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Remarks on fixed points of automorphisms and higher-order Weierstrass points in prime characteristic

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In this note we show how to use a theorem of Neeman [3] to get, for curves over algebraically closed fields of prime characteristic, the result of Farkas and Kra (section V.2.12 in [1]) asserting that if an automorphism has three or more fixed points then each of them is a q-Weierstrass point, for infinitely many values of q.

Notations.

Let X be a non-singular irreducible projective curve of genus $g \geq 2$, defined over an algebraically closed field of characteristic p. We denote by $\Phi_q : X \to \mathbb{P}^{d-1}$, where $d = g$ if $q = 1$ and $d = (2q-1)(g-1)$ if $q \geq 2$, the morphism given by a basis for the space of holomorphic q-differentials on X. To every point $P \in X$ there is a sequence $\epsilon_0(P) < \epsilon_1(P) < \ldots < \epsilon_{d-1}(P)$ of natural numbers, called the order-sequence of Φ_q at the point, which are the possible intersection multiplicities of the curve with the hyperplanes of $\mathbb{P}^{d-1}$ at the branch centered at P. For all but finitely many points P on X this sequence is the same. The generic sequence is denoted by $\epsilon_0 < \epsilon_1 < \ldots < \epsilon_{d-1}$ and is called the order-sequence of the morphism Φ_q. The finitely many points P on X whose order-sequence differs from the generic one are called q-Weierstrass points. The morphism Φ_q is called classical if $\epsilon_i = i$ for $0 \leq i \leq d-1$. (See Stöhr-Voloch [4]).

Remarks.

The first remark is essential in what follows and it says that Φ_q is classical, for infinitely many values of q. More precisely:

Remark 1. For an integer N relatively prime to the characteristic p and for each $0 \leq \mu \leq N-1$, the following set is infinite:

$$\left\{ q \in \mathbb{N} \;\middle|\; \begin{array}{l} q \equiv \mu (mod N) \text{ and} \\ \Phi_q \text{ is classical} \end{array} \right\}.$$

Proof: The theorem 2.7 of Neeman [3] says that given integers a, g and p, p prime, there exists an integer b divisible by a and a power p^m of p such that, for any curve X of genus g over a field of characteristic p, any morphism given by a divisor of degree $b + kp^m$, $k \in \mathbb{N}$, is classical. Taking $a = N(2g-2)$ and $k = l(2g-2)$, $l \in \mathbb{N}$, in this theorem, there exists an integer c such that any morphism given by a divisor of degree $(cN + lp^m)(2g-2)$, $l \in \mathbb{N}$, is classical.In particular, the morphism Φ_q is classical for $q = cN + lp^m$, $l \in \mathbb{N}$. Since N and p^m are relatively prime, we choose a natural number l_0 so that $l_0 p^m \equiv \mu(mod N)$ and take $l = l_0 + l_1 N$, $l_1 \in \mathbb{N}$.

Remark 2. Consider an automorphism of the curve X with order N relatively prime to the characteristic p. If it has three or more fixed points, then each of them is a q-Weierstrass point for infinitely many values of q with $q \equiv 1(mod N)$, and also for infinitely many values of q with $q \equiv 0(mod N)$.

Proof: The arguments used here are the ones in section V. 2.12 of Farkas-Kra [1]. (See also Cor. 3 in [2]). For fixed point P, the automorphism acts on the space of holomorphic q-differentials as the diagonal matrix (for a suitable choice of a basis):

$$diag\ (\xi^{q+\epsilon_0(P)}, \xi^{q+\epsilon_1(P)}, \ldots, \xi^{q+\epsilon_{d-1}(P)}),$$

where ξ is a primitive N-th root of unity.
Let $q > 1$ be such that Φ_q is classical. Passing to a power T of the automorphism, we assume that N is a prime number. Computing the dimension of the space of T-invariant holomorphic q-differentials, we get the equality:

$$\#\{q + \epsilon_i(P) \mid q + \epsilon_i(P) \equiv 0(mod N)\} = (2q-1)(g_T - 1) + \nu(T)[g(1 - \frac{1}{N})],$$

where g_T is the genus of the quotient curve X/T, $\nu(T)$ is the number of fixed points of T and the brackets $[\ldots]$ denote the integer part of ....
Suppose that the fixed point P is not a q-Weierstrass point (i.e. $\epsilon_i(P) = i$ for $i < d = (2q-1)(g-1)$) and that $q \equiv 0$ or $1(mod N)$. Using this in the equality above and Riemann-Hurwitz formula for $X \to X/T$, we get $\nu(T) \leq 2$.
Remark 2 now follows from Remark 1.

Remark 3. Consider an automorphism of the curve X whose order is a prime number N distinct from the characteristic p. Take $q \equiv 1(mod N)$ and let

$$\epsilon_0(P) < \epsilon_1(P) < \ldots < \epsilon_{d-1}(P)$$

be the order-sequence of Φ_q at a fixed point P for the automorphism. Denoting by $\tilde{g}$ the genus of the quotient curve, we have:

$$\tilde{g} = \#\left\{ \epsilon_i(P) \;\middle|\; \begin{array}{l} \epsilon_i(P) \geq d - g \text{ and} \\ \epsilon_i(P) \equiv -1(mod N) \end{array} \right\}.$$

Proof: This follows using the same arguments as before, after noticing that we always have (Φ_q being classical or not):

$$\epsilon_i(P) = i \text{ for } 0 \leq i \leq d - (g+1).$$

This remark gives a relation between the order-sequences at the fixed points.

Remark 4. Let G be an abelian group of automorphisms of the curve X with order N relatively prime to the characteristic p. Let $\tilde{X} = X/G$ be the quotient curve and $\pi : X \to \tilde{X}$ the projection. Denote by $R(\pi)$ the number of ramification points, that is $R(\pi) = \#\{P \in X | \tau(P) = P \text{ for some } \tau \in G \setminus id\}$. Assume that $R(\pi) \geq 3$. Then, each ramification point of π is a q-Weierstrass point for infinitely many values of q if one of the following conditions are satisfied:
a) there exist a fully ramified point (in this case G is cyclic).
b) the order N is an odd number.
c) the order N does not divide $2g$.

Proof: First of all we note that if there exists a point P on X with ramification index equals to $N/2$ and such that the only ramification points of $\pi_1 : X \to X/H$, where H is the subgroup of G that fixes the point P, are the point P and its conjugate under G, then it follows (from Riemann-Hurwitz formula) that N divides $2g$. The remark now follows by noticing that in the other cases each ramification point of π is a fixed point for some automorphism in G with three or more fixed points.

References

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