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Construction of families of curves from finite length graded modules

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We construct particular families of projective space curves starting from finite length graded modules, exhibiting several phenomena of specializations. The main tool is the Rao construction via a free resolution of the module.

0. Introduction and preliminaries

The main purpose of this paper is to show the power of liaison techniques for producing examples of phenomena in the classification of space projective curves. In particular, the starting points of all of these three applications are just variations on the Rao construction of a curve from a finite length graded $\mathbf{k}[x_0,x_1,x_2,x_3]$-module, where $\mathbf{k}$ is a field.

Section 1 is related to the existence of specializations and special families of space curves, concentrating on the variations of their cohomology groups; the conducting philosophy is that "every phenomenon allowed by semicontinuity is actually existing in some Hilbert scheme, soon or later".

In particular, in the first part of section 1 we study the possible configurations of specializations of families of space curves, and we concentrate for simplicity on the case of families with fixed speciality. The main result is more or less the following: every graph of submodules of a graded $\mathbf{k}[x_0,x_1,x_2,x_3]$-module of finite length can be realized (in a sense that will be made more precise) by a configuration of families of curves, with fixed speciality, in an irreducible component of a suitable Hilbert scheme, where the general element is an arithmetically Cohen-Macaulay curve. For a result in the opposite direction, see [2], section 3. In the second part of the section we perform a "simultaneous" specialization of curves with the same dimensions of the homogeneous components of the Hartshorne-Rao modules, but with all possible graded module structures.

In section 2 we study the existence of curves defined over an infinite field (non necessarily algebraically closed) $\mathbf{L}\subseteq\mathbf{k}$. The proof is easy but the results are new, as far as we know, and this shows again the power of these techniques.

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Throughout this paper $\mathbf{k}$ will be an algebraically closed field of characteristic zero, $S=\mathbf{k}[x_0,x_1,x_2,x_3]$, $\mathbf{P}^3_{\mathbf{k}}=$ Spec(S). A curve will be a closed locally Cohen-Macaulay generically locally complete intersection one-dimensional subscheme of $\mathbf{P}^3_{\mathbf{k}}$. To every curve C is associated its deficiency module, also called Hartshorne-Rao module, which is the graded S-module of finite length $M(C)=\oplus_i H^1(\mathbf{P}^3_{\mathbf{k}},\mathcal{I}_C(i))$. It is known ([10]) that two curves C and D are linked by liaison if and only if M(C) and M(D) are isomorphic up to shifting degrees and possibly dualizing. We say that C and D are in the same *even* liaison class if and only if M(C) and M(D) are isomorphic up to shifting degrees.

Moreover, given any S-module of finite length M, there exists an even liaison class whose curves have Hartshorne-Rao modules isomorphic (up to shifting) to M.

In [1] we defined an affine variety parametrizing all possible structures of graded S-module of finite length which are compatible with a given "graded" **k**-vector space structure. We recall here the definition. Let $M=\bigoplus_i M_i$ be a graded S-module of finite length, and let $(m_1,...,m_t)$ be a t-ple of non-negative integers such that $m_1>0$, $m_t>0$. We say that M is of type $(m_1,...,m_t)$ if, up to shifting degrees, $M_i=0$ if $i\leq 0$, $i>t$, and $\dim_{\mathbf{k}} M_i=m_i$, if $1\leq i\leq t$ (note that every homogeneous component of M has a structure of **k**-vector space). Let now V be the vector space $V=\bigoplus_{i=1}^{t} \mathbf{k}^{m_i}=\bigoplus_{i=1}^{t} V_i$, let $G=\{g\in S|\deg(g)=1\}\cup\{0\}$, and fix the canonical basis of V.

<u>Definition.</u> $\mathcal{V}=\mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t)=$

$$=\Big\{ f=(f_1,...,f_{t-1})\in \bigoplus_{i=1}^{t-1} \mathrm{Hom}(V_i\otimes G, V_{i+1}) \big| \forall g,h\in G, \forall \alpha\in V_i, \forall i,\ 1\leq i\leq t-2$$
$$f_{i+1}[f_i(\alpha\otimes g)\otimes h]=f_{i+1}[f_i(\alpha\otimes h)\otimes g]\Big\}$$

is called the <u>variety of module structures of finite length</u> over V of type $(m_1,...,m_t)$.

If $M=\bigoplus_{i=1}^{t} M_i$ is a graded S-module of type $(m_1,...,m_t)$, and $\mathcal{B}_i$ is a basis for the **k**-vector space M_i, then to M is associated an element of $\mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t)$ if we send each basis $\mathcal{B}_i$ to the canonical basis of V_i, since the multiplication is completely known if we know the vector space structure and the multiplication times element of G. Note that in this way we don't identify isomorphic module structures; nevertheless, every isomorphism class of structures is a locally closed irreducible subset of $\mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t)$. For other informations about the structure and the properties of $\mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t)$, see [1].

DEFINITION. A flat family $p:Z\to W$ of curves in $\mathbf{P}^3_{\mathbf{k}}$ ($Z\subset W\mathrm{x}\mathbf{P}^3_{\mathbf{k}}$, and p induced by the projection) is said to have fixed speciality if for all $t\in\mathbf{Z}$ and all $a\in W$, $b\in W$ the following equality is verified:

$$h^1(p^{-1}(a),\mathcal{O}_{p^{-1}(a)}(t)) = h^1(p^{-1}(b),\mathcal{O}_{p^{-1}(b)}(t)).$$

Since we are interested in the variations of the cohomology of curves we fix some notations. Let $F:\{0,1,2\}\mathrm{x}\mathbf{Z}\to\mathbf{N}$ be a function . If $(m_1,...,m_t)$ is a t-ple of non-negative integers such that $m_1>0$, $m_t>0$, we say that F is of type $(m_1,...,m_t)$ (in analogy with a previous definition) if there exists n such that

$$F(1,n+i)=m_i \quad \forall i,\ 1\le i\le t$$
$$F(1,v)=0 \quad \text{if } v\le n,\ v>n+t.$$

We denote

$$H_F=\{C \text{ curve} \mid F(i,s)=h^i(\mathbf{P}^3_{\mathbf{k}},\mathcal{J}_C(s)),\ \forall i,\forall s\},$$

and if $C\in H_F$, we say that F is the cohomology function of C. If $\mathcal{L}$ is an even liaison class, we denote $H_{F,\mathcal{L}}=H_F\cap\mathcal{L}$. It is known that this is an irreducible locally closed subset of the Hilbert scheme ([4], 2.2 and [1], 2.2).

DEFINITION. A cohomology function F of type $(m_1,...,m_t)$ is good if it satisfies the following condition: there exists a direct sum of line bundles L on $\mathbf{P}^3_{\mathbf{k}}$ and a vector bundle **K** over $\mathcal{V}(m_1,...,m_t)\mathrm{x}\mathbf{P}^3_{\mathbf{k}}$, with rank**K**=rankL+1, with the following properties:

1) $h^i(\mathbf{P}^3_{\mathbf{k}},K_\xi(s))=h^i(\mathbf{P}^3_{\mathbf{k}},K_\zeta(s))$
$\forall i,\ 0\le i\le 3,\ \forall s\in\mathbf{Z},\ \forall\xi,\zeta\in\mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t)$ ($K_\xi=\mathbf{K}_{\{\xi\}\mathrm{x}\mathbf{P}^3_{\mathbf{k}}}$)

2) for every ξ there is a morphism $\phi_\xi : L \to K_\xi$ whose dependency locus is a curve $Y_\xi \in H_F$, and $M(Y_\xi)$ is isomorphic to ξ.

In [2] it is proved that if F is good, then the irreducible components of H_F are in one to one correspondence with those of $\mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t)$, that the Buchsbaum curves of H_F lie in the intersection of all irreducible components of H_F, and that every curve $Y \in H_F$ specializes to a stick figure ([9]) through curves all in H_F. Moreover, for every t-ple $(m_1,...,m_t)$ there exist infinite good cohomologies of type $(m_1,...,m_t)$ (this was done by performing a "Rao construction with parameters").

1. Configurations of specializations

In an irreducible component of a Hilbert scheme we have a general cohomology, that is to say, there exists an open dense subset whose curve have the same cohomology. Besides these curves, one has special families, and one may ask for the configurations that these families may form. The natural answer is that the only obstruction to deal with should be represented by some semicontinuity condition. On this way, we show how to construct these configurations, starting from graphs of modules and morphisms. More precisely, we construct configurations of families with fixed speciality, and the graphs of modules is reproduced by the deficiency modules of the families. The exact statement is the following (a "path" is a composition of arrows, and "without loops" means without oriented loops, that is to say without oriented paths starting from a vertex and going back to the same vertex)

THEOREM 1.1.

Let $\mathcal{G}=\{G_i,\phi_k\}$ $,1\le i\le m$, *be a finite connected oriented commutative graph, without loops and with only one vertex where no arrow arrives, where the vertexes are finite-length graded* S-*modules and the arrows are surjective morphisms of graded* S-*modules. Then for infinitely many* (d,g) *there exists an irreducible component* U *of a suitable Hilbert scheme* $Hilb_{d,g}P^3_k$ *such that:*

a) the general element of U *is an arithmetically Cohen-Macaulay curve*

b) for every G_i *there is an irreducible flat family* V_i *of curves, having Hartshorne-Rao module isomorphic to* G_i *(suitably shifted)*

c) if there is a path from G_i *to* G_j, *then* V_i *is contained in the closure of* V_j

d) *all curves of all families have the same speciality*.

Proof.

Let H be the vertex where no arrow arrives, and $G_{j_1},\ldots,G_{j_s}$ the vertexes from which no arrow starts. We can consider another graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding the trivial module: we add to $\mathcal{G}$ $G_0=\{0\}$ and the maps $G_{j_k}\to G_0$.

Also $\mathcal{G}'$ is a finite connected oriented commutative graph, without loops; note that $\forall i$, $0\le i\le m$, there is a surjective map $H\to G_i$; let F_i be the submodule of H, kernel of this map.

We can associate to $\mathcal{G}'$ in a natural way another graph $\mathcal{G}'^*=\{F_i,\lambda_k\}$ of submodules of H; if there is an arrow $\phi_k:G_i\to G_j$, then $\lambda_k:F_i\to F_j$ is the corresponding injection.

We consider now a set of generators (as graded S-module) of F_i, $1\le i\le m$, and we extend it to a set of generators of H; we use these sets of generators for constructing free resolutions of F_i and H. In these resolutions, we consider the second syzygies modules ${}_2F_i$ and ${}_2H$. By sheafifying them, we get two locally free sheaves $\mathcal{F}_i$ and $\mathcal{H}$, and a morphism (by functoriality)

$$\gamma_i\colon \mathcal{F}_i \to \mathcal{H}$$

(for more details, see also [1]).

These locally free sheaves have the property that ([10])

$$\bigoplus_t H^1(\mathbf{P}^3_{\mathbf{k}},\mathcal{F}_i(t))=F_i,$$
$$\bigoplus_t H^1(\mathbf{P}^3_{\mathbf{k}},\mathcal{H}(t))=H,$$
$$H^2(\mathbf{P}^3_{\mathbf{k}},\mathcal{F}_i(t))=0=H^2(\mathbf{P}^3_{\mathbf{k}},\mathcal{H}(t)) \quad \forall t,$$

and that the map induced in cohomology

$$i\colon \bigoplus_t H^1(\mathbf{P}^3_{\mathbf{k}},\mathcal{F}_i(t)) \to \bigoplus_t H^1(\mathbf{P}^3_{\mathbf{k}},\mathcal{H}(t))$$

by γ_i is exactly the inclusion $F_i \to H$ (hence its cokernel is isomorphic to G_i), as in [2], prop.2.2.

Let us call

$$\mathcal{F}=\mathcal{H}\oplus\mathcal{F}_1\oplus\ldots\oplus\mathcal{F}_m,$$

and let us consider the morphisms

$$\psi_{m+1} =(0,0,\ldots\ldots,0)\colon\ \mathcal{F}\to\mathcal{H}$$
$$\psi_0 =(1_{\mathcal{H}},0,\ldots\ldots,0)\colon\ \mathcal{F}\to\mathcal{H}$$
$$\psi_1 =(0,\gamma_1,\ldots\ldots,0)\colon\ \mathcal{F}\to\mathcal{H}$$
$$\ldots\ldots$$
$$\psi_i =(0,0,\ldots,\gamma_i,\ldots,0)\colon\ \mathcal{F}\to\mathcal{H}$$
$$\ldots\ldots$$
$$\psi_m =(0,0,\ldots\ldots,\gamma_m)\colon\ \mathcal{F}\to\mathcal{H}.$$

It is possible to find a direct sum of line bundles $\mathcal{P}$ and morphisms $\sigma_0, \sigma_1,\ldots, \sigma_m, \sigma_{m+1}$, such that

$$\xi_{m+1} =\psi_{m+1}\oplus\sigma_{m+1}\colon \mathcal{F}\to\mathcal{H}'$$
$$\xi_0 =\psi_0\oplus\sigma_0\colon \mathcal{F}\to\mathcal{H}'$$
$$\xi_1 =\psi_1\oplus\sigma_1\colon \mathcal{F}\to\mathcal{H}'$$

....

$\xi_i \quad =\psi_i \oplus \sigma_i: \qquad \mathcal{F} \to \mathcal{H}'$

....

$\xi_m = \quad \psi_m \oplus \sigma_m: \qquad \mathcal{F} \to \mathcal{H}'$

(where $\mathcal{H}' = \mathcal{H} \oplus \mathcal{P}$) are injective morphisms of vector bundles. The maps induced in cohomology are yet the trivial map, the projection on H and the inclusions $F_i \to H$, $1 \leq i \leq m$.

Thanks to [2], lemma 2.1, there exists another direct sum of line bundle $\mathcal{K}$ and morphisms $\tau_0, \tau_1, \ldots, \tau_m, \tau_{m+1}$, such that

$\mu_{m+1} \quad =(\xi_{m+1}, \tau_{m+1}): \qquad \mathcal{F} \oplus \mathcal{K} \to \mathcal{H}'$

$\mu_0 \quad =(\xi_0, \tau_0): \qquad \mathcal{F} \oplus \mathcal{K} \to \mathcal{H}'$

$\mu_1 \quad =(\xi_1, \tau_1): \qquad \mathcal{F} \oplus \mathcal{K} \to \mathcal{H}'$

....

$\mu_i \quad =(\xi_i, \tau_i): \qquad \mathcal{F} \oplus \mathcal{K} \to \mathcal{H}'$

....

$\mu_m \quad =(\xi_m, \tau_m): \qquad \mathcal{F} \oplus \mathcal{K} \to \mathcal{H}'$

are morphisms whose dependency loci are curves $Y_{m+1}, Y_0, Y_1, \ldots, Y_m$. Note that their Hartshorne-Rao modules are, respectively, $M(Y_{m+1}) = H$, $M(Y_0) = \{0\}$, $M(Y_i) = {}^{H}/_{F_i} \cong G_i$ if $1 \leq i \leq m$ (this follows easily from the exact sequences

$$0 \to \mathcal{F} \oplus \mathcal{K} \to \mathcal{H}' \to \mathcal{J}_{Y_i}(\delta) \to 0).$$

Let us consider $\mathrm{Hom}(\mathcal{F} \oplus \mathcal{K}, \mathcal{H}')$: there is an open dense subset of morphisms having a two-codimensional dependency locus.

Now we are finally ready to define the families V_i. Let F_j be fixed, and let $F_{j_1}, \ldots, F_{j_p}$ be the vertexes of $\mathcal{G}'^*$ for which there is a path $F_{j_r} \to F_j$; we define V_j as the linear subspace of $\mathrm{Hom}(\mathcal{F} \oplus \mathcal{K}, \mathcal{H}')$ spanned by $\mu_{j_1}, \ldots, \mu_{j_p}$ and μ_j. Then there is an open dense subset of V_j consisting of morphisms having a two-codimensional dependency locus, hence a curve; let us call W_j the family of these curves. If μ is general in this open set, then the map induced in cohomology

$$\bigoplus_t H^1(\mathbf{P}^3_{\mathbf{k}}, \mathcal{F}\oplus\mathcal{K}(t)) \to \bigoplus_t H^1(\mathbf{P}^3_{\mathbf{k}}, \mathcal{H}'(t)) \cong H$$

has F_j as image (the injections $F_{j_r} \to H$ factorizes through F_j); hence its cokernel is isomorphic to G_j. $V_j \subseteq W_j$ will be the open subset consisting of curves having Hartshorne-Rao module isomorphic to G_j. It is clear that if there is a path from F_i to F_j, then W_i is contained in W_j, and hence V_i is contained in the closure of V_j.

Corresponding to the subspace spanned by $\mu_0,\ldots,\mu_m$ and μ_{m+1} there is an open subset consisting of arithmetically Cohen-Macaulay curves, with fixed cohomology and hence contained in one irreducible component of a suitable Hilbert scheme. All families F_j's are contained in its closure.

Note that since $H^2(\mathbf{P}^3_{\mathbf{k}}, \mathcal{H}'(t))=0$ $\forall t$, then all curves of all families have the same speciality.

In order to get infinite (d,g), it is enough to change the degrees of the line bundles appearing in $\mathcal{P}$ and $\mathcal{K}$.

It is clear that there is no restriction in assuming that the graph is without oriented loops. We can consider also graphs with many vertexes where no arrow arrives with the following

COROLLARY 1.2.

Let $\mathcal{G}=\{G_i,\phi_k\}$ $,1\le i\le m,$ *be a finite connected oriented commutative graph, without loops where the vertexes are finite-length graded* S-*modules and the arrows are surjective morphisms of graded* S-*modules. Then for infinitely many* (d,g) *there exists an irreducible component* U *of a suitable Hilbert scheme* $\mathrm{Hilb}_{d,g}\mathbf{P}^3_{\mathbf{k}}$ *such that:*

a) the general element of U *is an arithmetically Cohen-Macaulay curve*
b) for every G_i *there is a (possibly reducible) family* V_i *of curves, having Hartshorne-Rao module isomorphic to* G_i *(suitably shifted)*
c) if there is a path from G_i *to* G_j, *then the closure of every irreducible component of* V_j *contains an irreducible component of* V_j
d) *all curves of all families have the same speciality*.

Proof.

Let $H_1, ..., H_s$ be the vertexes where no arrow arrives, and for each one let $\mathcal{G}_i$ be the graph composed of all paths starting from H_i, and $\mathcal{G}_i$ '* be the associated graph of submodules of H_i. As in 1.1, we get locally free sheaves $\mathcal{H}_1$, $\mathcal{F}1_1$, $\mathcal{F}1_2$, ..., $\mathcal{F}1_{t(1)}$, $\mathcal{H}_2$, $\mathcal{F}2_1$, $\mathcal{F}2_2$, ..., $\mathcal{F}2_{t(2)}$,..., $\mathcal{H}_s$, $\mathcal{F}s_1$,$\mathcal{F}s_2$,...,$\mathcal{F}s_{t(s)}$. Note that for every i there can be more than one sheaf having the first cohmology module isomorphic to F_i (in fact, the number of locally free sheaves having first cohmology module isomorphic to F_i is equal to the number of H_j's for which there is a path from H_j to F_i).

We perform the same construction as in 1.1, but starting with morphisms

$$\psi_{i_j}: \mathcal{H}_1 \oplus .. \mathcal{H}_s \oplus \mathcal{F}1_1 \oplus .. \mathcal{F}1_{t(1)} \oplus \mathcal{F}2_1 \oplus ... \mathcal{F}2_{t(2)} \oplus \mathcal{F}s_1 \oplus .. \mathcal{F}s_{t(s)} \to$$
$$\to \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_s$$

where ψ_{i_j} restricted to $\mathcal{H}_k$, $k \neq i$, is the injection and restricted to $\mathcal{H}_i$ and to $\mathcal{F}l_m$, $(l,m) \neq (i,j)$ is the zero map, and restricted to $\mathcal{F}i_j$ is the map given by functoriality by the Rao construction. Then the proof follows as in 1.1.

REMARK 1.3.

Proposition 3.1 of [2] gives a result in the opposite direction.

REMARK 1.4.

If we consider vector bundles with $H^1(\mathbf{P}^3_k,\mathcal{F}_i(t))=0$ $\forall t$, and $\bigoplus_t H^2(\mathbf{P}^3_k,\mathcal{F}_i(t))=F_i$, one can perform a similar construction for families with fixed postulation. In a special case this was done in [6].

The next proposition shows how to specialize all liaison classes corresponding to a certain type (that is to say, corresponding to all modules of a fixed variety $\mathcal{V}_{\mathbf{P}3}(n_{i+1},\ldots,n_{i+h})$) to liaison classes of another type, say in the variety $\mathcal{V}_{\mathbf{P}3}(m_1,\ldots,m_t)$, at the same time. That is to say, we can find a Hilbert scheme where there are curves for every structure of $\mathcal{V}_{\mathbf{P}3}(n_i,\ldots,n_{i+h})$, and every such curve specializes to a curve whose Hartshorne-Rao module is in $\mathcal{V}_{\mathbf{P}3}(m_1,\ldots,m_t)$.

To prove this, first we need a lemma:

LEMMA 1.5.

Let $m_1,\ldots,m_t$ *be non-negative integers*, $m_1>0<m_t$. *Then there exists a locally free sheaf* $\mathbf{K}$ *on* $\mathbf{P}^3 x \mathcal{V}_{\mathbf{P}3}(m_1,\ldots,m_t)$ *such that* $\forall \xi \in \mathcal{V}_{\mathbf{P}3}(m_1,\ldots,m_t)$, *the graded module* $\bigoplus_i H^1(\mathbf{P}^3_k,\mathbf{K}_\xi(i))$ *is isomorphic to* ξ $(\mathbf{K}_\xi=\mathbf{K}_{\{\xi\}x\mathbf{P}3})$, *and the cohomology dimensions* $h^i(\mathbf{P}^3_k,\mathbf{K}_\xi(s))$ *don't depend on* ξ, *for every* i *and* s.

Proof.

The proof is contained in [1], proof of 3.1.

PROPOSITION 1.6.

Let $(m_1,...,m_t)$ *be a t-ple of non negative integers and* $(n_{i+1},...,n_{i+h})$ *be a h-ple of non-negative integers, with* $m_1>0<m_t$ *and* $n_{i+1}>0<n_{i+h}$ *as usual, and* $n_s \le m_s$, $\forall s$. *Then there exists a Hilbert scheme where there is a family* T *of curves, with fixed cohomology, such that:*

a) *for every* $\zeta \in \mathcal{V}_{\mathbf{P}^3}(n_{i+1},...,n_{i+h})$, *there exists a family* T_ζ *contained in* T *whose curves have Hartshorne-Rao module isomorphic to* ζ

b) *for every curve* $Y \in T$, *there exists an irreducible flat family to which* Y *belongs which contains a curve* Z_Y, *with Hartshorne Rao module* $\xi \in \mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t)$.

Proof.

Let $X=\mathbf{P}^3_\mathbf{k} \times \mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t) \times \mathcal{V}_{\mathbf{P}^3}(n_{i+1},...,n_{i+h})$, and let

$p: X \to \mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t) \times \mathcal{V}_{\mathbf{P}^3}(n_{i+1},...,n_{i+h})$

$q: X \to \mathbf{P}^3_\mathbf{k} \times \mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t)$

$r: X \to \mathbf{P}^3_\mathbf{k} \times \mathcal{V}_{\mathbf{P}^3}(n_{i+1},...,n_{i+h})$

be the projections. If $\mathbf{F}_m$ and $\mathbf{G}_n$ are the locally free sheaves on $\mathbf{P}^3_\mathbf{k} \times \mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t)$ and $\mathbf{P}^3_\mathbf{k} \times \mathcal{V}_{\mathbf{P}^3}(n_{i+1},...,n_{i+h})$ respectively, whose existence is given in 1.4, we denote by $\mathbf{F}=q^*(\mathbf{F}_m)$, $\mathbf{G}=r^*(\mathbf{G}_n)$.

We consider the relative *Hom* -sheaf ([3])

$$\mathcal{X}=\mathcal{H}om_{\mathcal{O}_X}(p;\mathbf{F},\mathbf{G});$$

for every $y=(\xi,\zeta) \in \mathcal{V}_{\mathbf{P}^3}(m_1,...,m_t) \times \mathcal{V}_{\mathbf{P}^3}(n_{i+1},...,n_{i+h})$, an element of $\mathcal{H}om_{\mathcal{O}_X}(p;\mathbf{F},\mathbf{G})_y$ is a morphism $\phi:\mathbf{F}_{m_\xi} \to \mathbf{G}_{n_\zeta}$. We consider the open subset $\mathcal{X}_0$ of the variety associated to $\mathcal{X}$ corresponding to surjective morphisms. Note that for every $\zeta \in \mathcal{V}_{\mathbf{P}^3}(n_{i+1},...,n_{i+h})$ it is always possible to find a $\xi\in \mathcal{V}_{\mathbf{P}^3}(m_1,\ldots,m_t)$ for which there is a surjective morphism $\mathbf{F}_{m_\xi}\to\mathbf{G}_{n_\zeta}$: simply, let $\xi=\zeta\oplus\tau$, where $\tau=\mathbf{k}^{m_1}\oplus\mathbf{k}^{m_2}\oplus..\oplus\mathbf{k}^{(m_{i+1}-n_{i+1})}\oplus..\oplus\mathbf{k}^{(m_{i+h}-n_{i+h})}\oplus\mathbf{k}^{m_{i+h+1}}\oplus\ldots\oplus\mathbf{k}^{m_t}$, with the trivial multiplicative structure. Then the natural surjection on the modules gives a surjection on the sheaves (by functoriality of the construction of Rao); more explicitely, note that $\mathbf{F}_{m_\xi}$ is nothing but $\mathbf{G}_{n_\zeta}\oplus$B, where **B** is a direct sum of twists of the cotangent sheaf and of line bundles ([5], 2.8)

Then we have the family $\mathcal{K}=(\{\mathbf{K}_{(\xi,\zeta,,\phi)}, j_{(\xi,\zeta,,\phi)})$ of the kernels of these morphisms; i.e. we have a family of exact sequences on $\mathbf{P}^3_{\mathbf{k}}$

$$0\to\mathbf{K}_{(\xi,\zeta,,\phi)}\to\mathbf{F}_{m_\xi}\to\mathbf{G}_{n_\zeta}\to 0$$

where the maps are $j_{(\xi,\zeta,,\phi)}$ and ϕ.

Following the proof of [2],2.2, the map induced in cohomology by $j_{(\xi,\zeta,,\phi)}$ has ζ as cokernel, and there exist two direct sums of line bundles **A** and **B** (that we can suppose that are not depending on ξ,ζ and ϕ, since all the parameter spaces we are dealing with are bounded) and morphisms $\pi_{(\xi,\zeta,,\phi)}$, $\sigma_{(\xi,\zeta,,\phi)}$, $\pi_{0,(\xi,\zeta,,\phi)}$, $\sigma_{0,(\xi,\zeta,,\phi)}$ such that

$J_{(\xi,\zeta,,\phi)}=(\,\pi_{(\xi,\zeta,,\phi)},\ j_{(\xi,\zeta,,\phi)}\oplus\sigma_{(\xi,\zeta,,\phi)}\,):\ \mathbf{K}_{(\xi,\zeta,,\phi)}\oplus\mathbf{A}\ \to\ \mathbf{F}_{m_\xi}\oplus\mathbf{B}$

$J_{0,(\xi,\zeta,,\phi)}=(\,\pi_{0,(\xi,\zeta,,\phi)},\ 0\oplus\sigma_{0,(\xi,\zeta,,\phi)}\,):\ \mathbf{K}_{(\xi,\zeta,,\phi)}\oplus\mathbf{A}\ \to\ \mathbf{F}_{m_\xi}\oplus\mathbf{B}$

are morphisms which drop rank in codimension two.

Now for every (ξ,ζ,ϕ) we consider the family of morphisms

$$\Lambda_{t;(\xi,\zeta,,\phi)}=t\cdot J_{(\xi,\zeta,,\phi)}+(1-t)\cdot J_{0,(\xi,\zeta,,\phi)},$$

with t belonging to a Zariski-open subset of **k** and $\Lambda_{t;(\xi,\zeta,,\phi)}$ dropping rank in codimension two, hence along a curve $Y_{t;(\xi,\zeta,,\phi)}$. We get by a standard computation on long exact sequences that $Y_{0;(\xi,\zeta,,\phi)}$ belongs to the liaison class corresponding to the deficiency module ξ, and that for $t\neq 0$ $Y_{t;(\xi,\zeta,,\phi)}$ belongs to the liaison class corresponding to ζ.

Note that all curves $Y_{t;(\xi,\zeta,,\phi)}$ have the same speciality, and even the same cohomology if $t\neq 0$.

<u>REMARK 1.7</u>. One may also try to patch together the constructions of 1.1 and 1.6, studying "graphs" of varieties of module structures.

2. Curves over a subfield of k.

In the sequel, we will suppose that $\mathbf{L}\subseteq\mathbf{k}$ is an infinite field, and we will assume for simplicity that the algebraic closure of $\mathbf{L}$ is $\mathbf{k}$. Of course, all statements will be true for every algebraically closed field $\mathbf{k}'$ containing $\mathbf{k}$. Let $T=\mathbf{L}[x_0,\dots,x_3]$.

<u>**THEOREM 2.1.**</u>

Let M *be a* T-*module of finite length and let* $\mathcal{M}$ *be the even liaison class of curves of* $\mathbf{P}^3_{\mathbf{k}}$, *with Hartshorne-Rao module isomorphic to* $M\otimes\mathbf{k}$ *(possibly shifted). Let* $Y\in\mathcal{M}$. *Then there exists* $X\in\mathcal{M}$ *defined over* $\mathbf{L}$ *such that* $F_X=F_Y$.

Proof.

Let us suppose that M is of type $(m_1,\dots,m_t)$, and let us consider the beginning of a free resolution of a suitable shift of M over $\mathbf{L}$:

$$\begin{array}{l} \cdots\cdots \to \bigoplus_{j=2}^{t+1} k_j T(-j) \to \bigoplus_{i=1}^{t} h_i T(-i) \to M \to 0 \\ \quad\nearrow \\ F \\ \nearrow \\ 0 \end{array}$$

(F are the second syzygies).

By flatness,

$$\begin{array}{l} \cdots\cdots \to \bigoplus_{j=2}^{t+1} k_j S(-j) \to \bigoplus_{i=1}^{t} h_i S(-i) \to M\otimes\mathbf{k} \to 0 \\ \quad\nearrow \\ G \\ \nearrow \\ 0 \end{array}$$

is a free resolution of $M\otimes\mathbf{k}$ over $\mathbf{k}$, and $G=F\otimes\mathbf{k}$.

By sheafifying one gets a vector bundle $\mathcal{F}$ over Spec(T) and a vector bundle $\mathcal{F}_{\mathbf{k}}$ over $\mathrm{Spec}(S)=\mathbf{P}^3_{\mathbf{k}}$.

Let now $Y\in\mathcal{M}$; there exists a direct sum of line bundles $\bigoplus_{m=1}^{r}\mathcal{O}_{\mathbf{P}^3_{\mathbf{k}}}(-a_m)$ and a morphism of vector bundles

$$\bigoplus_{m=1}^{r}\mathcal{O}_{\mathbf{P}^3_{\mathbf{k}}}(-a_m)\rightarrow\bigoplus_{n=1}^{s}\mathcal{O}_{\mathbf{P}^3_{\mathbf{k}}}(-b_n)\oplus\mathcal{F}_{\mathbf{k}}$$

whose cokernel is $\mathcal{J}_Y(\delta)$, for some δ. Since

$$\mathrm{Hom}\Big(\bigoplus_{m=1}^{r}\mathcal{O}_{\mathbf{P}^3_{\mathbf{k}}}(-a_m),\bigoplus_{n=1}^{s}\mathcal{O}_{\mathbf{P}^3_{\mathbf{k}}}(-b_n)\oplus\mathcal{F}_{\mathbf{k}}\Big)=$$
$$=\mathrm{Hom}\Big(\bigoplus_{m=1}^{r}\mathcal{O}_{\mathbf{P}^3_{\mathbf{L}}}(-a_m),\bigoplus_{n=1}^{s}\mathcal{O}_{\mathbf{P}^3_{\mathbf{L}}}(-b_n)\oplus\mathcal{F}\Big)\otimes\mathbf{k},$$

we have

$$\dim_{\mathbf{k}}\Big[\mathrm{Hom}\Big(\bigoplus_{m=1}^{r}\mathcal{O}_{\mathbf{P}^3_{\mathbf{k}}}(-a_m),\bigoplus_{n=1}^{s}\mathcal{O}_{\mathbf{P}^3_{\mathbf{k}}}(-b_n)\oplus\mathcal{F}_{\mathbf{k}}\Big)\Big]=$$
$$=\dim_{\mathbf{L}}\Big[\mathrm{Hom}\Big(\bigoplus_{m=1}^{r}\mathcal{O}_{\mathbf{P}^3_{\mathbf{L}}}(-a_m),\bigoplus_{n=1}^{s}\mathcal{O}_{\mathbf{P}^3_{\mathbf{L}}}(-b_n)\oplus\mathcal{F}\Big)\Big].$$

In fact, $\mathbf{k}$ is algebraic over $\mathbf{L}$, and hence it is limit of finite extensions. At every step one can apply a base change theorem ([8], p.255), and one can pass to the limit since everything is finitely presented ([7], 8).

Now,

$$\overline{\mathrm{Hom}\Big(\bigoplus_{m=1}^{r}\mathcal{O}_{\mathbf{P}^3_{\mathbf{L}}}(-a_m),\bigoplus_{n=1}^{s}\mathcal{O}_{\mathbf{P}^3_{\mathbf{L}}}(-b_n)\oplus\mathcal{F}\Big)}=$$
$$=\mathrm{Hom}\Big(\bigoplus_{m=1}^{r}\mathcal{O}_{\mathbf{P}^3_{\mathbf{k}}}(-a_m),\bigoplus_{n=1}^{s}\mathcal{O}_{\mathbf{P}^3_{\mathbf{k}}}(-b_n)\oplus\mathcal{F}_{\mathbf{k}}\Big)$$

and since in the second group there is an open dense subset of morphisms dropping rank in codimension 2, one can find a curve X which is the dependency locus of some morphism

$\phi \in \mathrm{Hom}(\bigoplus_{m=1}^{r} \mathcal{O}_{\mathbf{P}^3_{\mathbf{L}}}(-a_m), \bigoplus_{n=1}^{s} \mathcal{O}_{\mathbf{P}^3_{\mathbf{L}}}(-b_n) \oplus \mathcal{F})$.

Note moreover that $F_X = F_Y$.

COROLLARY 2.2.

With the hypothesis of 2.1, *every curve* $Y \in \mathcal{M}$ *has a specialization (with fixed cohomology) to a curve* $X \in \mathcal{M}$ *defined over* **L**.

Proof.

This is an easy consequence of 2.1 and of the irreducibility of $H_{F,\mathcal{M}}$.

LEMMA 2.3.

Let F *be a good cohomology. Then every curve* $Y \in H_F$ *specializes (with fixed cohomology) to a curve defined over* **L**.

Proof.

Let F be of type $(m_1,...,m_t)$. Let $\mathcal{B}$ be the Buchsbaum liaison class $\mathcal{L}_{m_1,...,m_t}$ (i.e., the corresponding Hartshorne-Rao module is nothing but the graded sum of vector spaces $\mathbf{k}^{m_1} \oplus ... \oplus \mathbf{k}^{m_t}$ with trivial multiplication). We can apply proposition 2.1 to the graded module $\mathbf{L}^{m_1} \oplus ... \oplus \mathbf{L}^{m_t}$ (where the multiplication is trivial too) in order to find a curve $X \in \mathcal{B}$ defined over **L**. But now we know that X is in the intersection of all the irreducible components of H_F, and hence the lemma follows.

COROLLARY 2.4.

In every liaison class there are curves which specialize (with fixed cohomology) to a curve defined over L.

Proof.

This is a consequence of lemma 2.3 and of the fact that good cohomologies exist (see [1], prop.3.1). A usual, from the existence of an infinite number of good cohomologies one gets infinite pairs (d,g) for which the thesis holds.

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