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REGULAR ORBITS OF HALL π-SUBGROUPS

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1. Introduction

Let V be a faithful KG-module and let H be a subgroup of G. When does V contain a regular H-orbit?

If H is a p-subgroup of the solvable group G and $O_p(G) = 1$, it has been proved by the first author that H always has a regular orbit on V, being $p = \mathrm{char}(K)$ odd ([1]).

This (not semisimple) result had Hall-Higman type applications : if G is a solvable group and p is an odd prime number, then $|G : O_{p'p}(G)|_p$ divides $b(P)$, where P is a Sylow-p-subgroup of G and (as in [4]) $b(P) = \max\{\chi(1) \mid \chi \in \mathrm{Irr}(P)\}$.

It is not in general true that any subgroup H of G has a regular orbit on V. However, if $O_\pi(G) = 1$, where π is the set of primes which divide $|H|$, we will show that H does have a regular orbit on V (when the group G is of odd order).

It is our aim to prove the following.

Theorem. *Let G be a group of odd order and let H be a Hall π-subgroup of G. Let V be a faithful G-module, over possibly different finite fields of odd π-characteristic. Assume that $V_{O_\pi(G)}$ is completely reducible. Then there exists $v \in V$ such that $C_H(v) \subseteq O_\pi(G)$.*

As in [1], we give the following character degree application.

Corollary. *Let G be a group of odd order and let H be a Hall π-subgroup of G. Then there exists $\alpha \in \mathrm{Irr}(H)$ such that $|G : O_{\pi'\pi}(G)|_\pi$ divides $\alpha(1)$. In particular, $|G : O_{\pi'\pi}(G)|_\pi \leq b(H)$.*

2. Proofs

Next Lemma is a powerful tool for looking for regular orbits.

Lemma. *Let G be a group of odd order having a faithful and irreducible quasiprimitive module V over a finite field of odd characteristic. Suppose that $F(G)$ is noncyclic. Then V contains at least two regular G-orbits.*

Proof. See Lemma 2.1. of [2].

Proof of the Theorem. Let G be a counterexample minimizing $dim_K(V)$.

(1) V is G-irreducible.

Proof. Let R be a Hall $\pi\prime$-subgroup of $O_{\pi\pi'}(G)$. If $h \in H - O_\pi(G)$, let $1 \neq Y(h)$ a Hall $\pi\prime$-subgroup of $[h, R]$.

If $h \in H - O_\pi(G)$, we claim that there exists an irreducible G-submodule $V(h)$ of V such that $Y(h)$ acts nontrivially on $V(h)$.

Since $V_{O_\pi(G)}$ is completely reducible and the fields have π-characteristic observe that $V_{O_{\pi\pi'}(G)}$ is also completely reducible.

Write $V_{O_{\pi\pi'}(G)} = V_1 \oplus \ldots \oplus V_t$, where the V_i's are the homogeneous components.

Since $Y(h) > 1$, suppose for instance that $Y(h)$ acts nontrivially on V_1.

Now consider the G-submodule $\sum_{x\in G} V_1 x$ and choose an irreducible G-submodule W of it. We claim that $V_1 \cap W > 0$. To prove this, let X be an irreducible $O_{\pi\pi'(G)}$-submodule of W. Since for every $x \in G$, the $V_1 x$'s are homogeneous components, it follows that $X \subseteq V_1 x$, for some $x \in G$. Since $Wx = W$, we will have that $V_1 \cap W > 0$.

Suppose now that $Y(h)$ acts trivially on $W \cap V_1$ and let Y be an irreducible $O_{\pi\pi'(G)}$-submodule of $W \cap V_1$. Therefore, since V_1 is a direct sum of modules isomorphic to Y, it follows that $Y(h)$ acts trivially on V_1.

This shows that $Y(h)$ acts nontrivially on $V(h) = W$, as claimed.

If $H = O_\pi(G)$ there is nothing to prove. Let $U = \sum_{h \in H - O_\pi(G)} V(h)$ a completely reducible G-submodule of V. If $U < V$, by induction, there exists $u \in U$ such that $C_{\bar{H}}(u) \subseteq O_\pi(\bar{G})$, where $\bar{G} = G/C_G(U)$. Let $C = C_G(U)$.

We claim that $H \cap C = O_\pi(G)$.

Let $h \in H \cap C - O_\pi(G)$. Since $h \in C$, it follows that h acts trivially on $V(h)$. Therefore, $[h, R] \subseteq kerV(h)$ and thus $Y(h)$ acts trivially on $V(h)$, which is a contradiction. This proves that $H \cap C \subseteq O_\pi(G)$. Now, $H \cap C \subseteq O_\pi(G) \cap C = O_\pi(C) \subseteq H \cap C$, as claimed.

Now we prove that $O_\pi(\bar{G}) = O_\pi(G)C/C$. Let $K/C = O_\pi(G)$.

Observe that $[K/CO_\pi(G), O_{\pi\pi'}(G)/CO_\pi(G)] = 1$.

If $h \in H \cap K - O_\pi(G)$, then $[h, R] \subseteq [K, O_{\pi\pi'}(G)] \subseteq O_\pi(G)C$. Since C contains the π'-subgroups of $O_\pi(G)C$, it follows that $Y(h) \subseteq C$, which is a contradiction. This proves that $K = O_\pi(G)C$, as we wanted.

Now, since $H \cap C = O_\pi(G)$ and $C_H(u) \subseteq O_\pi(G)C$, it follows that $C_H(u) \subseteq O_\pi(G)$ and we may assume that $U = V$.

Hence V is a completely reducible G-module.

If V is not irreducible, $V = V_1 \oplus V_2$, where each V_i is G-invariant. By induction, let $v_i \in V_i$ such that $C_{HC_G(V_i)/C_G(V_i)}(v_i) \subseteq O_\pi(G/C_G(V_i))$ and consider $v = v_1 + v_2$. Then $C_H(v) \subseteq O_\pi(G)$.

Thus, we may assume that V is an irreducible KG-module, for a finite field K with odd characteristic.

(2) V is a quasiprimitive G-module.

Proof. Let N be a normal subgroup of G and put $V_N = V_1 \oplus \ldots \oplus V_t$, where the V_i's are homogeneous N-modules. Moreover, V_i is an irreducible $N_G(V_i)$-module with $V_i^G = V$. Let H_1 be a Hall π-subgroup of $N_G(V_1)$ containing $N_H(V_1)$ and suppose that $t > 1$.

By induction, there exists $v \in V_1$ such that $C_{H_1 C_G(V_1)/C_G(V_1)}(v_1) \subseteq O_\pi(N_G(V_1)/C_G(V_1)) = S/C_G(V_1)$.

Now, G acts transitively on $\Omega = \{V_1, \ldots, V_t\}$ with kernel G_Ω.

By Corollary 1 of [3], we have that G/G_Ω has a regular orbit on the power set of Ω. Let $\Lambda \subseteq \Omega$ be a representative of such an orbit. Let $g_i \in G$ with $V_1 g_i = V_i$.

Let $w = x_1 + \ldots + x_t \in V$ be defined as follows: $x_i = vg_i$ if $V_i \in \Lambda$ and $x_i = -vg_i$ if $V_i \in \Omega - \Lambda$.

Observe that since G is a group of odd order and the characteristic of the field is odd, $ux \neq -u$ for all $x \in G$ and $u \in V$.

We see that $C_H(w) \subseteq O_\pi(G)$. Suppose that $h \in C_H(w)$ and write $V_j h = V_{\sigma(j)}$.Then $x_j h = x_{\sigma(j)}$. Since there is no $x \in G$ with $vx = -v$, we have that $V_{\sigma(j)} \in \Lambda$ if $V_j \in \Lambda$. This implies that $h \in G_\Omega$ (i.e., $\sigma(j) = j$ for all j). Consequently, $g_j h g_j^{-1} \in C_{N_G(V_1)}(v)$ for all j. Hence $h \in \bigcap_{j=1,\ldots,t} S^{g_j}$, which is a normal π-subgroup of G.

(3) $F(G)$ is cyclic.

Proof. If not, Lemma above gives us the existence of a regular orbit.

By Proposition (2.1) of [5], we may assume that $V = GF(q^n)^+$ and that $G \subseteq J$, where $J = C_n \ltimes M$, ($M = GF(q^n)^*$ acts as multiplications on V and $C_n = Gal(GF(q^n)/GF(q))$). Moreover, $G \cap M = F(G)$.

(4) There is no counterexample.

Proof. Let K and $< \sigma >$ be Hall 2'-subgroups of M and C_n, respectively, and let $G^* =< \sigma > \ltimes K$. Replacing G by some conjugate we may assume that $G \subseteq G^*$.

Also, observe that if the theorem is true for G^* it is also true for G. Hence, it is not loss to suppose $G = G^*$.

Replacing again by some conjugate, we may assume that $H =< \tau > \ltimes K_\pi$, where $< \tau >$ and K_π are Hall π-subgroups of $< \sigma >$ and K, respectively.

It suffices to show that $|\bigcup_{h\in H-O_\pi(G)} C_V(h)| < |V|$.

Write $H - O_\pi(G) = \bigcup_{j=1,\dots,t} x_j K_\pi$, where $x_j \in <\tau>$. Suppose that $\bigcup_{h\in H-O_\pi(G)} C_V(h) = V$. Then

$V^\# = \bigcup_{j=1,\dots,t}(\bigcup_{m\in K_\pi} C_{V^\#}(x_j m))$.

Since for each j, $\bigcup_{m\in K_\pi} C_{V^\#}(x_j m) = \{v \in V^\# \mid v^{x_j}/v \in K_\pi\}$ is a multiplicative subgroup of $V^\#$ (cyclic), it follows that

$V^\# = \bigcup_{m\in K_\pi} C_{V^\#}(x_j m)$, for some j.

Following notation and Proposition (1.3) of [5], if $N = \{x \in V^\# | N_{x_j}(x) = 1\}$, where N_σ is the norm map (i.e., $N_\sigma(y) = y\sigma(y)\dots\sigma^{o(\sigma)-1}(y)$), then $|N| = \frac{q^n-1}{q^{n/s}-1}$, where $s = o(x_j)$.

By Proposition (1.3) of [5], we will have that $N \subseteq K_\pi$. Then it follows that $|K_{\pi'}|$ divides $q^{n/s}-1$.

Since, by Galois Theory, $|C_{V^\#}(x_j)| = q^{n/s}-1$, we have that $K_{\pi'} \subseteq C_{V^\#}(x_j)$. Thus $< x_j, K_\pi > \triangleleft < x_j, K > \triangleleft\, G$.

Since $x_j \notin O_\pi(G)$, this is a contradiction.

Corollary. *Let G be a group of odd order and let H be a Hall π-subgroup of G. Then there exists $\alpha \in Irr(H)$ such that $|G : O_{\pi'\pi}(G)|_\pi$ divides $\alpha(1)$. In particular, $|G : O_{\pi'\pi}(G)|_\pi \le b(H)$.*

Proof. We may assume that $O_{\pi'}(G) = 1$. Let $N = O_\pi(G)$. Then, fairly standard arguments show that $C = C_G(F(N)/\Phi(N)) \subseteq N$. Write $V =$Irr$(F(N)/\Phi(N))$ and $\bar{G} = G/C$. Thus, $O_\pi(\bar{G}) = N/C$.

Now, V is a faithful $\bar{G}$-module such that $V_{O_\pi(\bar{G})}$ is completely reducible. By the Theorem, there exists $\lambda \in$ Irr(V) such that $C_H(\lambda) \subseteq N$. Let $\xi \in$ Irr($C_H(\lambda)|\lambda)$ and $\alpha = \xi^G \in$ Irr(H). Thus $|H : N|$ divides $\alpha(1)$, as wanted.

References

1. *A. Espuelas*, The Existence of regular orbits, *J. Algebra* **127**, No. 2, (1989), 259-268

2. *A. Espuelas*, Large character degrees of groups of odd order, To appear in Illinois J. Math.

3. *D. Gluck*, Trivial set stabilizers in finite permutation groups, *Can. J. Math.* **35**, (1983), 59-67

4. *I. M. Isaacs*, Character Theory of Finite Groups, New York, Academic Press, 1976

5. *A. Turull*, Supersolvable automorphism groups of solvable groups, *Math. Z.***183**, (1983), 47-73

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