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GAP ORDERS OF RATIONAL FUNCTIONS ON PLANE CURVES WITH FEW SINGULAR POINTS

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We prove that certain integers n cannot occur as degrees of linear series without base points on the normalization of a plane curve whose only singularities are a "small" number of nodes and ordinary cusps. As a consequence we compute the gonality of such a curve.

Introduction.

In the study of (smooth irreducible complete) algebraic curves it is interesting to know for which integers n there are linear series of degree n without base points (equivalently: rational functions of degree n) and, in particular, to compute the gonality of the curve, i.e. the least integer $n>0$ such that there is a linear series of degree n and positive dimension (necessarily without base points). One can see [CK] and [GR] for motivations and for some results in this direction.

In this paper we deal with a curve $\overline{C}$, which is the normalization of a plane singular curve C of degree d with only nodes or ordinary cusps as singularities.

Our main result shows that if the number of singular points is "small" , then certain integers cannot occur as degrees of linear series on $\overline{C}$ without base points. This is a

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partial answer to a problem posed by E.Ballico (see [BC], problem (5.b)), and implies that $\overline{C}$ is (d-2)-gonal, a result due to Coppens and Kato [CK].

Our statement is inspired by the above mentioned result of Coppens and Kato, where "small" is made clear. But our proof is of a completely different nature: indeed it uses a technique introduced in [GR], which is based on the study of the Hilbert function of certain zero-dimensional subscheme of $\mathbf{P}^2$; this has the advantage to put the problem in the natural framework of the theory of adjoints and to give the result in arbitrary characteristic.

Preliminaries and notation

We work over an algebraically closed field K of arbitrary characteristic.

If $X \subset \mathbf{P}^2$ is a zero-dimensional closed subscheme of degree $\partial(X)$, we denote by $H(X,i)$ the Hilbert function of X, by $\triangle H(X,i)=H(X,i)-H(X,i-1)$ its first difference for every $i>0$; and $H(X,0)=\triangle H(X,0)=1$.

Moreover we put $t=\max\{i \in N \mid \triangle H(X,i) \neq 0\}$ and $\alpha=\alpha(X)$ the least degree of a curve through X. We refer to [GR] for basic facts about the Hilbert function. For general notions on curves and linear series we refer to [H]. Moreover we shall use freely the following well known results on adjoints (see [F] and [K] for details): if C is an integral plane curve of degree d, with only nodes or ordinary cusps as singularities, the adjoints to C are just the curves passing through the singular points; and moreover the adjoints of degree d-3 cut out on the normalization of C the complete canonical series (outside of the "double point divisor", corresponding to the singular points).

Lemma 1.

Let $X\subset \mathbf{P}^2$ be a zero-dimensional subscheme such that $\partial(X) < [d/2]\ (d-[d/2])$ and

$\triangle H(X,d-2)\neq 0$, d an integer ≥ 4.

Then there is a curve Γ of $\mathbf{P}^2$ with the following properties:

i) $\deg \Gamma = k$, $1\leq k <[d/2]$
ii) $\partial(X\cap\Gamma)\geq k(d-k)$
iii) if $X \not\subset \Gamma$ and Y is any subscheme such that $X\cap\Gamma\subset Y\subset X$ and $\partial(Y) = \partial(X)-1$,

then $H(X,d-3)= H(Y,d-3)+1$.

Proof. Put $a=[d/2]$. If $\alpha\geq a$, then $\triangle H(X,a-1)=a$, and there exists an integer $\overline{i}$ such that

either $a\leq \overline{i} < d-2$ and $k= \triangle H(X, \overline{i}+1)=\triangle H(X, \overline{i}) < a$; otherwise

or $\triangle H(X,i)\geq a$ for every $a\leq i\leq d-3$;

or $a > \triangle H(X,\overline{i}) > \triangle H(X,\overline{i}) > \ldots > \triangle H(X,d-2) > 0$.

In each case we would have $\triangle H(X,i)\geq \triangle H(Z,i)$ for all i's, where Z is a complete intersection of type (a,d-a) and then $\partial(X) = \sum_{i=0}^{\infty}\triangle H(X,i)\geq a(d-a)$ (for more details see the proof of Theorem 3.1 in [GR]). If we denote by s the least integer $\geq a$ for which $\triangle H(X,i) = \triangle H(X,i+1)=k$ then there exists a curve Γ of degree k such that

$$\partial(X\cap\Gamma)= ks-k(k-3)/2+ \sum_{i=s+1}^{t}\triangle H(X,i) :=b$$

(see lemma 3.4 in [GR]).

An easy computation shows that $b\geq k(d-k)$ and then i) and ii) hold.

As $k< a\leq\alpha$ we have $X \not\subset \Gamma$. Let now Y be a subscheme such that $X\cap\Gamma\subset Y\subset X$ and $\partial(Y)=\partial(X)-1$; then $\triangle H(Y,i) \leq \triangle H(X,i)$ for every i and $\triangle H(Y,i) = \triangle H(X,i) -1$ for exactly one i.

By assumption

$$\triangle H(X,s)=...=\triangle H(X,s+h), \qquad h\geq 1;$$

it can be neither

$$\triangle H(Y,i) =\triangle H(X,i) -1 \qquad \text{if } s\leq i\leq s+h-1$$

(see the proof of lemma 3.5 in [GR]), nor

$\triangle H(Y,i) =\triangle H(X,i) -1$ if $i \geq s+h$, for otherwise we would have

$\partial(X\cap\Gamma)=\partial(Y\cap\Gamma)\leq b-1$, a contradiction.

Then $\triangle H(Y,i) =\triangle H(X,i) -1$ for some $i \leq s-1$, that is $H(Y,i) =H(X,i) -1$ for every $i \geq s-1$; in particular $H(Y,d-3)=H(X,d-3)-1$.

If $\alpha<a$ we have two cases

1) there is $\alpha\leq j\leq d-3$ such that $\triangle H(X,j+1) =\triangle H(X,j)=k<\alpha$

Then the previous argument applies.

2) No j as above exists. Then i) and ii) hold if we take $k=\alpha$ and iii) is void in this case.

Lemma 2

Let $X \subset \mathbf{P}^2$ be a zero-dimensional subscheme and assume X be the disjoint union of two subschemes X' and X''. Let $Z\subset X$ be a closed subscheme such that $X'\not\subset Z$.

Then there is a closed subscheme Y such that

i) $Z \subset Y\subset X$

ii) $\partial(X) = \partial(Y)+1$

iii) $X' \not\subset Y$

Proof Let U=spec(A) be an open affine subscheme of $\mathbf{P}^2$ containing X. The ideal I_X of X has a primary decomposition $I_X = q_1 \cap q_2 \cap \ldots \cap q_s$

and we may assume that $I_{X'} = q_1 \cap q_2 \cap \ldots \cap q_t$, $t \leq s$.

Since all the subschemes we consider are unmixed, we have $I_Z = q'_1 \cap q'_2 \cap \ldots \cap q'_s$

where $q_i \subset q'_i$ and either q'_i is primary or $q'_i = A$, for $i=1,\ldots,s$.

Since $X' \not\subset Z$ we may assume $q_1 \neq q'_1$. Then there is an ideal q , $q_1 \subset q \subset q'_1$

(possibly $q=A$) such that $l\ (q/q_1)=1$.

Let $J = q \cap q_2 \cap \ldots \cap q_s$, then the subscheme Y corresponding to J has the required properties.

Theorem 3

Let $C \subset \mathbf{P}^2$ be an integral curve of degree d whose only singularities are δ nodes or ordinary cusps. Let m,n positive integers and assume

1) $\delta + n < [d/2]\,(d - [d/2])$

2) $(m-1)\,d + 1 \leq n \leq md - 2m^2 - 1$.

Then every g^r_n on the normalization $\overline{C}$ of C has a base point.

Proof. We argue by contradiction. Let g^r_n be a base point free linear series. Then there is a divisor D in this series whose support is disjoint from the set of singular points. Let X' be the closed subscheme of $\mathbf{P}^2$ corresponding to D and let X" be the closed subscheme of $\mathbf{P}^2$ corresponding to the singular points with the reduced structure, and put $X=X' \cup X''$ (notice that $X' \cap X''=\emptyset$) Since $r \geq 1$ we have $\Delta H(X,d-2) \neq 0$ by Riemann-Roch and hence there is a curve Γ of degree k verifying i), ii), and iii) of lemma 1.

If $k \leq m-1$, since $n \geq (m-1)d+1$, we have $\Gamma \cap X \neq X'$ and hence, by lemma 2 there is a subscheme Y such that $X \cap \Gamma \subset Y \subset X$, $\partial(Y)=\partial(X)-1$ and $X' \not\subset Y$. Moreover, by lemma 1 iii) we have $H(X,d-3)-1=H(Y,d-3)$. Then if $D' < D$ is the divisor corresponding to $X' \cap Y$, we have $i(D)=i(D')+1$ and this shows that the point $P=D-D'$ is a base point for g^r_n, a contradiction.

If $k \geq m$, by lemma 1 ii) we have $\partial(C \cap \Gamma) \geq \partial(X \cap \Gamma) \geq k(d-k)$ whence

$$\partial(C \cap \Gamma) \geq 2[\, k(d-k) - n\,] + n$$

we will have an absurd if

$$2kd - 2k - n > kd$$

that is, by 2),
$$kd - 2k^2 - md + 2m^2 \geq 0$$

This is obvious if k=m. By (decreasing) induction on m we have

$$kd - 2k^2 - (m-1)d + 2(m-1)^2 = kd - 2k^2 - md + 2m^2 + d - 4m + 2 \geq 0$$

by the inductive hypothesis and because $d \geq 2m^2 + 2$ (otherwise no n verifying 2) exists).

Corollary 4 ([CK] Th.2.1)

Let the notation be as above. If $\delta+(d-3) < [d/2]\,(d-[d/2])$ and $\delta > 0$ then $\overline{C}$ is (d-2)-gonal.

Remark 5

If the characteristic of K is zero, we may assume, in the proof of theorem 3, that X is reduced (by Bertini); in this case lemma 2 is trivial. Not so in positive characteristic.

Remark 6

We observe that hypothesis 2) in the theorem cannot be weakened. Indeed it is not difficult to construct curves of degree d, with m^2 nodes and such that the curves of degree m through these nodes cut out a base point free linear series of degree $n = md - 2m^2$. And moreover it is obvious that for every curve there are base point free linear series of degree $(m-1)d$.

REFERENCES

[BC] Ballico E., Ciliberto C. (Eds) : Open problems in Algebraic Curves and Project ive Geometry Lect. Not.in Math. 1389 Berlin 1989

[CK] Coppens M., Kato T. : The gonality of smooth curves with plane models (preprint)

[F] Fulton W. Algebraic Curves , Benjamin , New York , 1969

[GR] Greco S., Raciti G. : The Luroth semigroup of plane algebraic curves (to appear on Pacific Journal of Mathematics)

[H] Hartshorne R. : Algebraic Geometry -GTM 52 Springer Verlag New York - Berlin 1977

[K] Keller O. : Vorlensungen uber algebraische Geometrie Leipsig, 1974

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