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ON THE IWASAWA INVARIANTS OF TOTALLY REAL NUMBER FIELDS

Hiroshi Yamashita

We shall show two sufficient conditions under which the Iwasawa invariants λ_k and μ_k of a totally real field k vanish for an odd prime l, based on the results obtained in [1], [3] and [4]. Let K_n be the composite of k and the l^n-th cyclotomic extension of the field Q of rational numbers. Let C_n be the factor group of the l-class group of K_n by a subgroup generated by ideals whose prime factors divide the principal ideal (l). Let ε_1 be an idempotent of the group ring $Z_l[\mathrm{Gal}(K_1/k)]$ defined in the below. We shall prove $\lambda_k = \mu_k = 0$ if there is a natural number n such that $\varepsilon_1 C_n$ vanishes, under additional conditions concerning ramifications in K_n/k.

1. Introduction. Let k denote a finite algebraic number field and l be an odd prime number. Let $K_n = k(\zeta_n)$ for an l^n-th root of unity ζ_n . Omitting the suffix, we use K to denote K_1. Set $\Delta = \mathrm{Gal}(K/k)$. Let k_n be the fixed field of K_n by Δ. Let l^{e_n} be the order of the l-Sylow subgroup of the ideal class group of k_n. It is known that the following asymptotic formula holds:

$$e_n \quad \sim \quad \lambda_k n + \mu_k l^n, \quad (n \to \infty),$$

where λ_k and μ_k are constants called the Iwasawa invariants of k. (See [6, Ch. 13].)

Let $Z_l[\Delta]$ denote the group ring of Δ over the ring Z_l of l-adic integers. Let ω be an l-adic character of Δ which satisfies $\zeta_l^\sigma = \zeta_l^{\omega(\sigma)}$ for every $\sigma \in \Delta$. Let ε_i be an idempotent in $Z_l[\Delta]$ associated with ω^i. Let A_n be the l-Sylow subgroup of the ideal class group of K_n and D_n be the subgroup generated by ideals whose prime factors divide (l). Denote by C_n the factor group A_n/D_n. $Z_l[\Delta]$ acts on A_n, C_n, D_n, canonically.

A finite algebraic number field k is called l-rational in [1] if k satisfies the following two conditions:

(1) The Leopoldt conjecture is valid for l.

(2) The Galois group $\mathrm{Gal}(M/k)$ of the maximal l-ramified abelian l-extension M of k is torsion free.

By virtue of Theorem 4 in [1] of l-rationality, we could prove the following theorem:

Theorem A. *Let k be a totally real number field such that no prime $\mathfrak{l} \mid (l)$ is decomposed completely in K/k. Then, if $\varepsilon_1 C_1 = \{1\}$, we have $\lambda_k = \mu_k = 0$.*

Let $\bar{H}_n$ be compositum of cyclic l-extensions of K_n which are embedded into cyclic l-extensions of arbitrary degree. Let $\bar{L}_n$ (resp. $\bar{L}_n'$) be the maximal unramified (resp. unramified l-decomposed) abelian l-extension of K_n. There are the canonical isomorphisms $\mathrm{Gal}(\bar{L}_n/K_n) \simeq A_n$ and $\mathrm{Gal}(\bar{L}_n'/K_n) \simeq C_n$. The mutual relations of the structures of their Galois groups are studied in [3] and [4]. We shall obtain the following theorem based on the results in [3] and [4]:

Theorem B. *Let k be a totally real number field. If there is a finite m satisfying the conditions* (i) *and* (ii) *below, the Leopoldt conjecture is valid for l in k_n for every n, and moreover, we have $\lambda_k = \mu_k = 0$.*

(i) *No prime $\mathfrak{l} \mid (l)$ is decomposed in K_{m+1}/K_m.*

(ii) $\varepsilon_1 C_{m+1} = \{1\}$.

2. Proof of the theorems. We shall prove Theorem A first. Let k be a totally real field which satisfies the assumptions of Theorem A. By virtue of (iv) of Theorem 4.1 of [1], we see k is l-rational. Hence, by (i) of that theorem, the Galois group $\mathrm{Gal}(N/k)$ of the maximal l-ramified l-extension N is a pro-finite l-group of rank 1. Since N is also the maximal l-ramified l-extension of k_n for every n and since $\mathrm{Gal}(N/k_n) \simeq Z_l$, we have $\mu_k = \lambda_k = 0$. q.e.d.

The key of the proof of Theorem B is the following lemma:

Lemma 1. *Suppose the conditions* (i) *and* (ii) *are valid for k. Then we have $\varepsilon_1 C_n = \{1\}$ for each $n \geq m$.*

Proof. Let g_n and h_n be the orderes of $\varepsilon_1(C_n/C_n^{\gamma-1})$ for a generator γ of $\mathrm{Gal}(K_n/K_m)$ and $\varepsilon_1 C_n$, respectively. Note $\varepsilon_1(C_n/C_n^{\gamma-1}) \simeq (\varepsilon_1 C_n)/(\varepsilon_1 C_n)^{\gamma-1}$, because $\mathrm{Gal}(K_n/k)$ is abelian. Hence $g_n = 1$ is equivalent to $h_n = 1$. Let E' be the l-unit group of K_m, and N_n be the norm map of K_n/K_m. Denote by $Z_l \otimes E'$ and $Z_l \otimes N_n K_n^\times$ the tensor products over the ring Z of integers, where $K_n^\times$ is the multiplicative group of K_n. Set

$$F_n = Z_l \otimes E'/(Z_l \otimes E' \cap Z_l \otimes N_n K_n^\times), \qquad D(n) = \oplus_{\mathfrak{l}|(l)} D_{\mathfrak{l}},$$

where $D_{\mathfrak{l}}$ are the decomposition groups of primes $\mathfrak{l}$ of K_m in K_n/K_m. Combining the homomorphism $\varepsilon_1 F_n \to \varepsilon_1 D(n)$ defined in Theorem III.2.16 in [3] with the canonical map $\varepsilon_1(E'/E'^{l^{n-m}}) \to \varepsilon_1 F_n$, we have the following homomorphism:

$$\varphi_n : \varepsilon_1 E'/E'^{l^{n-m}} \longrightarrow \varepsilon_1 D(n).$$

Simultaneously, we also have the following formula from Theorem III.2.16:

$$g_n = h_m \mid Coker \quad \varphi_n \mid .$$

We see $h_m = 1$ and φ_{m+1} is surjective, because of $g_{m+1} = 1$. Let $\pi_n : D(n) \to D(n)/D(n)^l \cong D(m+1)$ be the canonical map for $n > m$. Since $\pi_n \circ \varphi_n = \varphi_{m+1}$, we have φ_n is also surjective. Thus, $g_n = 1$. $h_n = 1$ follows from $g_n = 1$. q.e.d.

In [4], three parameters ρ, λ, μ of a family of finite $Z_l[\Delta]$-modules $\{X_n\}_{n\geq 1}$ are defined as follows. Let $x_{n,i}$ be the exponent of l of the order of $\varepsilon_i X_n$. When we have an asymptotic formula

$$x_{n,i} \sim \rho_i n l^n + \lambda_i n + \mu_i l^n, \quad (n \to \infty)$$

for $0 \leq i \leq |\Delta| - 1$, the parameters are defined to be l-adic characters

$$\rho = \sum \rho_i \omega^i, \quad \lambda = \sum \lambda_i \omega^i, \quad \mu = \sum \mu_i \omega^i.$$

Let $< \varphi, \psi >$ denote the symmetric scalar product of l-adic characters of Δ. Let H_n, L_n' and L_n be the maximal subfields of exponent l^n over K_n of $\bar{H}_n, \bar{L}_n'$ and $\bar{L}_n$, respectively. The parameters of $\{\mathrm{Gal}(L_n'/K_n)\}_{n\geq 1}$ (resp. $\{\mathrm{Gal}(L_n/K_n)\}_{n\geq 1}$) are denoted by λ_c (resp. λ_K) and μ_c (resp. μ_K). We note the ρ-parameters of these families vanish and $\lambda_k =< \lambda_K, 1_\Delta >$, $\mu_k =< \mu_K, 1_\Delta >$.

· *Proof of Theorem B.* Let E be the unit group of k_m and $\bar{E}$ be the closure in the direct product $\prod_{\mathfrak{l}|l} U_\mathfrak{l}$ of the local unit groups $U_\mathfrak{l}$ of the completions of k_m at places $\mathfrak{l}|l$. Set $F^{(n)} = E \cap \bar{E}^{l^n}$. Since the Kummer extensions of K_n by radicals $F^{(n)} K_n^{\times l^n}/K_n^{\times l^n}$ are subfields of L_n', the dual of the radicals is isomorphic to factor groups of $\varepsilon_1 C_n$. Hence, Lemma 1 implies $F^{(n)} \subset K_n^{\times l^n}$, that is, $F^{(n)} \subset E^{l^n}$. We obtain

$$E/E^{l^n} \simeq \bar{E}/\bar{E}^{l^n}, n = 1, 2, \cdots.$$

This implies the Z-rank of E equals the Z_l-rank of $\bar{E}$. Hence the Leopoldt conjecture is valid for every k_n for $n \leq m$. Replacing k_m with k_t in this argument, we see the Leopoldt conjecture is valid for k_t for every $t > m$.

Let ρ_H, λ_H, μ_H be the parameters of the family $\{\mathrm{Gal}(H_n/K_n)\}$ in the tables I, II of [4]. Let $\Delta_\mathfrak{p}$ be the decomposition groups in K/k of $\mathfrak{p}|\infty$. Let χ_∞ be the sum of induced characters $Ind_{\Delta_\mathfrak{p}}^{\Delta} 1_{\Delta_\mathfrak{p}}$ of the unit characters $1_{\Delta_\mathfrak{p}}$ of $\Delta_\mathfrak{p}$ for every $\mathfrak{p}|\infty$. From the tables, we see $\rho_H = \omega\chi_\infty$, $< \lambda_H, 1_\Delta >=< \lambda_c, \omega >$ and $< \mu_H, 1_\Delta >=< \mu_c, \omega >$. Since $\Delta_\mathfrak{p}$ contains the complex conjugation $J \in \Delta$, we have $< \omega\chi_\infty, 1_\Delta >= 0$. We also see $\bar{H}_n \supset \bar{L}_n$ from the table in [3,Ch.I,1.3.c], and hence we have $H_n \supset L_n$. Therefore, we have the following consequence:

$$< \rho_H, 1_\Delta >= 0,$$

$$< \lambda_c, \omega >=< \lambda_H, 1_\Delta >\geq< \lambda_K, 1_\Delta >= \lambda_k,$$

$$< \mu_c, \omega >=< \mu_H, 1_\Delta >\geq< \mu_K, 1_\Delta >= \mu_k.$$

Lemma 1 proves $0 =< \lambda_c, \omega >= \lambda_k$ and $0 =< \mu_c, \omega >= \mu_k$. q.e.d.

Remark. We give a sketch of an alternative proof of $\lambda_k = \mu_k = 0$, based on the Iwasawa theory. Let M be the maximal l-ramified abelian l-extension of $k_\infty = \cup_{n\geq 1} k_n$ and set $H = \cup_{n\geq 1}\bar{H}_n$. Let s_n be the number of divisors $\mathfrak{l}(l)$ of k_n decomposed completely in K_n. Set $\lambda_l = \max\{ s_n; n \geq 1\}$. By the table in [3,Ch.I,1.3.c], we have $M \supset H \supset k_\infty$, and moreover, since the Leopoldt conjecture is valid, we have $\mathrm{Gal}(M/H) \simeq Z_p^{\lambda_l}$. Let X be the Galois group of the maximal l-ramified abelian l-extension of k_∞ and A (resp. D) be the inductive limit of A_n (resp. D_n) with respect to the natural maps. By virtue of Lemma 1, we see $\varepsilon_1 A = \varepsilon_1 D$. Let I_n be subgroups of the ideal groups of K_n generated by $\mathfrak{l}(l)$. Let Q_l be the field of quotients of Z_l. We have $\varepsilon_1 I_n/I_m \simeq \varepsilon_1 D_n/D_m$ from the isomorphism (10) of [5,p.480] when k is abelian over Q. We could show this isomorphism is valid for a totally real k with a similar argument as in the proof of (10) in [5]. The inductive limit of $\varepsilon_1 I_n/I_m$ for $n > m$ with respect to the natural maps is isomorphic to $(Q_l/Z_l)^{\lambda_l}$. Since there is $n_1 \geq m$ such that $\varepsilon_1 D_n^{l^{n-n_1}} = \varepsilon_1 D_{n_1} \supset \varepsilon_1 D_m$ holds for $n > n_1$, we have $\varepsilon_1 D \simeq (Q_l/Z_l)^{\lambda_l}$. By the Iwasawa theory, we have a pairing on $X \times \varepsilon_1 A$ (see [2,Sect.2]). Hence, $X \simeq Z_l^{\lambda_l}$. This means $\mathrm{Gal}(H/k_\infty)$ is of finite order. Since $\bar{L}_n \subset \bar{H}_n$, we have $\lambda_k = \mu_k = 0$.

3. Examples. Let k be the maximal real subfield of an f-th cyclotomic field K such that $l \mid f, l^2 \nmid f$. Denote by λ_l the number of divisors defined in the above remark, and by I_n the subgroup of the ideal group of K_n which is also defined in the above. $\varepsilon_1 C_n$ is the minus part C_n^-. Set $l^{e_n} = |A_n^-|$. We have $l^{\lambda_l(n-1)} \mid |D_n^-|$, because $I_n^-/I_1^- \simeq D_n^-/D_1^-$. Hence $|C_n^-| \leq l^{\lambda_l(e_n-n+1)}$.

Therefore, by virtue of Theorem A and B, we have $\lambda_k = 0$ if $\lambda_l = e_1 = 0$ or if $\lambda_l = e_2$. We examine the table in [6] of relative class numbers of f-th cyclotomic fields and obtain the following examples:

(1) $\lambda_l = e_1 = 0$

$l = 3$ $f =$ 12, 15, 21, 84, 51, 75, 123, 129, 147
$l = 5$ $f =$ 15, 35, 45, 65, 105, 135, 115
$l = 7$ $f =$ 28, 35, 56, 77, 119

$l = 11$	$f =$	$33, 44, 99$
$l = 13$	$f =$	65
$l = 17$	$f =$	$51, 119, 153$
$l = 23$	$f =$	$92, 115$
$l = 29$	$f =$	87
$l = 41$	$f =$	123

(2) $\lambda_l = e_2 \geq 1$

$l = 3$	$\lambda_l = 1$	$f =$	$24, 48, 60, 33, 132, 204, 165, 300$
$l = 3$	$\lambda_l = 2$	$f =$	$120, 264$
$l = 3$	$\lambda_l = 4$	$f =$	240
$l = 5$	$\lambda_l = 1$	$f =$	$20, 40, 60$
$l = 5$	$\lambda_l = 2$	$f =$	120
$l = 7$	$\lambda_l = 1$	$f =$	$21, 84$

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Hiroshi Yamashita
Kanazawa Women's College
10, Sue-machi
Kanazawa 920-13
Japan

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