here we understand* $Q_{w_{j-1},i-n_{k+1-j}}$ *to be* 0, *if* $i<n_{k+1-j}$ *and its algebra structure is as in theorem (4.25).*

For the rest of this section we discuss modular invariant theory and extended Dyer-Lashof algebras.

There is a well known injection $i^*:\ H^*(G)\longrightarrow H^*(A)^{W_G(A)}$ induced from the inclusion $i:\ A\longrightarrow G$, where $\Sigma_{p^n,p}\le G\le \Sigma_{p^n}$, $A=\prod_{i=1}^n E_i$, and $W_G(A)$ is the Weyl subgroup of A in G (see Quillen's Theorem in [11]). The image of this map has been studied by Huynh in [11]. We recall his result:

Theorem 4.27. *(Huynh [11])*
a) Im $i^*(A,\ \Sigma_{p^n,p})\equiv E(W_1,\ldots,W_n)\otimes P[V_1,\ldots,V_n]$
b) Im $i^*(A,\ \Sigma^t_{p^n,p})\equiv E(\eta_n W_1,\ldots,\eta_n W_n)\otimes P[\eta_n V_1,\ldots,\eta_n V_n]$
c) Im $i^*(A,\ G)\equiv$ Im $i^*(A,\Sigma_{p^n,p})\cap H^*(A)^{W_G(A)}$
Here $\Sigma_{p^n,p}=\Sigma_{p^{n-1},p}\int\Sigma_p$. $\Sigma^t_{p^n,p}=\Sigma_p\int\Sigma_{p^{n-1},p}$, *and* $W_i=M_{i,i-1}L_{i-1}^{\frac{p-3}{2}}$.

Proposition 4.28. *a) Let* S *be the subalgebra of* $(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n])^{U_n}$ *generated by the following elements:* $\{V_i,\ W_i,\ \text{for } i=1,\ldots,n\}$. *Then*

$$S\cap(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n])^{P_n(N)}=S^{P_n(N)}.$$

Here $S^{P_n(N)}$ *is the dual to the extended Dyer-Lashof coalgebra of length n denoted by* $RN[n]$ *and isomorphic to a subalgebra of* $H^*(A)^{P'_n(N)}$.
b) Let S' *be the subalgebra of* $(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n])^{U^t_n}$ *generated by the following elements:* $\{\eta_n V_i,\ \eta_n W_i\mid i=1,\ldots,n\}$. *Then* $S'\cap(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n])^{\eta_n P_n(N)\eta_n}\equiv S^{P'_n(N)}$ *as algebras over the Steenrod algebra. Here* $S^{P_n(N)'}$ *is the dual to the extended Dyer-Lashof coalgebra of length n denoted by* $R'N[n]$ *and isomorphic to a subalgebra of* $H^*(A)^{P_n(N)}$.

The following corollary discuss the relation between coalgebras of homology operations and mod$p$-homologies of appropriate subgroups of Σ_{p^n}.

Corollary 4.29. *Let* $i(A,G_n)$ *denote the inclusion between the named subgroups, then*
(Im $i^*(A,G_n))^*$ *injects into* $H_*(G_n;\mathbf{Z}/p\mathbf{Z})$, *where the second asterisk denotes the dual.*
Hence: (Im $i^*(A,G_n))^*\ >\longrightarrow H_*(G_n;\mathbf{Z}/p\mathbf{Z})$, implies monomorphisms :

$$a)\ U[n] >\longrightarrow H_*(\Sigma_{p^n,p};\mathbf{Z}/p\mathbf{Z}),$$
$$b)\ R\mathcal{N}[n] >\longrightarrow H_*(G_n;\mathbf{Z}/p\mathbf{Z}),$$
$$c)\ R'\mathcal{N}[n] >\longrightarrow H_*(G'_n;\mathbf{Z}/p\mathbf{Z}).$$

Here $G_n = \Sigma_{p^{n_1}} \int \cdots \int \Sigma_{p^{n_\ell}}$ and $G'_n = \Sigma_{p^{n_\ell}} \int \cdots \int \Sigma_{p^{n_1}}$.

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