Werk

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Attractors which are Homeomorphic to Compact Abelian Groups

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Let $\mu : Y \approx Y$ be a self-homeomorphism on a manifold Y and $B \subseteq Y$ a compact subset with $\mu(B) = B$. B is an attractor, if it has a neighborhood U such that for every neighborhood V there exists a number N with $\mu^n(U) \subseteq V$ for $n \geq N$. We are interested in the special case where B has the homeomorphism type of a compact abelian group, because in this way we obtain examples generalizing the solenoid attractor and Anosov diffeomorphisms on tori. Furthermore, assume an attractor B contains a non periodic point $x \in B$ whose orbit $\Omega_x : \{\mu^n(x) | n \in \mathbb{Z}\}$ is dense in B. The compact space B carries a natural uniform structure, which can be transported to $\mathbb{Z}$ by the bijective map $n \mapsto \mu^n(x)$. This turns $\mathbb{Z}$ into a uniform space, such that the translations are uniformly continuous, and we now want to investigate the special case where this uniform structure is that of a topological group. Then the group law on $\Omega_x \approx \mathbb{Z}$ can be extended over B turning B into a compact abelian group, and we must have $\forall y \in B : \mu(y) = y + x$. We will see that in this case the structure of B can be completely classified.

We let $A \in M_n(\mathbb{Z})$ be an $n \times n$-matrix with integer entries and $\det A \neq 0$. By $\mathbf{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ we denote the n-dimensional torus and observe that our matrix A induces a group epimorphism $A : \mathbf{T}^n \twoheadrightarrow \mathbf{T}^n$; let $\mathcal{S}_A$ be the inverse limit of the tower $\left\{\mathbf{T}^n \xleftarrow{A} \mathbf{T}^n \xleftarrow{A} \mathbf{T}^n \xleftarrow{A} \cdots\right\}$. $\mathcal{S}_A$ is a compact abelian group; we think of it as "n-dimensional solenoid"[1]. We set $G_A := \bigcup_{k=0}^{\infty} A^{-k}\mathbb{Z}^n \subset \mathbb{Q}^n$ and observe that the character group of $\mathcal{S}_A$ is G_{A^t} with the discrete topology, where A^t denotes the transpose of the matrix A. The shift map $\sigma_A : \mathcal{S}_A \approx \mathcal{S}_A$ maps an element $(t_1, t_2, \ldots)$ of the inverse limit to $(At_1, t_1, t_2, \ldots)$.

For $A \in GL_n(\mathbb{Z})$ we get $\mathcal{S}_A = \mathbf{T}^n$, $G_A = \mathbb{Z}^n$ and σ_A is the Anosov diffeomorphism determined by A. For $n = 1$ and $A = (2)$ we get the dyadic solenoid $\mathcal{S}_2$, and $G_2 \subset \mathbb{Q}$ is the group of fractions whose denominator is a power of 2.

On $\mathbb{R}^n$ we use the maximum norm $\|(t_1, \ldots t_n)\| = \max_j |t_j|$; the corresponding matrix norm is given by $\|a_{jk}\| = \max_j \sum_k |a_{jk}|$.

[1] The reader should be warned that this term is used in the literature for a variety of different concepts

Lemma 1. *If $\|A^{-1}\| < 1$ then G_A is dense in $\mathbb{R}^n$. More precisely: For each $\mathbf{t} \in \mathbb{R}^n$ and each $k \geq 1$ there exist $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ with $\|\mathbf{t} - A^{-k}\mathbf{a} - \mathbf{b}\| \leq \|A^{-1}\|^k$ and $\|A^{-k}\mathbf{a}\| \leq 1$.*

Proof. At first we take $\mathbf{a}' \in \mathbb{Z}^n$ with $\|A^k\mathbf{t} - \mathbf{a}'\| \leq 1$, then $\mathbf{b} \in \mathbb{Z}^n$ with $\|A^{-k}\mathbf{a}' - \mathbf{b}\| \leq 1$ and set $\mathbf{a} := \mathbf{a}' - A^k\mathbf{b}$. We get $\|\mathbf{t} - A^{-k}\mathbf{a} - \mathbf{b}\| = \|A^{-k}(A^k\mathbf{t} - \mathbf{a}')\| \leq \|A^{-k}\| \leq \|A^{-1}\|^k$. □

We observe that the condition $\|A^{-1}\| < 1$ cannot be omitted: For

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{1}$$

we get $\|A^{-1}\| = 1$, $G_A = \mathbb{Z} \times G_2$ and $\mathcal{S}_A = \mathbf{T} \times \mathcal{S}_2$.

Proposition 1. *The shift map $\sigma_A : \mathcal{S}_A \approx \mathcal{S}_A$ is chaotic in the sense of Devaney [2, §1.8, Def.8.5] if $\|A^{-1}\| < 1$.*

Proof. By [5, Thm.36] it suffices to show that $A : \mathbf{T}^n \twoheadrightarrow \mathbf{T}^n$ is chaotic.
i) Periodic orbits are dense in $\mathbf{T}^n$: Let $\mathbf{t} \in \mathbb{R}^n$ and $\varepsilon > 0$ be given; we have to find $k > 1$, $\mathbf{t}' \in \mathbb{R}^n$ and $\mathbf{a} \in \mathbb{Z}^n$ with $\|\mathbf{t} - \mathbf{t}'\| \leq \varepsilon$ and $A^k\mathbf{t}' = \mathbf{t}' + \mathbf{a}$. Since $A^{-k} \mapsto 0$ there is k with $\left\|(I - A^{-k})^{-1}\right\| \leq 2$ and $2\|A^{-1}\|^k \leq \varepsilon$. Lemma 1 provides us with $\mathbf{a} \in \mathbb{Z}^n$ such that $\|(I - A^{-k})\mathbf{t} - A^{-k}\mathbf{a}\| \leq \|A^{-1}\|^k$. With $\mathbf{t}' := (I - A^{-k})^{-1} A^{-k}\mathbf{a}$ all requirements are satisfied.
ii) $\mathbf{T}^n$ contains a dense orbit: For $k \geq 1$ we denote by $V_k \subset \mathbb{Z}^n$ the finite set of all elements $\mathbf{a}$ with $\|A^{-k}\mathbf{a}\| \leq 1$ and order the *disjoint* union of all the V_k in a sequence $\mathbf{a}_1, \mathbf{a}_2, \ldots$; for each k we denote by $\mu(k)$ the number with $\mathbf{a}_k \in V_{\mu(k)}$. We set $\nu(k) := \sum_{j=1}^{k} \mu(j)$, define

$$\mathbf{t} := \sum_{k=1}^{\infty} A^{-\nu(k)}\mathbf{a}_k \in \mathbb{R}^n \tag{2}$$

and claim that the image of $\mathbf{t}$ in $\mathbf{T}^n$ generates a dense orbit under A. For prescribed $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ we take $k \geq 1$ with $\|A^{-1}\|^k / (1 - \|A^{-1}\|) \leq \varepsilon/2$ and, using Lemma 1, $\mathbf{a}_m \in V_k$, $\mathbf{b} \in \mathbb{Z}^n$ with $\|x - A^{-k}\mathbf{a}_m - \mathbf{b}\| \leq \|A^{-1}\|^k \leq \varepsilon/2$. With $\mathbf{b}' := \mathbf{b} - \sum_{j=1}^{m-1} A^{\nu(m-1)-\nu(j)}\mathbf{a}_j \in \mathbb{Z}^n$ we get:

$$\left\|\mathbf{x} - A^{\nu(m-1)}\mathbf{t} - \mathbf{b}'\right\| = \left\|\mathbf{x} - A^{-\mu(m)}\mathbf{a}_m - \sum_{j=m+1}^{\infty} A^{\nu(m-1)-\nu(j)}\mathbf{a}_j - \mathbf{b}\right\| \leq$$
$$\|\mathbf{x} - A^{-k}\mathbf{a}_m - \mathbf{b}\| + \|A^{-1}\|^k / (1 - \|A^{-1}\|) \leq \varepsilon.$$

iii) Devaney's definition also specifies that $A : \mathbf{T}^n \twoheadrightarrow \mathbf{T}^n$ must have sensitive dependence on initial conditions, but by [1] this follows from (i) and (ii). □

We observe in particular that for $\|A^{-1}\| < 1$ σ_A generates a dense orbit on $\mathcal{S}_A$. On the other hand, for A as in (1) we get $\sigma_A = \mathrm{id} \times \sigma_2$, and because of the identity mapping in the first factor such a homeomorphism cannot create a dense orbit and hence is not chaotic.

Remark 1. $\mathcal{S}_A$ contains a closed subgroup isomorphic to $\mathcal{D}_A := \varprojlim_k \mathbb{Z}^n/A^k\mathbb{Z}^n$; a monomorphism $\varphi : \mathcal{D}_A \rightarrowtail \mathcal{S}_A$ is constructed by mapping an element of $\varprojlim_k \mathbb{Z}^n/A^k\mathbb{Z}^n$ with coordinates $\mathbf{m}_k$ to the element of $\varprojlim_k \mathbb{T}^n$ with coordinates $A^{-k}\mathbf{m}_k$. The entire group $\mathcal{S}_A$ can be reconstructed from $\mathcal{D}_A$ in the following way: There is a quotient map $\Phi : \mathcal{D}_A \times I^n \twoheadrightarrow \mathcal{S}_A$ given by $\Phi((\mathbf{m}_k), \mathbf{t}) := \left(A^{-k}(\mathbf{t} + \mathbf{m}_k)\right)$. For $j = 1, \ldots n$ we let $\mathbf{e}_j \in \mathbb{Z}^n$ be the j-th unit vector and $\psi_j : \mathcal{D}_A \approx \mathcal{D}_A$ the homeomorphism mapping an element with coordinates $\mathbf{m}_k$ to the element with coordinates $\mathbf{m}_k + \mathbf{e}_j$. Φ imposes on $\mathcal{D}_A \times I^n$ the identifications $(\xi, \mathbf{t}) \sim (\psi_j(\xi), \mathbf{t}')$, where $\mathbf{t} = (t_1, \ldots t_n) \in I^n$ is a vector with $t_j = 1$ and $\mathbf{t}'$ is the same vector except that $t'_j = 0$.

Example 1. Let $A \in M_n(\mathbb{Z})$ with $|\det A| \geq 2$ be given; we want to construct a diffeomorphism on an analytic manifold having $\mathcal{S}_A$ as an attractor.

We consider $\mathbb{C}^n$ as normed unitary commutative algebra with maximum norm and with product defined coordinatewise, i.e. $(z_1, \ldots z_n) \cdot (u_1, \ldots u_n) = (z_1u_1, \ldots z_nu_n)$. Let $p : \mathbb{R}^n \to \mathbb{C}^n$ be the homomorphism from the additive group of $\mathbb{R}^n$ to the multiplicative group of invertible elements of $\mathbb{C}^n$ given by $p(t_1, \ldots t_n) := \left(e^{2t_1\pi i}, \ldots e^{2t_n\pi i}\right)$.

The group $\mathbb{Z}^n$ operates on $X := \mathbb{C}^n \times \mathbb{R}^n$ by

$$\mathbf{m}(\mathbf{z}, \mathbf{t}) := (\mathbf{z}, \mathbf{t} + \mathbf{m}); \tag{3}$$

$\mathbf{m} \in \mathbb{Z}^n$, $\mathbf{z} \in \mathbb{C}^n$, $\mathbf{t} \in \mathbb{R}^n$. There is a homeomorphism $\lambda : X \approx X$ defined by

$$\lambda(\mathbf{z}, \mathbf{t}) := (\vartheta p(\mathbf{t})(\mathbf{1} + \mathbf{z}), A\mathbf{t}); \tag{4}$$

$\vartheta < 1$ is a parameter whose value will be specified later. We observe:

$$\lambda(\mathbf{m}(\mathbf{z}, \mathbf{t})) = (A\mathbf{m})\lambda(\mathbf{z}, \mathbf{t}). \tag{5}$$

(5) allows to extend our group operation over $G_A \supset \mathbb{Z}^n$ by

$$\mathbf{g}(\mathbf{z}, \mathbf{t}) := \lim_{k \to \infty} \lambda^{-k}\left((A^k\mathbf{g})\,\lambda^k(\mathbf{z}, \mathbf{t})\right), \tag{6}$$

because the limit is attained for $k = r$ if $A^r\mathbf{g} \in \mathbb{Z}^n$.

We will need an explicit formula for this group operation, and as auxiliary construction we consider the homomorphism $L_r : \mathbb{R}^n \to \mathbb{C}^n$ defined by

$$L_r(\mathbf{t}) := p\left(-\sum_{k=0}^{r-1} A^k\mathbf{t}\right) \tag{7}$$

for $r \geq 0$. For $\mathbf{g} \in G_A$ the sequence $L_r(\mathbf{g})$ is stationary for large r and we can define

$$L(\mathbf{g}) := \lim_{r\to\infty} L_r(\mathbf{g}). \tag{8}$$

Furthermore we set

$$P_{\mathbf{g}}(\mathbf{t}) := \sum_{k=0}^{\infty} \vartheta^{-k} L_k(\mathbf{t}) \left(\mathbf{1} - L\left(-A^k\mathbf{g}\right)\right) \tag{9}$$

for $\mathbf{g} \in G_A$ and $\mathbf{t} \in \mathbb{R}^n$; the series terminates. By direct verification one shows:

$$L(\mathbf{m}) = \mathbf{1} \qquad \text{for } \mathbf{m} \in \mathbb{Z}^n \tag{10}$$

$$L(\mathbf{h}+\mathbf{g}) = L(\mathbf{h})L(\mathbf{g}) \tag{11}$$

$$L(A\mathbf{g}) = p(\mathbf{g})L(\mathbf{g}) \tag{12}$$

$$P_{\mathbf{m}}(\mathbf{t}) = 0 \qquad \text{for } \mathbf{m} \in \mathbb{Z}^n \tag{13}$$

$$P_{\mathbf{h}+\mathbf{g}}(\mathbf{t}) = P_{\mathbf{g}}(\mathbf{t}) + L(-\mathbf{g})P_{\mathbf{h}}(\mathbf{t}+\mathbf{g}) \tag{14}$$

$$P_{A\mathbf{g}}(A\mathbf{t}) = \vartheta p(\mathbf{t}) \left(P_{\mathbf{g}}(\mathbf{t}) + L(-\mathbf{g}) - \mathbf{1}\right). \tag{15}$$

These conditions imply that we can *define* a group operation of G_A on X by

$$\mathbf{g}(\mathbf{z},\mathbf{t}) := \left(L(\mathbf{g})\left(\mathbf{z} + P_{\mathbf{g}}(\mathbf{t})\right), \mathbf{t}+\mathbf{g}\right). \tag{16}$$

From equation (15) we derive

$$\lambda\left(\mathbf{g}(\mathbf{z},\mathbf{t})\right) = (A\mathbf{g})\lambda(\mathbf{z},\mathbf{t}) \tag{17}$$

for all $\mathbf{g} \in G_A$ and $(\mathbf{z},\mathbf{t}) \in X$, and this means that the operation (16) coincides with (6).

Let us suppose we are given $\mathbf{g} \in G_A$ with $A^{m+1}\mathbf{g} \in \mathbb{Z}^n$ but $A^m\mathbf{g} \notin \mathbb{Z}^n$. With $\alpha := \det A$ (remember $|\alpha| \geq 2$) we get $A^m\mathbf{g} \in \alpha^{-1}\mathbb{Z}^n \setminus \mathbb{Z}^n$ and hence: $\|P_{\mathbf{g}}(\mathbf{t})\| \geq \vartheta^{-m} \left\|L_m(\mathbf{t})\left(\mathbf{1} - L\left(-A^m\mathbf{g}\right)\right)\right\| - \sum_{k=0}^{m-1} \vartheta^{-k} \left\|L_k(\mathbf{t})\left(\mathbf{1} - L\left(-A^k\mathbf{g}\right)\right)\right\|$ $\geq \vartheta^{-m} \left\|\mathbf{1} - p\left(A^m\mathbf{g}\right)\right\| - 2\sum_{k=0}^{m-1} \vartheta^{-k} \geq \vartheta^{-m} \left|1 - e^{2\alpha^{-1}\pi i}\right| - 2\frac{\vartheta^{-m}-1}{\vartheta^{-1}-1}$ and therefore

$$\|P_{\mathbf{g}}(\mathbf{t})\| \geq \beta\vartheta^{-m} \qquad \text{for } \mathbf{g} \in A^{-m-1}\mathbb{Z}^n \setminus A^{-m}\mathbb{Z}^n; \tag{18}$$

especially

$$\|P_{\mathbf{g}}(\mathbf{t})\| \geq \beta \quad \text{for } \mathbf{g} \in G_A \setminus \mathbb{Z}^n \tag{19}$$

with $\beta := \left|1 - e^{2\alpha^{-1}\pi i}\right| - \frac{2\vartheta}{1-\vartheta}$. Choosing ϑ small enough we can assume

$$\beta > \frac{2\vartheta}{1-\vartheta} > 0. \tag{20}$$

From (19) it follows, that the action of G_A on X is properly discontinuous and hence that the quotient map $q : X \to X/G_A$ is a covering projection [7, Ch.2,Sec.6,Thm.7]; in particular $Y := X/G_A$ is a real analytic manifold. Y is an Eilenberg-MacLane space of type $(G_A, 1)$, because X is contractible and G_A is the group of covering transformations of q. Since the matrix A operates

as isomorphism on $G_A \subset \mathbb{Q}^n$, equation (17) shows that λ induces an analytic diffeomorphism $\mu : Y \approx Y$ by passing to the quotients.

Let $\varrho : \mathbb{R}^n \to \mathbb{C}^n$ be the map

$$\varrho(\mathbf{t}) := \sum_{k=1}^{\infty} \vartheta^k L_k \left(-A^{-k}\mathbf{t}\right), \tag{21}$$

then

$$\|\varrho(\mathbf{t})\| \leq \frac{\vartheta}{1-\vartheta}, \tag{22}$$

$$\varrho(A\mathbf{t}) = \vartheta p(\mathbf{t})\left(1 + \varrho(\mathbf{t})\right), \tag{23}$$

$$\lambda\left(\varrho(\mathbf{t}), \mathbf{t}\right) = \left(\varrho(A\mathbf{t}), A\mathbf{t}\right). \tag{24}$$

Condition (24) implies that $\Gamma := \left\{(\varrho(\mathbf{t}), \mathbf{t}) \,\middle|\, \mathbf{t} \in \mathbb{R}^n\right\}$ satisfies $\lambda(\Gamma) = \Gamma$. We set $\varepsilon(\mathbf{z}, \mathbf{t}) := \|\mathbf{z} - \varrho(\mathbf{t})\| \geq d\left((\mathbf{z}, \mathbf{t}), \Gamma\right)$. From (23) we conclude $\varepsilon\lambda(\mathbf{z}, \mathbf{t}) = \vartheta\varepsilon(\mathbf{z}, \mathbf{t})$ and hence $\lim_{k\to\infty} d\left(\lambda^k(\mathbf{z}, \mathbf{t}), \Gamma\right) = 0$ uniformly on each set of the form

$$U_R := \left\{(\mathbf{z}, \mathbf{t}) \in X \,\middle|\, \|\mathbf{z} - \varrho(\mathbf{t})\| \leq R\right\}. \tag{25}$$

This means that Γ is an "attractor" of λ.[2]

It is our intention to show that $B := \overline{q(\Gamma)} \subseteq Y$ is an attractor of $\mu : Y \approx Y$ homeomorphic to $\mathcal{S}_A$. To this end we construct a map $\Omega : \mathcal{D}_A \times I^n \to X$ by

$$\Omega(\xi, \mathbf{t}) := \left(\sum_{k=1}^{\infty} \vartheta^k L_k \left(-A^{-k}\left(\mathbf{t} + \mathbf{m}_k\right)\right), \mathbf{t}\right), \tag{26}$$

where $\mathbf{m}_k$ is the coordinate of ξ in $\mathbb{Z}^n / A^k\mathbb{Z}^n$. We claim that Ω is an embedding; if this were false we could find two points $\xi \neq \xi'$ with $\Omega(\xi, \mathbf{t}) = \Omega(\xi', \mathbf{t})$. Let k be the least integer, such that the coordinates of ξ and ξ' satisfy $\mathbf{m}'_k - \mathbf{m}_k \notin A^k\mathbb{Z}^n$, and proceed as in equation (18) to show $\|\Omega(\xi, \mathbf{t}) - \Omega(\xi', \mathbf{t})\| \geq \vartheta^m \left|1 - e^{2\alpha^{-1}\pi i}\right| > 0$. This contradicts the assumption.

If $\xi_{\mathbf{m}} \in \mathcal{D}_A$ denotes the point with all coordinates equal to $\mathbf{m}$, then we get

$$\Omega\left(\xi_{\mathbf{m}}, \mathbf{t}\right) = \left(\varrho(\mathbf{t} + \mathbf{m}), \mathbf{t}\right) = (-\mathbf{m}) \cdot \left(\varrho(\mathbf{t} + \mathbf{m}), \mathbf{t} + \mathbf{m}\right) \in G_A \cdot \Gamma. \tag{27}$$

Since $\Delta := \Omega\left(\mathcal{D}_A \times I^n\right)$ is compact and the points of the form $\xi_{\mathbf{m}}$ are dense in $\mathcal{D}_A$ we conclude

$$G_A \cdot \Delta = \overline{G_A \cdot \Gamma} = q^{-1}\left(B\right). \tag{28}$$

B is an attractor of μ, because Δ is contained in U_R when R is large enough. From (28), (19) and (20) it follows that $q\Omega : \mathcal{D}_A \times I^n \twoheadrightarrow B$ is the quotient map with the same identifications as in Remark 1, hence there is a homeomorphism $h : B \approx \mathcal{S}_A$ with $hq\Omega = \Phi$. (27) and (24) imply $hq\lambda\left(\varrho(\mathbf{t}), \mathbf{t}\right) = \sigma_A hq\left(\varrho(\mathbf{t}), \mathbf{t}\right)$ and therefore $\mu_B = h^{-1}\sigma_A h$.

[2] This is the only case where we consider *non compact* attractors

Remark 2. The reader should note that our approach starts with more general assumptions than R.F. Williams's work [8], the main difference being that our attractors need not be expanding; the dynamical systems under consideration need not even be differentiable. The attractor $\mathcal{S}_A$ fails to be expanding if A has an eigenvalue ≤ 1.

Example 2. We want to take a closer look at Example 1 in the case $n = 1$ and $A = (2)$, and we will show that the homeomorphism $\mu : Y \approx Y$, $Y = \mathbb{C} \times \mathbb{R}/G_2$, is topologically conjugate to a homeomorphism $\mu' : U \approx U$ of an open subset $U \subset S^3$, such that $B' := S^3 \setminus U$ is another copy of the dyadic solenoid $\mathcal{S}_2$. Furthermore, $\mu' : U \approx U$ can be extended over S^3 turning B' into a repellor.

By (20) we can find a number γ with

$$\frac{\vartheta}{1-\vartheta} < \gamma < \frac{1}{2}\beta; \tag{29}$$

then (19) implies

$$|P_g(t)| \geq \beta > 2\gamma \text{ for } g \in G_2 \setminus \mathbb{Z}. \tag{30}$$

We set

$$\Sigma := \{(u,t) \in \mathbb{C} \times \mathbb{R} \big| |u| \leq \gamma\} \tag{31}$$

and observe:

$$\lambda(\Sigma) \subseteq \Sigma^\circ, \tag{32}$$

$$\bigcup_{n=0}^{\infty} \lambda^{-n}(\Sigma) = \mathbb{C} \times \mathbb{R}, \tag{33}$$

$$\Sigma \cap g\Sigma = \emptyset \text{ unless } g \in \mathbb{Z}. \tag{34}$$

(34) follows from (30), and (33) holds because the "attractor" $\Gamma = \{(\varrho(t), t) \big| t \in \mathbb{R}\}$ lies in the interior of Σ. We define a map $h : \Sigma \to D^2 \times S^1$ by $h(u,t) := (\gamma^{-1}u, p(t))$. (34) implies that h induces a homeomorphism $\bar{h} : q(\Sigma) \approx D^2 \times S^1$, and from (32) and (33) we infer:

$$\mu q(\Sigma) \subseteq q(\Sigma)^\circ \text{ and } \bigcup_{n=0}^{\infty} \mu^{-n} q(\Sigma) = Y. \tag{35}$$

$f := \bar{h}\mu\bar{h}^{-1} : D^2 \times S^1 \rightarrowtail D^2 \times S^1$ is given by

$$f(u,z) := \left(\vartheta z\left(\gamma^{-1} + u\right), z^2\right), \tag{36}$$

and (35) implies that Y is homeomorphic to the limit of the direct sequence

$$D^2 \times S^1 \xrightarrow{f} D^2 \times S^1 \xrightarrow{f} \cdots. \tag{37}$$

Under this homeomorphism our μ corresponds to the homeomorphism, which operates on each copy of $D^2 \times S^1$ by f.

The sphere $S^3 = \{(u,v) \in \mathbb{C}^2 \big| |u|^2 + |v|^2 = 1\}$ is the union of two solid tori

$$T_1 = \{(u,v) \in S^3 \big| |u| \leq |v|\}\,, \tag{38}$$

$$T_2 = \{(u,v) \in S^3 \big| |u| \geq |v|\}\,; \tag{39}$$

T_1 and T_2 are mapped onto each other by the automorphism $\tau : S^3 \approx S^3$, $\tau(u,v) := (v,u)$. We identify $D^2 \times S^1$ with T_1 by means of the homeomorphism $\varphi : D^2 \times S^1 \approx T_1$, $\varphi(u,z) := \left(1+|u|^2\right)^{-1/2}(u,z)$, and claim:

(*) There is a homeomorphism $\psi : T_1 \setminus \operatorname{Int} f(T_1) \approx T_1 \setminus \operatorname{Int} f(T_1)$ with $\psi = f\tau$ on ∂T_1 and $\psi = \tau f^{-1}$ on $f(\partial T_1)$.

When (*) is proved we can define a homeomorphism $F : S^3 \approx S^3$ by

$$F := \begin{cases} f & \text{on } T_1, \\ \psi\tau & \text{on } T_2 \setminus \operatorname{Int} \tau f(T_1), \\ \tau f^{-1}\tau & \text{on } \tau f(T_1), \end{cases} \tag{40}$$

and since the direct sequence

$$T_1 \hookrightarrow F^{-1}(T_1) \hookrightarrow F^{-2}(T_1) \hookrightarrow \cdots \tag{41}$$

is isomorphic to

$$T_1 \xrightarrow{F} T_1 \xrightarrow{F} \cdots \tag{42}$$

and hence to (37) we get $Y \approx \bigcup_{n=0}^{\infty} F^{-n}(T_1) = S^3 \setminus \bigcap_{n=0}^{\infty} F^{-n}(T_2)$. Under this homeomorphism μ corresponds to the restriction of F, and since $F^{-1} = \tau f \tau$ on T_2 we have $B' = S^3 \setminus \bigcup_{n=0}^{\infty} F^{-n}(T_1) = \bigcap_{n=0}^{\infty} F^{-n}(T_2) = \tau\left(\bigcap_{n=0}^{\infty} f^n(T_1)\right) \approx \mathcal{S}_2$.

Finally, to give a sketch of proof for (*), let us consider the linked circles ω and ν in S^3 as displayed in the top left picture of Figure 1. We view ω and ν as thick lines, i.e. as solid tori. We identify ν with T_2 and ω with $f(T_1)$. Following the ambient isotopies pictured in Figure 1 we can construct a homeomorphism $\psi : S^3 \approx S^3$ interchanging ω and ν, and the restriction of Ψ to $T_1 \setminus \operatorname{Int} f(T_1)$ (i.e. the link complement of ω and ν) evidently interchanges the two boundary components ∂T_1 and $f(\partial T_1)$. To show that the operation of Ψ there does indeed coincide with $f\tau$ and τf^{-1}, respectively, one must construct a cellular decomposition of $T_1 \setminus \operatorname{Int} f(T_1)$ and analyze Ψ on the cells. The details are too cumbersome to be included here.

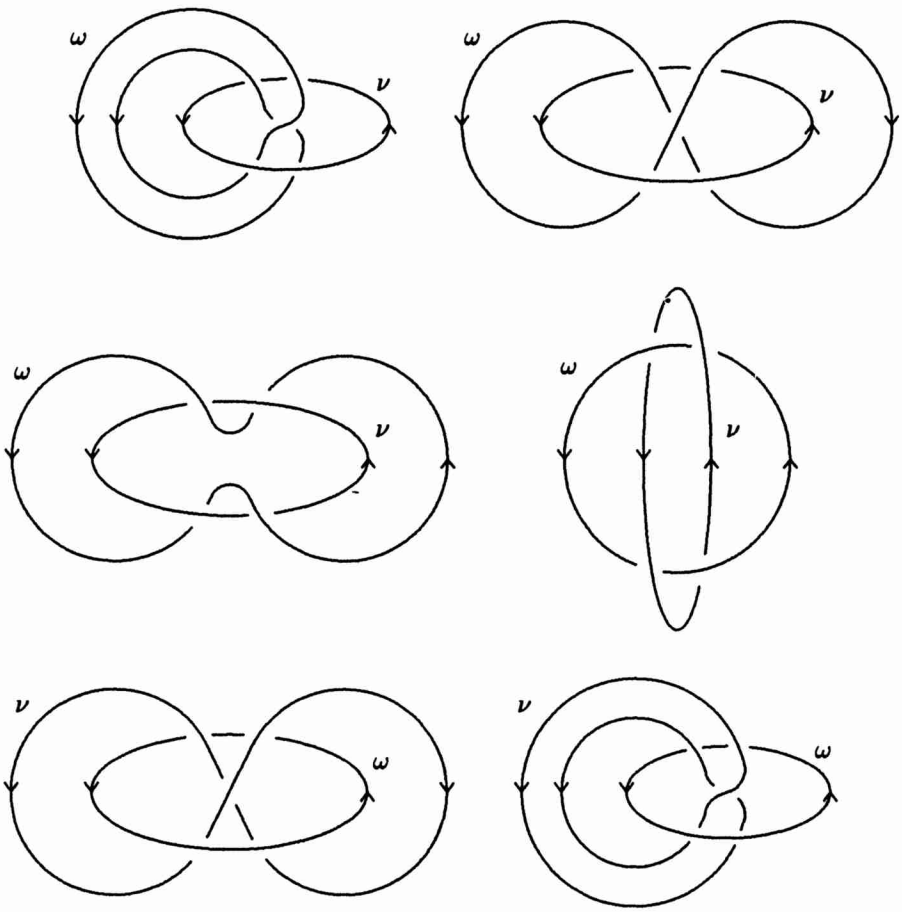

Figure 1

Problem. Find an analytic expression for the homeomorphism $F : S^3 \approx S^3$ constructed in equation (40).

Remark 3. The homeomorphism $\lambda : \mathbb{C}^n \times \mathbb{R}^n \approx \mathbb{C}^n \times \mathbb{R}^n$ may be replaced by a different one with similar properties, for instance

$$\lambda'(\mathbf{z}, \mathbf{t}) := (\vartheta(\mathbf{z} + p(\mathbf{t})), A\mathbf{t}) . \tag{4'}$$

With this choice Example 1 becomes slightly simpler, and in Example 2 it leads to the embedding $f' : D^2 \times S^1 \rightarrowtail D^2 \times S^1$ defined by

$$f'(u, z) := \left(\vartheta\left(u + \gamma^{-1} z\right), z^2\right) . \tag{36'}$$

(36′) is commonly used in treatments of the solenoid attractor ([2,p.202] or [5,p.160]). We observe that the embedding (36) differs from (36′) by an internal twist of the torus $f\left(D^2 \times S^1\right)$. (36′) *cannot* be extended to a homeomorphism on S^3.

The following Lemma is proved in [3]:

Lemma 2. *We consider an attractor B of a self-homeomorphism $\mu : Y \approx Y$ on a manifold and denote by $\varphi := \check{H}^*(\mu_{|B}) : \check{H}^*(B) \to \check{H}^*(B)$ the induced homomorphism of Čech cohomology groups with integer coefficients. Then $\check{H}^*(B)$ contains a finitely generated, φ-invariant subgroup G with $\bigcup_{n=1}^{\infty} \varphi^{-n}(G) = \check{H}^*(B)$.*

Theorem 1. *a) We suppose that B is a connected attractor of a homeomorphism $\mu : Y \approx Y$ on a manifold Y, and that B is homeomorphic to a compact abelian group. Then there is $A \in M_n(\mathbb{Z})$ for some $n \geq 1$ with $\det A \neq 0$ such that $B \approx \mathcal{S}_A$ and the shape morphisms induced by σ_A and $\mu_{|B}$ coincide.*
b) For any $A \in M_n(\mathbb{Z})$ with $\det A \neq 0$ there is a real analytic manifold Y and an analytic diffeomorphism $\mu : Y \approx Y$ such that μ has an attractor homeomorphic to $\mathcal{S}_A$ and the operation of μ-on this attractor coincides with σ_A.

Proof. a) If B is homeomorphic to the compact, connected abelian group Γ, then $\check{H}^1(B, \mathbb{Z})$ is isomorphic to the character group $\chi(\Gamma)$, which is torsion free [4, p.254]. Let $\varphi : \chi(\Gamma) \to \chi(\Gamma)$ be the automorphism corresponding to $\check{H}^1(\mu_{|A})$. By Lemma 2 $\chi(\Gamma)$ contains a φ-invariant, finitely generated and therefore free subgroup F with $\chi(\Gamma) = \bigcup_{k=0}^{\infty} \varphi^{-k} F$. If $F \approx \mathbb{Z}^n$ and $A \in M_n(\mathbb{Z})$ is the matrix representing the restriction of φ to F, then $\chi(\Gamma) \approx G_A$. Now Pontrjagin duality implies $\Gamma \approx \mathcal{S}_{A^t}$. Under the isomorphism $\chi(\Gamma) \approx G_A$ the automorphism φ transforms into matrix multiplication by A. Now it follows from [4, Ch.III,§3,Thm.1] that σ_{A^t} and $\mu_{|B}$ induce the same morphism in the ordinary shape category.

b) For $\det A = \pm 1$ we have $\mathcal{S}_A = \mathbf{T}^n$ and σ_A is multiplication by A, hence we can set $Y := \mathbf{T}^n \times \mathbb{R}$ and $\mu(\mathbf{t}, r) := (A\mathbf{t}, r/2)$. The case $|\det A| \geq 2$ is settled by Example 1. □

We remember that at the beginning our assumption was more restrictive than in Theorem 1, because we assumed that the homeomorphism operates on the attractor by group translation. An example for this situation is the following: We consider the manifold $Y := \mathbf{T}^n \times \mathbb{R}$ and the homeomorphism $\mu(\mathbf{t}, r) := (\mathbf{t} + \mathbf{t}_0, r/2)$. Evidently $\mathbf{T}^n \times \{0\}$ is an attractor of μ, and if $\mathbf{t}_0$ is properly chosen, then the orbits are dense in B. The following Theorem shows that this example covers the general situation:

Theorem 2. *We suppose that B is a connected attractor of a homeomorphism $\mu : Y \approx Y$ on a manifold Y, that B is homeomorphic to a compact abelian group and that μ operates on B by group translation $\mu(b) = b + b'$, where $b' \in B$ generates a dense subgroup. Then B is isomorphic to a torus $\mathbf{T}^n$.*

Proof. From Theorem 1 we know $B \approx \mathcal{S}_A$ for some $A \in M_n(\mathbb{Z})$ and $\mu_{|B}$ coincides with σ_A in the shape category. A group translation on $\mathcal{S}_A$ induces the identical shape morphism, but σ_A induces the identical shape morphism only if A is the identity matrix. □

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