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A SIMPLE CHARACTERIZATION OF ALMOST UNIFORM CONVERGENCE BY STOCHASTIC CONVERGENCE

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Abstract. Motivated by Egorov's theorem and the characterization of the equivalence of P-stochastic convergence and P-almost convergence by the property of the probability distribution P to be purely atomic and concentrated on a countable number of pairwise disjoint P-atoms (cf. [1], p. 68), it is proved that P-stochastic resp. P-almost convergence is equivalent to P-almost uniform convergence (cf. [2], p. 89/90) if and only if P is purely atomic and concentrated on a finite number of pairwise disjoint P-atoms. Furthermore, this property of P is equivalent to the condition that any P-stochastic convergent sequence admits a P-almost uniform convergent subsequence. Finally a proof is given that P is purely atomic and concentrated on a finite number of pairwise disjoint P-atoms if and only if there does not exist a purely finitely additive {0,1}-valued probability charge, which vanishes for all P-zero sets.

Let P denote a probability distribution on a σ-algebra $\mathfrak{F}$ of subsets of a set Ω, such that there exists a decreasing sequence $A_n \in \mathfrak{F}$, $n \in \mathbb{N}$, satisfying $P(A_n) > 0$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Then the sequence X_n, $n \in \mathbb{N}$, of random variables defined by $X_n = nI_{A_n}$ converges pointwise but not P-almost uniformly, i.e. there does not exist a sequence $A_n \in \mathfrak{F}$, $n \in \mathbb{N}$, described above, if P-stochastic resp. P-almost convergence implies P-almost uniform convergence. Furthermore, a sequence $A_n \in \mathfrak{F}$, $n \in \mathbb{N}$, of the type described above does not exist if and only if P is purely atomic and concentrated on a finite number of pairwise disjoint P-atoms. This follows from the decomposition $P = \alpha P_1 + (1-\alpha)P_2$, $0 \leq \alpha \leq 1$, where P_1 is atomless and P_2 is purely atomic with a countable number of pairwise disjoint atoms. Conversely, if P is concentrated on a finite number of pairwise disjoint atoms, then P-almost uniform convergence follows from P-stochastic convergence, if one takes into consideration, that a real valued $\mathcal{S}$-measurable function is P-a.e. constant on a P-atom. Finally, since the sequence X_n, $n \in \mathbb{N}$, defined by $X_n = nI_{A_n}$, where $A_n \in \mathcal{S}$, $n \in \mathbb{N}$, is decreasing and satisfies $\bigcap_{n=1}^{\infty} A_n = \emptyset$ and $P(A_n) > 0$, $n \in \mathbb{N}$, does not admit a subsequence, which converges P-almost uniformly, the following result has been proved:

Theorem. The following conditions are equivalent for a probability distribution P:

(i) P-stochastic convergence implies P-almost uniform convergence;

(ii) P-almost convergence implies P-almost uniform convergence;

(iii) any P-stochastic convergent sequence admits a P-almost uniformly convergent subsequence;

(iv) P is purely atomic and concentrated on a finite number of pairwise disjoint P-atoms.

Remark.

1. Condition (iv) is equivalent to the property of P, that there does not exist some {0,1}-valued, purely finitely additive probability charge, which vanishes for all P-zero sets (cf. [3], p. 187 - 189 for a different proof). Clearly (iv) implies that there does not exist some {0,1}-valued, purely finitely additive probability charge vanishing for all P-zero sets. For the converse direction one might start from a decreasing sequence $A_n \in \mathcal{S}$, $n \in \mathbb{N}$, satisfying $\bigcap_{n=1}^{\infty} A_n = \emptyset$ and $P(A_n) > 0$, $n \in \mathbb{N}$. Then $B_n = A_n^c \cap A_{n-1}$, $n \in \mathbb{N}$, $(A_0 = \Omega)$, are pairwise disjoint subsets belonging

to the underlying σ-algebra $\mathcal{S}$ of subsets of Ω and satisfy $\bigcup_{n=1}^{\infty} B_n = \Omega$. Furthermore, since $P(A_n) > 0$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} A_n = \emptyset$ is valid, one might assume that $B_n \neq \emptyset$ holds true for all $n \in \mathbb{N}$. Let now Q^* stand for the probability charge defined on the σ-algebra $\mathcal{S}^*$ generated bv the B_n, $n \in \mathbb{N}$, which satisfies $Q^*(B_n) = 0$, $n \in \mathbb{N}$. An extension of Q^* to some $\{0,1\}$-valued probability charge Q on $\mathcal{S}$, which vanishes for all P-zero sets, proves the converse direction.

2. A similar reasoning yields the characterization of an algebra $\mathcal{A}$ of subsets of a set Ω, which has the property that any finite, nonnegative and finitely additive set function is already σ-additive, by the condition, that there does not exist a countable number of pairwise disjoint and nonvoid sets $A_1, A_2, \ldots$, belonging to $\mathcal{A}$ and satisfying $\bigcup_{n=1}^{\infty} A_n = \Omega$. There exist infinite algebras of subsets of a set Ω of this type as the following example shows:

<u>Example.</u> Choosing $\Omega = \mathbb{N}$ and generated $\mathcal{A}$ by all singletons $\{n\}$, $n \in \mathbb{N} \setminus \{1\}$, i.e. $\mathcal{A}$ consists of all finite subsets of $\mathbb{N} \setminus \{1\}$ and their complements. Then $A_n \in \mathcal{A}$, pairwise disjoint and nonvoid, $n \in \mathbb{N}$, satisfying $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$, leads to the fact that A_n is a finite subset of $\mathbb{N} \setminus \{1\}$, $n \in \mathbb{N}$, which is a contradiction to $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$.

The importance of this example can be illustrated by introducing the algebra $\mathcal{A}'$ generated by $\mathcal{A}$ and $\{1\}$, i.e. $\mathcal{A}' = \{A \subset \mathbb{N}\colon\ A \text{ or } A^c \text{ finite}\}$ and the probability measure P on $\mathcal{A}'$ concentrated on $\{1\}$ resp. the purely finitely additive probability charge Q on $\mathcal{A}'$, which vanishes for all singletons $\{n\}$, $n \in \mathbb{N}$. Then P and Q coincides on $\mathcal{A}$, i.e. in general a probability measure P on an algebra $\mathcal{A}$ of subsets of a set Ω might have extensions to the algebra $\mathcal{A}'$ generated by $\mathcal{A}$ and some subset A of Ω, which are not σ-additive probability charges on $\mathbb{A}'$. This effect cannot occur, if $\mathbb{A}$ is a σ-algebra of subsets of a set Ω and $\mathbb{A}'$ denotes the σ-algebra generated by $\mathbb{A}$ and some subsets A of Ω. A proof of this assertion follows from the fact that $\mu(A \cap A_1 + A^c \cap A_2) = P^*(A \cap A_1) + P^*(A^c \cap A_2)$, $A_j \in \mathbb{A}$, $j = 1,2$, where P^* denotes the outer measure of P and $\{A_1 \cap A + A^c \cap A_2 : A_j \in \mathbb{A},\ j = 1,2\} = \mathbb{A}'$ is valid, defines a finite measure μ on $\mathbb{A}'$ satisfying $P' \le \mu$ for all probability measures P' on $\mathbb{A}'$, which are equal on $\mathbb{A}$ to P. Furthermore, there exists probability measures P' of this type, namely $P'(A \cap A_1 + A^c \cap A_2) = P^*(A \cap A_1) + P_*(A^c \cap A_2)$ (resp. $P'(A \cap A_1 + A^c \cap A_2) = P_*(A \cap A_1) + P^*(A^c \cap A_2)$), $A_j \in \mathbb{A}$, $j = 1,2$, where P_* denotes the inner measure of P.

References

[1] Lukacs, E.: Stochastic Convergence. Academic Press, New York, 1975

[2] Halmos, P.: Measure Theory. Van Nostrand, New York, 1961

[3] Plachky, D.: A characteristic property of atomic measures with finitely many atoms resp. atomless measures. Contributions to Stochastics, Ed. W. Sendler, Physica, Heidelberg, 1987

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