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Rationality of a certain formal power series attached to local densities of quadratic forms

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We will prove the rationality of the power series attached to local densities defined by the author (Tôhoku Math.J. 44(1992), 211-218).

1 Introduction

Let p be a prime number. Let A and B be non-degenerate symmetric matrices of degrees m and n ($m \geq n \geq 1$), respectively, with entries in the ring $\mathbf{Z}_p$ of p-adic integers. We define a local density $\alpha_p(B, A)$ by

$$\alpha_p(B, A)$$

$$= \lim_{e \to \infty} p^{(-mn+<n>)e} \#\{\bar{X} \in M_{mn}(\mathbf{Z}_p)/p^e M_{mn}(\mathbf{Z}_p);\ {}^tXAX \equiv B \bmod p^e\},$$

where $< n >= n(n+1)/2$ and $M_{mn}(\mathbf{Z}_p)$ denotes the set of (m,n)-matrices with entries in $\mathbf{Z}_p$. For the matrices A and B stated above, and an l-tuple $\mathfrak{n} = (n_1, ..., n_l)$ of positive integers $n_1, .., n_l$ such that $n_1 + ... + n_l = n$, we define a formal power series $P(B, A; x_1, ..., x_l) = P(B, A, \mathfrak{n}; x_1, ..., x_l)$ by

$$P(B, A, \mathfrak{n}; x_1, ..., x_l) = \sum_{r_1, ..., r_l = 0}^{\infty} \alpha_p(B[\mathrm{diag}(p^{r_1} E_{n_1}, ..., p^{r_l} E_{n_l})], A) x_1^{r_1} ... x_l^{r_l},$$

where for an (m,n)-matrix C and an (m,m)-matrix D we write $D[C] = {}^tCDC$, and for square matrices $A_1, ..., A_r$ we often write

$$\mathrm{diag}(A_1, ..., A_r) = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_r \end{pmatrix}.$$

It is an important problem to consider the rationality of this power series (cf. Kitaoka [Ki]). When B is expressed as $B = \mathrm{diag}(B_1, ..., B_l)$ with B_i a square matrix of degree n_i $(i = 1, .., l)$, the rationality of this series follows from Böcherer and Sato [BS, Theorem 1]. Their method is based on the theory of p-adic integrals due to Denef [Den]. In this paper, we show the rationality of this series for an arbitrary B in a different method from theirs:

Theorem 1.1 *Let A and B be as above. Then $P(B, A; x_1, ..., x_l)$ is a rational function of $x_1, ..., x_l$ over the field $\mathbf{Q}$ of rational numbers, and its denominator is a product of polynomials of the form*

$$1 - p^e x_1^{e_1} .... x_l^{e_l}$$

with integers $e, e_1, ..., e_l$ such that $e_1, .., e_l \geq 0$.

We explain how to prove Theorem 1.1. Let $\mathbf{C}$ be the field of complex numbers. In Section 2, for the A, B and $\mathbf{n}$ stated above, we introduce a function $\zeta(B, A; Z, w) = \zeta(B, A, \mathbf{n}; Z, w)$ on the set

$$D = \{(Z, w) = ((z_{ij})_{1 \leq i \leq j \leq n}), (w_i)_{1 \leq i \leq l}) \in \mathbf{C}^{<n>+l}; \mid z_{ij} \mid < 1, \mid w_i \mid < 1\},$$

which is called a *generalized Igusa local zeta function* attached to A and B, and was defined in Katsurada [Ka] (cf. Section 2). As is shown by [Ka], the series $P(B, A; x_1, ..., x_l)$ can be realized as a coefficient of the Laurent expansion of $\zeta(B, A; Z, px_1 \prod (p^{-1} z_{ij})^{-4n_1}, ..., px_l \prod (p^{-1} z_{ij})^{-4n_l})$ with respect to (z_{ij}). Thus the rationality of $P(x_1, ..., x_l)$ follows from that of $\zeta(B, A; Z, w)$ by an elementary complex analytic method (cf. Proposition 3.1). Our method of proving rationality of $P(B, A; x_1, ..., x_l)$ is indirect in comparison to [BS]. However, since $\zeta(B, A; Z, px_1 \prod (p^{-1} z_{ij})^{-4n_1}, ..., px_l \prod (p^{-1} z_{ij})^{-4n_l})$ can be expressed explicitly, an explicit result on the denominator of $P(B, A; x)$ can be obtained by our method. For example, see Proposition 4.6. A generalization of this result will be published in a subsequent paper.

2 Generalized Igusa local zeta functions

For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m, n)-matrices with entries in R. Let $Sym(n; R)$ be the set of all symmetric matrices of degree n with

entries in R, and put

$$GL(n;R) = \{A \in M_{mn}(R); \det A \in R^*\},$$

where $\det A$ denotes the determinant of a square matrix A, and R^* denotes the unit group of R. Let $\mathbf{Z}$ be the ring of rational integers, and $\mathbf{Q}_p$ the field of p-adic numbers. Let $\{x_{ij} \ (1 \leq i \leq m, 1 \leq j \leq n), x_1, ..., x_l\}$ be variables over $\mathbf{Q}_p$. We then denote by X the matrix (x_{ij}), and put $x = (x_1, ..., x_l)$. Let $\mathbf{n}$ be an l-tuple $(n_1, ..., n_l)$ of positive integers such that $n_1 + ... + n_l = n$. Define a symmetric matrix $(g_{ij}(X, x))$ by

$$(g_{ij}(X, x)) = A[X] - B[\mathrm{diag}(x_1 E_{n_1}, ..., x_l E_{n_l})].$$

For a set S and non-negative integers n and l, put

$$S^{<n>+l} = \{((a_{ij})_{1 \leq i \leq j \leq n}, a_1, ..., a_l); a_{ij}, a_k \in S\}.$$

We simply write an element $((a_{ij})_{1 \leq i \leq j \leq n}, (a_i)_{1 \leq i \leq l})$ of $S^{<n>+l}$ as $((a_{ij}), (a_i))$ if no confusion arises. For a subset $\{a_k\}_{k \in T}$ of a commutative ring S indexed by a set T, we also write $\sum_{k \in T} a_k$, and $\prod_{k \in T} a_k$ simply as $\sum_k a_k$, and $\prod_k a_k$, respectively. In particular for an element (a_{ij}) of $S^{<n>}$, we often simply write $\sum_{1 \leq i \leq j \leq n} a_{ij}$, and $\prod_{1 \leq i \leq j \leq l} a_{ij}$ as $\sum a_{ij}$, and $\prod a_{ij}$, respectively. Further let ord_p be the normalized additive valuation of $\mathbf{Q}_p$, and put $| \ v \ |_p = p^{-\mathrm{ord}_p(v)}$ for $v \in \mathbf{Q}_p$. Then we define a function $Z(B, A, \mathbf{n}; (s_{ij}), s_1, ..., s_l)$ on the set

$$\mathbf{C}_+^{<n>+l} = \{((s_{ij}), s_1, ..., s_l) \in \mathbf{C}^{<n>+l}; \mathrm{Re}\ s_{ij} > 0, \ \mathrm{Re}\ s_i > 0\}$$

by

$$Z(B, A, \mathbf{n}; (s_{ij}), s_1, ..., s_l) = \int_{\mathbf{Z}_p^l \times M_{mn}(\mathbf{Z}_p)} \prod | \ g_{ij} \ |_p^{s_{ij}} \prod_{i=1}^{l} | \ x_i \ |_p^{s_i} | \ dx \ || \ dX \ |,$$

where $| \ dx \ |$ (resp. $| \ dX \ |$) denotes the normalized Haar measure on $\mathbf{Q}_p^l$ (resp. $M_{mn}(\mathbf{Q}_p)$) (for the precise definition, see Katsurada [Ka]). We call this function a generalized Igusa local zeta function attached to A and B. Put $z_{ij} = p^{-s_{ij}}$ $(1 \leq i \leq j \leq n), w_i = p^{-s_i}$ $(1 \leq i \leq l)$. We often write $Z = (z_{ij}), w = (w_i)$. Then $Z(B, A; (s_{ij}), s_1, ..., s_l)$ can be regarded as a function of Z and w. Thus we write $\zeta(B, A, \mathbf{n}; Z, w) = Z(B, A, \mathbf{n}; (s_{ij}), s_1, ..., s_l)$. We simply write $\zeta(B, A, \mathbf{n}; Z, w) = \zeta(B, A; Z, w)$. Then we have

Theorem 2.1.([Ka, Theorem 1]) *Put*

$$N(B) = \{(k_{ij}) \in \mathbf{Z}^{<n>}; \ \min k_{ij} \geq 2\mathrm{ord}_p(2 \det B) + 1\}.$$

Then in the region

$$E = \{((z_{ij}), x_1, ..., x_l) \in \mathbf{C}^{<n>+l}; 0 <| z_{ij} |< 1 \ (1 \leq i \leq j \leq n),$$
$$0 <| px_k \prod (p^{-1}z_{ij})^{-4n_k} |< 1 \ (1 \leq i \leq j \leq n, 1 \leq k \leq l)\},$$

we have

$$\zeta(B, A; Z, px_1 \prod (p^{-1}z_{ij})^{-4n_1}, ..., px_l \prod (p^{-1}z_{ij})^{-4n_l})$$
$$= \sum_{(k_{ij}) \in \mathbf{Z}^{<n>} \setminus N(B)} P((k_{ij}); x_1, ..., x_l) \prod (p^{-1}z_{ij})^{k_{ij}}$$
$$+(1-p^{-1})^{<n>+l} P(B, A, \mathbf{n}; x_1, ..., x_l) \sum_{(k_{ij}) \in N(B)} \prod (p^{-1}z_{ij})^{k_{ij}},$$

where $P((k_{ij}); x_1, ..., x_l)$ *is a convergent power series of* $x_1, ..., x_l$ *for each element* (k_{ij}) *of* $\mathbf{Z}^{<n>} \setminus N(B)$.

To obtain a more explicit representation of this function, we recall some basic results on resolution following Deshomme [Des]. For a while, let K be an arbitrary field. From now on, we use a bold face $\mathbf{D}$ to denote the effective divisor of a non-singular variety U associated with a locally principal closed subscheme D of U, and vice versa. Let x_{ij} $(1 \leq i \leq m, 1 \leq j \leq n), x_i$ $(1 \leq i \leq l)$ be variables over K. We simply write $X = (x_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$, and $x = (x_i)_{1 \leq i \leq l}$. We identify $M_{mn}(K) \times K^l$ with the set $\mathsf{A}(K)$ of K-valued points of the $(mn+l)$-dimensional affine space $\mathsf{A} = \mathrm{Spec}\ K[X, x]$. Let A and B be non-degenerate symmetric matrices of degrees m and n $(m \geq n \geq 1)$, respectively, with entries in K. Let $(g_{ij}(X, x))$ be the symmetric matrix stated above. Let $\mathbf{G}_{ij}$ (resp. $\mathbf{X}_i$) be the divisor of A defined by the equation $g_{ij}(X, x) = 0$ (resp. $x_i = 0$), and put

$$\mathbf{D} = \sum_{1 \leq i \leq j \leq n} \mathbf{G}_{ij} + \sum_{i=1}^{l} \mathbf{X}_i,$$

and $f = f(B, A, \mathbf{n}, (X, x)) = \prod_{1 \leq i \leq j \leq n} g_{ij} \prod_{1 \leq i \leq l} x_i$. Let (Y, h) be a resolution for f over K in the sense of [Des]. Let $\{\mathbf{E}_k\}_{k \in T}$ be the set of all irreducible components of $h^*(\mathbf{D})$. Put

$$h^*(\mathbf{G}_{ij}) = \sum_k N(i, j; k)\mathbf{E}_k,$$
$$h^*(\mathbf{X}_i) = \sum_k N(i; k)\mathbf{E}_k,$$

and

$$h^*(\wedge_{1 \leq i \leq m, 1 \leq j \leq n} dx_{ij} \wedge \wedge_{i=1}^{l} dx_i) = \sum_k (\nu(k) - 1)\mathbf{E}_k.$$

Then for any $k \in T$ we have $\sum_{1\leq i\leq j\leq n} N(i,j;k) + \sum_{1\leq i\leq l} N(i;k) \geq 1, \nu(k) \geq 1$. Let $K = \mathbf{Q}_p$. Then by Deshomme [Des, Theorem 2.5.1], we have:

Theorem 2.2 *Under the above notation and assumptions we have*

$$\zeta(B, A; Z, w) = \frac{R(Z, w)}{\prod_{i,j} z_{ij}^{L(i,j)} \prod_i z_i^{M(i)} \prod_k (1 - \prod_{i,j} z_{ij}^{N(i,j;k)} \prod_i z_i^{N(i;k)} p^{-\nu(k)})},$$

where $\prod_{i,j}$ *(resp.* $\prod_i, \prod_k$ *) means* $\prod_{1\leq i\leq j\leq n}$ *(resp.* $\prod_{1\leq i\leq l}, \prod_{k\in T}$ *),* $L(i,j), M(i)$ *are non-negative integers, and* $R(Z,w)$ *is a polynomial of* z_{ij}, z_i.

3 Proof of the main result

Let $\mathbf{Z}^+$ be the set of non-negative integers. Let M be a finite set. Let $\{l(k)\}_{1\leq k\leq r}$, $\{m(i,j)\}_{i\in M, 1\leq j\leq r}$ be two sets of integers. From now on put $z = (z_1, ..., z_r)$ and $x = (x_1, ..., x_s)$. To prove the main result, we need the following proposition:

Proposition 3.1. *Let the notation be as above. Let* $\{g_i(x)\}_{i\in M}$ *be a subset of* $\mathbf{Q}[x]$. *Put*

$$g(z, x) = \frac{1}{\prod_{k=1}^r z_k^{l(k)} \prod_{i\in M} (1 - g_i(x) \prod_{k=1}^r z_k^{m(i,k)})}$$

and let

$$g(z, x) = \sum_{e_1,...,e_r=-\infty}^{\infty} P(e_1, ..., e_r; x) z_1^{e_1} ... z_r^{e_r}$$

be the Laurent expansion of $g(z,x)$ *with respect to* $z_1, ..., z_r$ *in the region* $0 < |\, g_i(x) \prod_{k=1}^r z_k^{m(i,k)} \,| < 1 \quad (i \in M)$. *Then there exist a set* $\{h(i,k)\}_{i\in T, k\in M}$ *of integers with a finite set* T, *and a polynomial* $R(e_1, ..., e_r; x)$ *in* x *over* $\mathbf{Q}$ *such that*

$$P(e_1, ..., e_r; x) = \frac{R(e_1, ..., e_r; x)}{\prod_i (1 - \prod_{k\in M} g_k(x)^{h(i,k)})}.$$

This is quite elementary. But for the sake of completeness, we give a proof.

Lemma 3.2. *Let* $e_1, .., e_r$ *be positive integers. Put*

$$F(z, X_1, ..., X_r) = \frac{1}{(1 - z^{e_1} X_1) \prod_{i=2}^r (1 - z^{-e_i} X_i)}.$$

Let

$$F(z, X_1, ..., X_r) = \sum_{e=-\infty}^{\infty} P_e(X_1, ..., X_r) z^e$$

be the Laurent expansion of $F(z, X_1, ..., X_{r+1})$ *with respect to* z *in* $0 <| z^{e_1} X_1 |< 1, 0 <| z^{-e_i} X_i |< 1$ $(2 \leq i \leq r)$. *Then* $P_e(X_1, ..., X_r)$ *is a rational function of* $X_1, ..., X_r$ *over* $\mathbf{Q}$ *with denominator* $(1 - X_1^{e_2} X_2^{e_1}) ....(1 - X_1^{e_r} X_r^{e_1})$. *In particular for any integer* $e \geq 0$, *we have*

$$P_e(X_1, ..., X_r) = \frac{\sum_{n_2, ..., n_r} X_1^{(e_2 n_2 + ... + e_r n_r + e)/e_1} X_2^{n_2} ... X_r^{n_r}}{(1 - X_1^{e_2} X_2^{e_1}) ....(1 - X_1^{e_r} X_r^{e_1})}$$

where $n_2, ..., n_r$ *run over all non-negative integers such that* $0 \leq n_i \leq e_1 - 1$ $(i = 2, ..., r)$, *and* $e_2 n_2 + ... + e_r n_r + e \equiv 0 \bmod e_1$.

Proof. Define a formal power series $P_e(X_1, ..., X_r)$ by

$$P_e(X_1, ..., X_r) = \sum_{e_1 m_1 - e_2 m_2 + ... - e_r m_r = e} X_1^{m_1} ... X_r^{m_r}.$$

Then $P_e(X_1, ..., X_r)$ is a convergent power series and we have

$$F(z, X_1, ..., X_r) = \sum_{e=-\infty}^{\infty} z^e P_e(X_1, ..., X_r)$$

for $0 <| z^{e_1} X_1 |< 1, 0 <| z^{-e_i} X_i |< 1$ $(2 \leq i \leq r)$. Further we have

$$P_e(X_1, ..., X_r) = \sum_{m_2, ..., m_r} X_1^{(e_2 m_2 + ... + e_r m_r + e)/e_1} X_2^{m_2} ... X_r^{m_r} - \sum_{m_2', ..., m_r'} X_1^{(e_2 m_2' + ... + e_r m_r' + e)/e_1} X_2^{m_2'} ... X_r^{m_r'}, \tag{3.1}$$

where $m_2, ..., m_r$ (resp. $m_2', ..., m_r'$) run over all non-negative integers such that $e_2 m_2 + ... + e_r m_r + e \equiv 0 \bmod e_1$ (resp. $e_2 m_2' + ... + e_r m_r' + e < 0, e_2 m_2' + ... + e_r m_r' + e \equiv 0 \bmod e_1$). Let $P_e'(X_1, ..., X_r)$ and $P_e''(X_1, ..., X_r)$ denote the first and second sum on the right hand-side of (3.1), respectively. Then $P_e''(X_1, ..., X_r)$ belongs to $1/X_1^e \mathbf{Q}[X_1, X_2, ..., X_r]$. Further we have

$$P_e'(X_1, ..., X_r) = \sum_{m_2', ..., m_r'} X_1^{(e_2 m_2' + ... + e_r m_r' + e)/e_1} X_2^{m_2'} ... X_r^{m_r'} \times \sum_{m_2, ..., m_r = 0}^{\infty} X_1^{e_2 m_2 + ... + e_r m_r} X_2^{e_1 m_2} ... X_r^{e_1 m_r}, \tag{3.2}$$

where $m_2', ..., m_r'$ run over all non-negative integers $m_2', ..., m_r'$ such that $0 \leq m_i' \leq e_1 - 1$ $(i = 2, ..., r)$, $e_2 m_2' + ... + e_r m_r' + e \equiv 0 \bmod e_1$. Clearly the second sum on the right hand side of (3.2) equals

$$\frac{1}{(1 - X_1^{e_2} X_2^{e_1})....(1 - X_1^{e_r} X_r^{e_1})}.$$

Thus $P_e(X_1, ..., X_r)$ is a rational function of $X_1, ..., X_r$ over $\mathbf{Q}$ with denominator $X_1^e(1-X_1^{e_2}X_2^{e_1})....(1-X_1^{e_r}X_r^{e_1})$. On the other hand, $P_e(X_1, ..., X_r)$ is holomorphic in the neighbourhood of $X_1 = ... = X_r = 0$. Thus the first part of the assertion can be proved. Further if $e \geq 0$, we have $P_e''(X_1, ..., X_r) = 0$. Thus the second part of the assertion holds.

Corollary. *Let $e_1, .., e_r, e_{r+1}, .., e_s$ be positive integers. Put*

$$F(z, X_1, ..., X_s) = \frac{1}{z^{e_0 - 1} \prod_{i=1}^{r}(1 - z^{e_i} X_i) \prod_{i=r+1}^{s}(1 - z^{-e_i} X_i)}.$$

Let

$$F(z, X_1, ..., X_s) = \sum_{e=-\infty}^{\infty} P_e(X_1, ..., X_s) z^e$$

be the Laurent expansion of $F(z, X_1, ..., X_s)$ with respect to z in $0 <| z^{e_i} X_i |< 1$ $(1 \leq i \leq r)$, $0 <| z^{-e_i} X_i |< 1$ $(r+1 \leq i \leq s)$. Then for any $e \geq 0$, $P_e(X_1, ..., X_s)$ is a rational function of $X_1, ..., X_s$ over $\mathbf{Q}$, and its denominator is of the form:

$$\prod_{i=1}^{r} \prod_{k=r+1}^{s} (1 - X_i^{e_k} X_k^{e_i}).$$

Proof. By an elementary calculation, we can show that there exist polynomials $\{Q_i(X_1, ..., X_s)\}_{i=1}^{r}$ over $\mathbf{Q}$ such that

$$z^{1-e_0} F(z, \beta_1, ..., \beta_s)$$

$$= \sum_{i=1}^{r} \frac{Q_i(\beta_1, ..., \beta_s)}{\prod_{1 \leq k \leq r, k \neq i}(\beta_i^{e_k} - \beta_k^{e_i})(1 - z^{e_i}\beta_i)(1 - z^{-e_{r+1}}\beta_{r+1})...(1 - z^{-e_s}\beta_s)}$$

for any $\beta_1, ..., \beta_s \in \mathbf{C}$ such that $\beta_i^{e_k} \neq \beta_k^{e_i}$ $(1 \leq i \neq k \leq r)$. Thus by the lemma, there exist rational functions $\{P_e(X_1, ..., X_s)\}_{e \in \mathbf{Z}}$ such that

$$P_e(X_1, ..., X_s) = \frac{R_e''(X_1, ..., X_s)}{\prod_{i=1}^{r} \prod_{1 \leq k \leq r, k \neq i}(X_i^{e_k} - X_k^{e_i}) \prod_{i=r+1}^{s} \prod_{k=1}^{r}(1 - X_i^{e_k} X_k^{e_i})}$$

with $R''_e(X_1, ..., X_s) \in \mathbf{Q}[X_1, ..., X_s]$, and

$$F(z, \beta_1, ..., \beta_s) = \sum_{e=-\infty}^{\infty} P_e(\beta_1, ..., \beta_s) z^e \tag{3.3}$$

for $\beta_1, ..., \beta_s \in \mathbf{C}$ such that $0 <| \beta_j |^{1/e_j} <| \beta_i |^{-1/e_i}$ $(1 \leq i \leq r, r+1 \leq j \leq s)$, and $\beta_i^{e_k} \neq \beta_k^{e_i}$ $(1 \leq i \neq k \leq r)$. On the other hand, $P_e(X_1, ..., X_s)$ is holomorphic in the neighbourhood of $X_1 = ... = X_s = 0$. Thus $R''_e(X_1, ..., X_s)$ is divided by $\prod_{i=1}^{r} \prod_{1 \leq k \leq r, k \neq i} (X_i^{e_k} - X_k^{e_i})$, and (3.3) also holds for $\beta_i^{e_k} = \beta_k^{e_i}$ with some $1 \leq i \neq k \leq r$ by continuity. Thus the assertion holds.

Proof of Proposition 3.1. We prove the assertion by induction on r. Let the notation be as in Proposition 3.1. Put

$$M' = \{i \in M; m(i,1) > 0\}, M'' = \{i \in M; m(i,1) < 0\},$$

and

$$M''' = \{i \in M; m(i,1) = 0\}.$$

Define a rational function of z_1, and X_i $(i \in M' \cup M'')$ by

$$Q(z_1, (X_i)_{i \in M' \cup M''}) = \frac{1}{z_1^{l(1)} \prod_{i \in M' \cup M''} (1 - X_i z_1^{m(i,1)})}.$$

Then by Proposition 3.4, there exist a family $\{Q_e((X_i)_{i \in M' \cup M''})\}_{e \in \mathbf{Z}}$ of rational functions with denominator

$$\prod_{i \in M', i' \in M''} (1 - X_{i'}^{m(i,1)} X_i^{-m(i',1)})$$

such that

$$Q(z_1, (X_i)_{i \in M' \cup M''}) = \sum_{e=-\infty}^{\infty} Q_e((X_i)_{i \in M' \cup M''}) z_1^e$$

in $0 <| z_1^{m(i,1)} X_i |< 1$ $(i \in M' \cup M'')$. Put

$$z' = (z_2, ..., z_r), Y_i(z', x) = g_i(x) \prod_{k=2}^{r} z_k^{m(i,k)} \quad (i \in M' \cup M''),$$

and

$$R(z', x) = \frac{1}{\prod_{k=2}^{r} z_k^{l(k)} \prod_{i \in M'''} (1 - g_i(x) \prod_{k=2}^{r} z_k^{m(i,k)})}.$$

Then we have

$$Q(z_1, (Y_i(z', x))_{i\in M'\cup M''}) = \sum_{e=-\infty}^{\infty} Q_e((Y_i(z', x))_{i\in M'\cup M''})z_1^e \tag{3.4}$$

in $0 <| z_1^{m(i,1)}Y_i(z', x) |< 1$ $(i \in M' \cup M'')$, and

$$g(z, x) = Q(z_1, (Y_i(z', x))_{i\in M'\cup M''})R(z', x). \tag{3.5}$$

Let $r = 1$. Then the assertion follows directly from (3.4) and (3.5). Assume that the assertion holds for $r - 1$. For each $e \in \mathbf{Z}$, put

$$g_e(z', x) = \sum_{e_2,\dots,e_r=-\infty}^{\infty} P(e, e_2\dots, e_r; x)z_2^{e_2}\dots z_r^{e_r}.$$

Then by (3.4) and (3.5), we have

$$g_e(z', x) = R(z', x)Q_e((Y_i(z', x))_{i\in M'\cup M''}).$$

Thus $g_e(z', x)$ is a rational function of $z_2, \dots, z_r, x_1, \dots, x_s$ over $\mathbf{Q}$ and its denominator is

$$\prod_{i=2}^{r} z_i^{l''(i)} \prod_{i\in M', i'\in M''} (1 - a_{i,i'}(x) \prod_{k=2}^{r} z_k^{m(i,i';k)}) \prod_{i\in M'''} (1 - g_i(x) \prod_{k=2}^{r} z_k^{m(i,k)})$$

where $l''(i) \in \mathbf{Z}^+$, $a_{i,i'}(x) = g_i(x)^{-m(i',1)}g_{i'}(x)^{m(i,1)}$, and

$$m(i, i'; j) = m(i', j)m(i, 1) - m(i', 1)m(i, j).$$

Thus by the inductive hypothesis, $P(e, e_2\dots, e_r; x)$ is a rational function of $x_1, \dots, x_s$ over $\mathbf{Q}$ with the required property. Thus the assertion holds.

Proof of Theorem 1.1. Let A and B be non-degenerate symmetric matrices of degrees m and n $(m \geq n \geq 1)$, respectively, with entries in $\mathbf{Z}_p$. Then by Theorems 2.1, 2.2, and Proposition 3.1, $P(B, A; x_1, \dots, x_l)$ is a rational function of $x_1, \dots, x_l$ over $\mathbf{Q}$, and every factor of its denominator is

$$1 - p^e x_1^{e_1}\dots x_l^{e_l}$$

with integers $e, e_1, \dots, e_l$ such that $e_1, .., e_l \geq 0$ or

$$x_1^{e_1}\dots x_l^{e_l}$$

with $e_1, \dots, e_l \geq 0$. On the other hand, $P(B, A; x_1, \dots, x_l)$ is holomorphic in some neighborhood of $x_1 = \dots = x_l = 0$. Thus the assertion holds.

4 An example

In this section, using Theorem 2.1, we give an explicit formula for $P(B, A; x)$ when $\deg B = 1$, and A is unimodular (cf. Proposition 4.6.) Though it is merely a reformulation of Siegel [S, Hilfssatz 16], using the method in this section, we can generalize it to the case where $\deg B$ is greater than 1, the detail of which will be published in a subsequent paper. Our method can be applied for every prime p. But for simplicity, from now on, we assume $p \neq 2$.

Let $A = (a_{\alpha\beta})_{1\leq\alpha,\beta\leq m} \in Sym(m; \mathbf{Z}_p)$, $B = b \in \mathbf{Z}_p$, and $\{X_\alpha\ (1 \leq \alpha \leq m), x\}$ be variables over $\mathbf{Q}_p$. Put $X = (X_\alpha)_{1\leq\alpha\leq m}$, and

$$g(X, x) = \sum_{\alpha,\beta=1}^{m} a_{\alpha,\beta} X_\alpha X_\beta - bx^2.$$

The following two lemmas can be easily proved.

Lemma 4.1. *Let the notation be as above. Assume that A is unimodular.Let $\Xi = (\Xi_1, ..., \Xi_m)$, and ξ be elements of $\mathbf{Z}_p^m$ and $\mathbf{Z}_p$,respectively such that*

$$g(\Xi, \xi) \equiv 0 \mod p,$$

and

$$\Xi_i \not\equiv 0 \mod p \text{ with some } i.$$

Put

$$u = g(\Xi + pX, \xi + px)/p.$$

Then the map Φ of $\mathbf{Z}_p^{m+1}$ to itself defined by

$$(x, X) \longmapsto (x, u, X_1, ..., X_{i-1}, X_{i+1}, ..., X_m)$$

is a measure prerserving analytic homeomorphism.

Lemma 4.2 *Let $y_1, ..., y_r$ be variables over $\mathbf{Q}_p$, and $| dy_1....dy_r |$ be the normalized Haar measure of $\mathbf{Q}_p^r$ induced by the differential $dy_1 \wedge .... \wedge dy_r$ on $\mathbf{Q}_p^r$. Let T be a subset of $\{1, ..., r\}$. Then we have*

$$\int_{\mathbf{Z}_p^r} \prod_{i\in T} | y_i |^{N_i s_i + \nu_i - 1} | dy_1...dy_r | = (1 - p^{-1})^{\#T} \prod_{i\in T} \frac{1}{1 - p^{-N_i s_i - \nu_i}}.$$

For an element D of $Sym(r, \mathbf{Z}_p)$ and $d \in \mathbf{Z}_p$, put

$$a'(d, D) = \#\{\bar{X} \in (\mathbf{Z}_p/p\mathbf{Z}_p)^r; X \not\equiv \mathbf{0} \mod p, \text{ and } D[X] \equiv d \mod p\},$$

and

$$\beta_p(d, D) = p^{-r+1}a'(d, D).$$

Proposition 4.3 *In additions to the notation and the assumption at the begining of this section, assume that* $b \in \mathbf{Z}_p^*$ *or* $p\mathbf{Z}_p^*$. *Then we have*

$$\zeta(B, A; z, w)$$
$$= \frac{(1-p^{-1})^2}{1-p^{-m-1}z^2w} \times \left\{ \frac{p^{-1}z}{1-p^{-1}z}(\beta_p(b, A) + \frac{p^{-1}w\beta_p(0, A)}{1-p^{-1}w}) + h(w) \right\}$$

with $h(w) \in \mathbf{Q}[w, (1-p^{-1}w)^{-1}]$.

Proof. For each $(\Xi, \xi) \in \mathbf{Z}_p^m \times \mathbf{Z}_p$, put

$$X_0((\Xi, \xi)) = \{(X, x) \in \mathbf{Z}_p^m \times \mathbf{Z}_p; (X, x) \in (\Xi, \xi) + ((p\mathbf{Z}_p)^m \times p\mathbf{Z}_p)\}.$$

For $(\Xi, \xi) \in \mathbf{Z}_p^m \times \mathbf{Z}_p$, put

$$\zeta((\Xi, \xi); z, w) = p^{-m-1} \int_{\mathbf{Z}_p^m \times \mathbf{Z}_p} | g(\Xi + pX, \xi + px) |_p^s | \xi + px |_p^t | dx | | dX |$$

where $z = p^{-s}, w = p^{-t}$, and $| dx |$ (resp. $| dX |$) denotes the normalized Haar measure on $\mathbf{Q}_p$ (resp. $\mathbf{Q}_p^m$) induced by the differential dx (resp. $dX_1 \wedge \ldots \wedge dX_m$). Then we have

$$\zeta(B, A; z, w)$$

$$= \sum_{(\bar{\Xi}, \bar{\xi}) \in (\mathbf{Z}_p/p\mathbf{Z}_p)^m \times \mathbf{Z}_p/p\mathbf{Z}_p} \int_{X_0((\Xi, \xi))} | g(X, x) |_p^s | x |_p^t | dx | | dX |$$
$$= \sum_{(\bar{\Xi}, \bar{\xi}) \in (\mathbf{Z}_p/p\mathbf{Z}_p)^m \times \mathbf{Z}_p/p\mathbf{Z}_p} \zeta((\Xi, \xi); z, w).$$

If $(\Xi, \xi) = (\mathbf{0}, 0)$, we have

$$Z((\Xi, \xi); z, w) = p^{-m-1}z^2wZ(B, A; z, w).$$

Assume that $\Xi \not\equiv \mathbf{0}$, and

$$g(\Xi, \xi) \equiv 0 \bmod p.$$

Then by Proposition 4.1 we have

$$\zeta((\Xi, \xi); z, w) = p^{-m-1} \int_{\mathbf{Z}_p^m \times \mathbf{Z}_p} | pu |_p^s | \xi + px |_p^t | dx | | du | | d\hat{X} |,$$

where $| d\hat{X} | = | dX_1 \dots dX_{i-1} dX_{i+1} \dots dX_m |$ with some i. Thus by Lemma 4.2 we have

$$\zeta((\Xi, \xi); z, w) = \begin{cases} \dfrac{p^{-m+1}(1-p^{-1})^2(p^{-1}z)(p^{-1}w)}{(1-p^{-1}z)(1-p^{-1}w)} & \text{if } \xi \equiv 0 \bmod p \\ \dfrac{p^{-m}(1-p^{-1})(p^{-1}z)}{1-p^{-1}z} & \text{otherwise.} \end{cases}$$

We note that if $\xi \not\equiv 0 \bmod p$, and $g(\Xi, \xi) \equiv 0 \bmod p$, we have $\Xi \not\equiv \mathbf{0} \bmod p$. Thus we have

$$\sum_{(\bar{\Xi}, \bar{\xi})} \zeta((\Xi, \xi); z, w) = \frac{(1-p^{-1})^2(p^{-1}z)p^{-m+1}a'(b, A)}{1-p^{-1}z},$$

where $(\bar{\Xi}, \bar{\xi})$ runs over all elements of $(\mathbf{Z}_p/p\mathbf{Z}_p)^m \times \mathbf{Z}_p/p\mathbf{Z}_p$ such that $\xi \not\equiv 0 \bmod p$, and $g(\Xi, \xi) \equiv 0 \bmod p$. Similarly we have

$$\sum_{(\bar{\Xi}, \bar{\xi})} \zeta((\Xi, \xi); z, w) = \frac{(1-p^{-1})^2(p^{-1}z)(p^{-1}w)p^{-m+1}a'(0, A)}{(1-p^{-1}z)(1-p^{-1}w)},$$

where $(\bar{\Xi}, \bar{\xi})$ runs over all elements of $(\mathbf{Z}_p/p\mathbf{Z}_p)^m \times \mathbf{Z}_p/p\mathbf{Z}_p$ such that $\xi \equiv 0, \Xi \not\equiv 0 \bmod p$ and $g(\Xi, \xi) \equiv 0 \bmod p$. Finally if $g(\Xi, \xi) \not\equiv 0 \bmod p$, we have

$$\zeta((\Xi, \xi); z, w) = p^{-m-1} \int_{\mathbf{Z}_p^m \times \mathbf{Z}_p} | \xi + px |_p^t | \, dx \, || \, dX \, | \, .$$

Thus $\zeta((\Xi, \xi); z, w)$ belongs to $\mathbf{Q}[w, (1-p^{-1}w)^{-1}]$. Thus the assertion is proved.

By Theorem 2.1, Lemma 3.2, and Proposition 4.3, we have

Proposition 4.4. Let the notation and the assumptions be as above. Then we have

$$P(B, A; x) = \frac{1}{1-p^{-m+2}x} \times (\beta_p(b, A) + \frac{x\beta_p(0, A)}{1-x}).$$

Let χ be the quadratic character of $\mathbf{Z}_p$ mod p. For $b \in \mathbf{Z}_p$ and $U \in GL_m(\mathbf{Z}_p)$, define $\varepsilon(U)$, and $\varepsilon(b, U)$ by $\varepsilon(U) = \chi((-1)^{[m/2]} \det U)$, and $\varepsilon(b, U) = (-\chi(b))^m \chi((-1)^{[m/2]} \det U)$, respectively. Then, as is well known, we have

Lemma 4.5. (c.f. Siegel [S, Hilfssatz 12]). *Let $d \in \mathbf{Z}_p$ and $A \in GL_m(\mathbf{Z}_p) \cap Sym(m; \mathbf{Z}_p)$. Then we have*

$$\beta_p(d, A) = \begin{cases} 1 - \varepsilon(d, A)p^{-[m/2]} & \text{if } d \in \mathbf{Z}_p^* \\ (1 - \varepsilon(A)p^{-[m/2]})(1 + \varepsilon(A)p^{-[(m-1)/2]}) & \text{if } d \in p\mathbf{Z}_p. \end{cases}$$

It follows from Proposition 4.3 and Lemma 4.5 that

Corollary. *Let $b \in \mathbf{Z}_p^*$ or $\in p\mathbf{Z}_p^*$, and $A \in GL_m(\mathbf{Z}_p) \cap Sym(m;\mathbf{Z}_p)$. Then we have*

$$P(b,A;x) = \begin{cases} \dfrac{(1-\varepsilon(b,A)p^{-[m/2]})(1+\varepsilon(b,A)p^{-[(m-1)/2]}x)}{(1-x)(1-p^{-m+2}x)} & \text{if } b \in \mathbf{Z}_p^* \\ \dfrac{(1-\varepsilon(A)p^{-[m/2]})(1+\varepsilon(A)p^{-[(m-1)/2]})}{(1-x)(1-p^{-m+2}x)} & \text{if } b \in p\mathbf{Z}_p^*. \end{cases}$$

Define a formal power series $Q(b,A;x)$ by

$$Q(b,A;x) = \sum_{r=0}^{\infty} \alpha_p(p^r b, A)x^r$$

which is a special case of the power series defined in [Ki]. We note that

$$Q(b,A;x) = P(b,A;x^2) + xP(pb,A;x^2).$$

Thus we have

Proposition 4.6.([S,Hilfssatz 16]) *Let $b \in \mathbf{Z}_p^*$. Then we have*

$$Q(b,A;x) = \begin{cases} \dfrac{(1-\varepsilon p^{-m/2})}{(1-x)(1-\varepsilon p^{-m/2+1}x)} & \text{if } m \text{ is even} \\ \dfrac{(1+\eta p^{-(m-1)/2})(1-\eta p^{-(m-1)/2}x)}{(1-x)(1-p^{-m+2}x^2)} & \text{if } m \text{ is odd,} \end{cases}$$

where $\varepsilon = \chi((-1)^{-m/2}\det A)$, and $\eta = \chi((-1)^{-(m-1)/2}b\det A)$.

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