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THE SINGULAR SET OF AN ENERGY MINIMIZING MAP FROM B^4 TO S^2

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The singular set of an energy minimizing map from a four dimensional domain to S^2 consists locally of a finite set and a finite union of Hölder continuous curves.

0. Introduction.

Much is now known about the singular set of an energy minimizing map from a 3 dimensional domain to the 2 sphere $\mathbf{S}^2$. In the domain, it consists of isolated points [**SU1**]. At each such isolated point, there is a unique tangent map [**S1**] which is classified [**BCL**], and the asymptotic behavior of the minimizer is well-studied [**S1**], [**S2**], [**GW**].

For a 4 dimensional domain, the Schoen-Uhlenbeck work [**SU1**] implies that the singular set of an energy minimizing map is of Hausdorff dimension one. Here (5.1), for the case of maps of a 4 dimensional domain into $\mathbf{S}^2$, we show that *the singular set is locally a union of a finite set and a finite family of Hölder continuous closed curves having at most a finite number of crossings*. In his important recent work on the uniqueness of cylindrical tangent maps [**S4**], L. Simon has shown that these curves are actually $C^{1,\alpha}$ smooth away from the crossings and locally of finite length. The present work, being independent of these uniqueness results and involving some miscellaneous results on minimizing tangent maps, should be of independent interest.

In §3 we discuss the general structure and compactness of homogeneous minimizers from $\mathbf{B}^4$ to $\mathbf{S}^2$. These serve as tangent maps for general minimizers. Unfortunately, few examples are known. The most important are the cylindrical maps $\omega \circ \left(\frac{p}{|p|}\right)$ corresponding to an

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orthogonal rotation ω of $\mathbf{S}^2$ and an orthogonal projection $p : \mathbf{R}^4 \to \mathbf{R}^3$. These exhibit the line singularity $p^{-1}\{0\}$. A point singularity is given by the homogeneous extension $H\left(\frac{x}{|x|}\right)$ of the Hopf map $H : \mathbf{S}^3 \to \mathbf{S}^2$ [GC]. In §6 we also show that the family of those homogeneous energy minimizing maps from $\mathbf{B}^4$ to $\mathbf{S}^2$ which are singular only at the origin form a compact subfamily in H^1. In particular, their restrictions to $\mathbf{S}^3$ have their Hopf invariants universally bounded. In §4 we control the location of singularities by combining the well-known "persistence of singularities" for minimizers with the "persistence of regular points" which is special to the problem here. We conclude that the nonisolated singular point set is, at each sufficiently small scale, close in Hausdorff distance to some line segment. With this we can apply in §5 the Reifenberg Topological Disk Lemma [**R**], [**M**, 10.5] to obtain the desired structure of Hölder continuous curves. For notational convenience we here treat only domains in $\mathbf{R}^4$. One can easily adapt the discussion to handle general Riemannian domains. For such modifications, see e.g. [**HL2**, §7].

An important remaining question concerns the uniqueness of the tangent map at a crossing point of the singular set. Here we have not ruled out the possibility of the singular curves spiralling slowly into the crossing point.

The authors appreciate discussions with L. Simon concerning his recent works [**S3**], [**S4**].

1. Preliminaries.

Here we collect for frequent reference various important results concerning energy minimizing maps. We use the following notations for open balls:

$$\mathbf{B}_r^m(a) = \{x \in \mathbf{R}^m : |x-a| < r\},\ \mathbf{B}_r^m = \mathbf{B}_r^m(0),\ \mathbf{B}^m = \mathbf{B}_1^m,$$

and occasionally delete the m when its value is clear from the context.

Suppose $u : \Omega \to N$ is an energy minimizing map where Ω is a domain in $\mathbf{R}^m$ and N is a compact Riemannian submanifold of some Euclidean space.

(1.1) [**SU1**, 2.4] (Monotonicity) $r^{2-m} \int_{\mathbf{B}_r} |\nabla u|^2 dx \le s^{2-m} \int_{\mathbf{B}_s} |\nabla u|^2 dx$ whenever $\overline{\mathbf{B}}_r \subset \mathbf{B}_s \subset \Omega$ and equality holds if and only if u is <u>homogeneous</u> i.e. $u(x) \equiv u\left(\frac{sx}{|x|}\right)$. The <u>density</u> $\Theta_u(a) = \lim_{r \downarrow 0} r^{2-m} \int_{\mathbf{B}_r(a)} |\nabla u|^2 dx$ exists and is upper semi-continuous in a.

(1.2) [**SU1**, 2.5,4.7] For $a \in \Omega$, any sequence $r_i \downarrow 0$ contains a subsequence s_i so that the scaled maps $v_i = u(a + s_i(\cdot))|\mathbf{B}_1$ converge strongly in H^1 to some homogeneous map v, called a <u>tangent map</u> of u at a.

(1.3) [**SU1**, Th.2] The <u>singular set</u> $\operatorname{Sing} u$ of discontinuities of u is closed and coincides with $\{a \in \Omega : \Theta_u(a) > \epsilon_0\}$ for some positive constant $\epsilon_0 = \epsilon_0(\Omega, N)$. It is empty for $m \le 2$, discrete in Ω for $m = 3$, and has Hausdorff dimension $\le m-3$ for $m \ge 4$. The restriction of u to $\Omega \sim \operatorname{Sing} u$ is smooth.

(1.4) [**SU1**, 4.6] A sequence u_i of energy minimizers that is weakly convergent in H^1 to a map u_∞ is actually strongly convergent in H^1.

(1.5) [**SU1**, 4.5,4.6] Any limit point $a \in \Omega$ of a sequence $a_i \in \operatorname{Sing} u_i$ is a singular point of u_∞, and the convergence of u_i to u_∞ is uniform on each compact subset of $\Omega \sim \operatorname{Sing} u_\infty$.

(1.6) If $\operatorname{Sing} v = \{0\}$ for some tangent map v of u at a, then, a is, by 1.2, 1.3, and 1.5, an isolated point of $\operatorname{Sing} u$. Moreover, in this case v is unique [**S1** §8].

(1.7) [**BCL**, 1.2] In case $m = 3$ and $N = \mathbf{S}^2$, a nonconstant tangent map must have the form $\omega\left(\frac{x}{|x|}\right)$ for some orthogonal rotation ω of $\mathbf{R}^3$.

(1.8) [**HKL** §3] In case N is simply connected, there is a constant $D_0 = D_0(m, N)$ so that $r^{2-m} \int_{\mathbf{B}_r(a)} |\nabla u|^2 dx \le D_0$ whenever $\mathbf{B}_r(a) \subset \Omega$.

(1.9) [**HL1**, 6.4] In case N is simply connected, a limit u_∞ of energy minimizers is itself energy minimizing.

2. Energy minimizing maps independent of a variable.

2.1 LEMMA. *Suppose $u \in H^1(\mathbf{B}^m, N)$ and $w(y,z) = u(y)$ for all $(y,z) \in \mathbf{B}^m \times \mathbf{R}$. Then u is energy minimizing if and only if $w|(\mathbf{B}^m \times [0,R])$ is energy minimizing for all $R > 0$.*

PROOF: If u is energy minimizing, and $\tilde{w} \in H^1(\mathbf{B} \times \mathbf{R}, N)$ has trace $\tilde{w}|\partial(\mathbf{B} \times \mathbf{R}) \equiv w|\partial(\mathbf{B} \times \mathbf{R})$, then, for almost all $z \in [0,R]$, the minimality of u implies

$$\int_{\mathbf{B}\times\{z\}} |\nabla \tilde{w}|^2 dy \ge \int_{\mathbf{B}\times\{z\}} |\nabla_{\tan} \tilde{w}|^2 dy \ge \int_{\mathbf{B}\times\{z\}} |\nabla u|^2 dy.$$

Integrating from $y = 0$ to $y = R$ gives

$$\int_{\mathbf{B}\times[0,R]} |\nabla \tilde{w}|^2 dy \ge \int_{\mathbf{B}\times[0,R]} |\nabla w|^2 dy.$$

So w is minimizing.

Conversely, if u is not minimizing, then there exists a $\tilde{u} \in H^1(\mathbf{B}, N)$ and $\eta > 0$ so that $\tilde{u}|\partial\mathbf{B} \equiv u|\partial\mathbf{B}$ and

$$\eta + \int_{\mathbf{B}} |\nabla \tilde{u}|^2 dy < \int_{\mathbf{B}} |\nabla u|^2 dy.$$

Let

$$f(y,z) = \begin{cases} u\left(-\frac{y}{z}\right) & \text{for } -1 \le z \le -|y| \\ u\left(\frac{y}{|y|}\right) & \text{for } |z| < |y| \\ \tilde{u}\left(\frac{y}{z}\right) & \text{for } |y| < z \le 1, \end{cases}$$

and choose $R > 4 + \eta^{-1} 2 \int_{\mathbf{B}\times[-1,1]} |\nabla f|^2 dydz$. With

$$g(y,z) = \begin{cases} f(y, z-1) & \text{for } 0 \le z \le 2 \\ \tilde{u}(y) & \text{for } 2 \le z \le R-2 \\ f(y, -z+R-1) & \text{for } R-2 \le z \le R, \end{cases}$$

we see that $g|\partial(\mathbf{B}\times\mathbf{R}) \equiv w|\partial(\mathbf{B}\times\mathbf{R})$ and

$$\int_{\mathbf{B}\times[0,R]} |\nabla g|^2 dydz = 2 \int_{\mathbf{B}\times[-1,1]} |\nabla f|^2 dydz + (R-4)\int_{\mathbf{B}} |\nabla \tilde{u}|^2 dy$$

$$\le 2 \int_{\mathbf{B}\times[-1,1]} |\nabla f|^2 dydz + (R-4)\left(\int_{\mathbf{B}} |\nabla u|^2 dy - \eta\right)$$

$$< (R-4)\int_{\mathbf{B}} |\nabla u|^2 dy < \int_{\mathbf{B}} |\nabla w|^2 dydz.$$

hence w is not minimizing. ∎

From this Lemma and 1.7, we obtain the

2.2 COROLLARY. *Suppose $w : \mathbf{B}^4 \to \mathbf{S}^2$ is energy minimizing. If w is homogeneous and independent of the last variable (i.e. $w(y,z) = w\left(\frac{y}{|y|}, 0\right)$), then either w is a constant or $w(y,z) = \omega\left(\frac{y}{|y|}\right)$ for some orthogonal rotation ω of $\mathbf{R}^3$.*

2.3 REMARK: Since $\omega\left(\frac{y}{|y|}\right)$ has energy 8π on the unit ball $\mathbf{B}^3$, $w|\mathbf{B}^4 \cap \{z = \pm r\}$ has energy $8\pi\sqrt{1-r^2}$. Inasmuch as $\frac{\partial w}{\partial z} \equiv 0$, Fubini's theorem implies that w has total energy $8\pi \int_{-1}^{1} \sqrt{1-r^2}dr = 4\pi^2$.

3. Homogeneous energy minimizing maps from B^4 to S^2.

3.1 THEOREM. *There exist positive constants D_0, N_0, d_0, C, α and, for $\epsilon > 0$, numbers $\beta = \beta(\epsilon) \in (0, \epsilon]$ and $\gamma = \gamma(\epsilon) \in [8\beta, \frac{d_0}{8}]$ so that any homogeneous energy minimizing map $v : \mathbf{B}_2^4 \to \mathbf{S}^2$ satisfies the following:*

(1) $\int_{\mathbf{B}} |\nabla v|^2 dx \le D_0$,

(2) *$\mathbf{S}^3 \cap \operatorname{Sing} v$ consists of an even number, not exceeding N_0, of points separated by distances at least d_0.*

(3) *For each point $a \in \mathbf{S}^3 \cap \operatorname{Sing} v$, there is the asymptotic estimate*

$$\left| v(x) - \omega_a \left[\frac{p_a(x-a)}{|p_a(x-a)|}\right] \right| \le C|x-a|^\alpha$$

for some orthogonal rotation ω_a of $\mathbf{R}^3$ and orthogonal projection $p_a : \mathbf{R}^4 \to \mathbf{R}^3$ with $p_a(a) = 0$.

(4)

$$r^{-2} \int_{\mathbf{B}_r(b)} |\nabla v|^2 dx < 4\pi^2 + \epsilon$$

whenever $a \in \mathbf{S}^3 \cap \operatorname{Sing} v$, $b \in \mathbf{B}$, $|a - \frac{b}{|b|}| < \beta$, and $r \in (0, 2|b|\gamma]$.

PROOF: The energy bound (1) follows from 1.8 because $\mathbf{S}^2$ is simply connected.

An argument similar to the proof of Lemma 2.1 with a cylindrical domain replaced by a conical domain shows that $v|\mathbf{S}^3$ is locally almost minimizing in the sense that there exists a constant c so that for all $b \in \mathbf{S}^3$ and $\rho > 0$

$$\int_{\mathbf{S}^3 \cap \mathbf{B}_\rho(b)} |\nabla_{\tan} v|^2 d\mathcal{H}^3 \leq c\rho + \int_{\mathbf{S}^3 \cap \mathbf{B}_\rho(b)} |\nabla_{\tan} h|^2 d\mathcal{H}^3$$

whenever $h \in H^1(\mathbf{S}^3 \cap \mathbf{B}_\rho(b), \mathbf{S}^2)$ agrees with v on $\mathbf{S}^3 \cap \partial\mathbf{B}_\rho(b)$. Here C is also independent of v.

For $a \in \mathbf{S}^3 \cap \operatorname{Sing} v$, we find, by the argument of **[SU** §**5]**, 1.9, and Corollary 2.2 that any tangent map of v at a is energy minimizing and in the form $\omega_a \circ \left(\frac{p_a}{|p_a|}\right)$ as above. The argument of **[SU]** or 1.6 implies that there are no other singularities in some $\mathbf{S}^3$ neighborhood of a. For the almost minimizing map $v|\mathbf{S}^3$, the asymptotic theory of **[S]** and **[GW]** now gives the uniqueness of the tangent map and the asymptotic estimate (3). Since ω_a is simply a rotation, the degree of each singularity of $v|\mathbf{S}^3$ is ± 1. Moreover since the total degree of $v|\mathbf{S}^3$ is zero, half of the singularities are of degree $+1$ and half are of degree -1.

The constants C, α above are, in principle, computable (See **[GW]**). On the other hand, to obtain d_0 we resort to a compactness argument. First we note that the almost minimizing property of $v|\mathbf{S}^3$ and the argument of **[HKL** §**3]** or else the minimizing property of v, 1.8, and the elementary observation

$$\mathbf{B}_r(a) \subset \{x : |a - \frac{x}{|x|}| < \sin^{-1} r \text{ and } 1 - r < |x| < 1 + r\} \subset \mathbf{B}_{\sqrt{5r}}(a),$$

leads to the absolute energy density bound

$$\rho^{-1} \int_{\mathbf{S}^3 \cap \mathbf{B}_\rho(a)} |\nabla_{\tan} v|^2 d\mathcal{H}^3 \leq E_0 \qquad \text{for all } \rho > 0.$$

Using this bound, we can now argue as in **[AL]** or **[HL2]** to obtain d_0. That is, we first assume, for contradiction, that there existed no such d_0 and find a sequence of homogeneous energy minimizing maps $v_i : \mathbf{B}^4 \to \mathbf{S}^2$ along with distinct points a_i, b_i in $\mathbf{S}^3 \cap \operatorname{Sing} v_i$ so that the numbers $r_i = \frac{1}{2}|a_i - b_i|$ approach 0 as $i \to \infty$. Rotating $\mathbf{B}^4$, we may assume that each $b_i = (0,1)$. Letting $w_i(x) = w_i((0,1) + r_i x)$ for $x \in \mathbf{B}^3 \times \{0\}$, we infer from the above energy density bound that, after passing to a subsequence, w_i converges weakly in $H^1(\mathbf{B}^3 \times \{0\})$ to a map w. Using the almost minimality of v_i and arguments in the proofs of 1.4 and 1.9, we check that the convergence is actually strong in H^1 and that w is energy minimizing. Moreover applying the monotonicity equality to each v_i as in [**SU1**, 2.5], we also find that w is homogeneous. In particular, $\operatorname{Sing} w \subset \{(0,0)\}$. However, by construction, a singularity of w_i converges, after passing to a subsequence, to a point $a \in \partial \mathbf{B}^3_{\frac{1}{2}}(0) \times \{0\}$, which must be, by 1.5, a singularity of v. This contradiction gives d_0. We readily obtain N_0 as a constant $.d_0^{-3}$.

To prove (4) for a particular v, we use (3) and 2.3 to first choose $\gamma = \gamma(\epsilon) < \frac{1}{4}d_0$ so that

$$\gamma^{-2} \int_{\mathbf{B}_\gamma(a)} |\nabla v|^2 dx < 4\pi^2 + \frac{1}{2}\epsilon$$

whenever $a \in \mathbf{S}^3 \cap \operatorname{Sing} v$ and then choose $\beta < \frac{\gamma}{8}$ so that

$$\gamma^{-2} \int_{\mathbf{B}_\gamma(b)} |\nabla v|^2 dx < 4\pi^2 + \epsilon$$

whenever $|b - a| < \frac{1}{2}\beta$ for some $a \in \mathbf{S}^3 \cap \operatorname{Sing} v$. Inequality (4) then follows by the homogeneity of v and the monotonicity inequality. To see that $\beta(\epsilon)$ and $\gamma(\epsilon)$ can be chosen independent of v, we may use the uniform lower bound d_0 and the compactness of the set of homogeneous minimizers in a contradiction argument as before. ∎

3.2 LEMMA. *(Compare [**A**, 2.26]) Suppose $v : \mathbf{B}^{m+1} \to N$ is homogeneous and energy minimizing. Then $\Theta_v(0,1) \le \Theta_v(0,0)$ with equality if and only if $v(y,z)$ is independent of z.*

PROOF: This follows essentially from the argument of [**A**, 2.26] which, for the reader's convenience, we will repeat here.

The homogeneity of v implies that the energy density ratio

$$r^{1-m} \int_{\mathbf{B}_r(0,0)} |\nabla v|^2 dx = \Theta_v(0,0) \qquad \text{for all } r > 0.$$

From this and the monotonicity inequality (1.1) applied with center $(0,1)$

$$\begin{aligned}\Theta_v(0,1) &\le \rho^{1-m} \int_{\mathbf{B}_\rho(0,1)} |\nabla v|^2 dx \le \sigma^{1-m} \int_{\mathbf{B}_\sigma(0,1)} |\nabla v|^2 dx \\ &\le \lim_{r\to\infty} r^{1-m} \int_{\mathbf{B}_r(0,1)} |\nabla v|^2 dx \le \lim_{r\to\infty} r^{1-m} \int_{\mathbf{B}_{r+1}(0,0)} |\nabla v|^2 dx \\ &= \lim_{r\to\infty} \left(\frac{r}{(r+1)}\right)^{1-m} \Theta_v(0,0) = \Theta_v(0,0)\end{aligned}$$

for $0 < \rho < \sigma$. Here equality would imply, by the monotonicity identity [**HL1**], that v is, homogeneous also about the point $(0,1)$, i.e.

$$v(y,z) = v(ty,\ tz + 1 - t) \qquad \text{for all } t > 0.$$

Then for $z \neq 1$, we could set $t = (1-z)^{-1}$ to find that

$$v(y,z) = v(ty,0) = v(y,0)$$

and also note that $v(y,1) = v(2y,2) = v(2y,0) = v(y,0)$. Thus $v(y,z)$ would be independent of z. ∎

3.3 COROLLARY. *If $v : \mathbb{B}^4 \to \mathbb{S}^2$ is homogeneous and energy minimizing and if $a \in \mathbb{S}^3 \cap \operatorname{Sing} v$, then $4\pi^2 = \Theta_v(a) \leq \Theta_v(0)$ with equality if and only if $v = \omega \circ \left(\frac{p}{|p|}\right)$ for some rotation ω of $\mathbf{R}^3$ and orthogonal projection $p : \mathbf{R}^4 \to \mathbf{R}^3$ with $p(a) = 0$.*

PROOF: We may rotate to have $a = (0,1)$ and apply 3.3 and 2.2. ∎

4. Isolated and nonisolated singularities.

4.1 DEFINITIONS:
$\operatorname{Sing}_0 u = \{b \in \operatorname{Sing} u : \mathbb{S}^3 \cap \operatorname{Sing} v = \emptyset$ for some tangent map v of u at $b\}$,
$\operatorname{Sing}_1 u = \operatorname{Sing} u \sim \operatorname{Sing}_0 u$.

By 1.6 one may change "some" to "every" in the definition of $\operatorname{Sing}_0 u$. Also by 1.1 and 1.5, each point of $\operatorname{Sing}_0 u$ is an isolated point of $\operatorname{Sing} u$. Below in 4.4 we will verify that these isolated points actually have no accumulation point in Ω. First we show how, in a strong sense, the points of $\operatorname{Sing}_1 u$ are not isolated, and, in fact, the singular set of a minimizer approximates at each small scale the singular set of a homogeneous minimizer as considered in 2.1.

4.2 LEMMA. *For every $\epsilon > 0$, there is a positive $\delta = \delta(\epsilon)$ so that if $u : \mathbb{B}^4_2 \to \mathbb{S}^2$ is energy minimizing, $(\overline{\mathbb{B}}_1 \sim \mathbb{B}_{\frac{1}{2}}) \cap \operatorname{Sing} u \neq \emptyset$, $0 \in \operatorname{Sing}_1 u$, and*

$$2^{-2} \int_{\mathbb{B}_2} |\nabla u|^2 dx < 4\pi^2 + \delta,$$

then, for some rotation ω of $\mathbf{R}^3$ and some orthogonal projection $p : \mathbf{R}^4 \to \mathbf{R}^3$,

$$\left\| u - \omega \circ \left(\frac{p}{|p|}\right) \right\|_{H^1(\mathbb{B}^4_{\frac{3}{2}})} < \epsilon,$$
$$\overline{\mathbb{B}} \cap \operatorname{Sing} u \subset \{x : \ \operatorname{dist}(x, p^{-1}\{0\}) < \epsilon\},$$

and, for every $z \in [-1,1]$, $(\mathbb{B}^3_\epsilon \times \{z\}) \cap \operatorname{Sing}_1 u \neq \emptyset$; hence,

$$\overline{\mathbb{B}} \cap p^{-1}\{0\} \subset \{x : \ \operatorname{dist}(x, \operatorname{Sing}_1 u) < \epsilon\}.$$

PROOF: Suppose the theorem is false for some $\epsilon > 0$. Then there is, for each positive integer i, an energy minimizing map $u_i : \mathbf{B}_2^4 \to \mathbf{S}^2$ with a singularity in $\mathbf{B}_1 \sim \mathbf{B}_{\frac{1}{2}}$ that satisfies

$$2^{-2}\int_{\mathbf{B}_2} |\nabla u_i|^2 dx \le 4\pi^2 + i^{-1},$$

but that does not satisfy one of the above three conclusions. From the strong compactness of minimizers 1.4, 1.9, we may, after passing to a subsequence, assume that

(1) $$\|u_i - v\|_{H^1(\mathbf{B}_2^4)} \to 0 \text{ as } i \to \infty$$

for some energy minimizing map $v : \mathbf{B}_2^4 \to \mathbf{S}^2$. By lower semi-continuity of energy,

$$2^{-2}\int_{\mathbf{B}_2} |\nabla v|^2 dx \le \liminf_{i\to\infty} 2^{-2}\int_{\mathbf{B}_2} |\nabla u_i|^2 dx \le 4\pi^2.$$

On the other hand, by monotonicity 1.1, the above strong H^1 convergence, and 3.3,

$$2^{-2}\int_{\mathbf{B}_2^4} |\nabla v|^2 dx \ge \Theta_v(0) = \lim_{r\to 0} r^{-2}\int_{\mathbf{B}_2} |\nabla v|^2 dx$$

$$\ge \limsup_{r\to 0} \lim_{i\to\infty} r^{-2}\int_{\mathbf{B}_2} |\nabla u_i|^2 dx \ge \Theta_{u_i}(0) \ge 4\pi^2.$$

The resulting equality here and 1.1 imply that $v(x) \equiv v\left(\frac{x}{|x|}\right)$. Moreover, $(\overline{\mathbf{B}_1} \sim \mathbf{B}_{\frac{1}{2}} \cap \operatorname{Sing} v \ne \emptyset$ by 1.3. Thus,

(2) $$v = \omega \circ \left(\frac{p}{|p|}\right)$$

for a suitable rotation ω and projection p by Corollary 3.3.

For notational convenience we now assume $p(y,z) = y$. The small energy regularity theorem 1.3 implies that, for i sufficiently large, $u_i|(\overline{\mathbf{B}}_{\frac{3}{2}}^4 \sim (\mathbf{B}_\epsilon^3 \times \mathbf{R}))$ is continuous, in particular,

(3) $$\overline{\mathbf{B}} \cap \operatorname{Sing} u_i \subset \{x : \operatorname{dist}(x, p^{-1}\{0\}) < \epsilon\}.$$

On the set $\overline{\mathbf{B}}_{\frac{3}{2}}^4 \sim (\mathbf{B}_\epsilon^3 \times \mathbf{R})$, the u_i converge, by 1.5, uniformly to $\omega\left(\frac{y}{|y|}\right)$; hence, for all $z \in [-1,1]$ and all i sufficiently large,

$$\deg u_i \mid \partial\mathbf{B}_\epsilon^3 \times \{z\} = \deg v \mid \partial\mathbf{B}_\epsilon^3 \times \{z\} = \pm 1$$

Elementary topology now implies that, each $u_i|(\partial\mathbf{B}^3_\epsilon \times \{z\})$ does not admit any continuous extension to $\mathbf{B}^3_\epsilon \times \{z\}$; in particular,

$$(\mathbf{B}^3_\epsilon \times \{z\}) \cap \operatorname{Sing} u_i \neq \emptyset.$$

To obtain the stronger conclusion that $(\mathbf{B}^3_\epsilon \times \{z\}) \cap \operatorname{Sing}_1 u_i \neq \emptyset$, we observe that, otherwise, $(\mathbf{B}^3_\epsilon \times \{z\}) \cap \operatorname{Sing} u_i$ would be contained in $\operatorname{Sing}_0 u_i$. Then, by 1.6, $(\mathbf{B}^3_\epsilon \times \{z\}) \cap \operatorname{Sing} u_i$ would be a finite set $\{a_1, \ldots, a_j\}$ and, for a small $\sigma > 0$, u_i would be continuous on the topological 3-ball formed from $\mathbf{B}^3_\epsilon \times \{z\}$ by replacing each flat 3-balls $\mathbf{B}^3_\sigma(a_i) \times \{z\}$ by the upper hemisphere $\partial^+\mathbf{B}^4_\sigma(a_i)$. But this would be inconsistent with $u_i|\partial\mathbf{B}^3_\epsilon \times \{z\}$ having degree 1. Thus, for all i sufficiently large and all $z \in [-1, 1]$,

$$(\mathbf{B}^3_\epsilon \times \{z\}) \cap \operatorname{Sing}_1 u_i \neq \emptyset,$$

which along with (1), (2), and (3) gives the desired contradiction. ∎

4.3 THEOREM. *For every $\epsilon > 0$, there is a positive $R = R(\epsilon)$ and, for each energy minimizing map $u : \mathbf{B}^4 \to \mathbf{S}^2$ with $0 \in \operatorname{Sing}_1 u$, a positive even integer $k \leq N_0$, so that to each $r \in (0, R]$ is associated a homogeneous energy minimizing map $v_r : \mathbf{B}^4 \to \mathbf{S}^2$ such that $\mathbf{S}^3 \cap \operatorname{Sing} v_r$ has exactly k points,*

$$\|u(r\cdot) - v_r\|_{H^1(\mathbf{B})} < \epsilon,$$
$$(\overline{\mathbf{B}}_r \sim \mathbf{B}_{\frac{r}{2}}) \cap \operatorname{Sing} u \subset \{x : \operatorname{dist}(x, \operatorname{Sing} v_r) < r\epsilon\},$$

and, for every $a \in \mathbf{S}^3 \cap \operatorname{Sing} v_r$ and $s \in \left[\frac{1}{2}r, r\right]$, $\partial\mathbf{B}_s \cap \mathbf{B}_\epsilon(sa) \cap \operatorname{Sing}_1 u \neq \emptyset$; hence,

$$(\overline{\mathbf{B}}_r \sim \mathbf{B}_{\frac{r}{2}}) \cap \operatorname{Sing} v_r \subset \{x : \operatorname{dist}(x, \operatorname{Sing}_1 u) < r\epsilon\}.$$

PROOF: If we can establish the above inclusions for all sufficiently small r for some such homogeneous minimizer v_r whenever $\epsilon < \frac{d_0}{8}$, then, from Theorem 3.1(2), we would see that card $(\mathbf{S}^3 \cap \operatorname{Sing} v_r)$ is uniquely determined by $\partial\mathbf{B}_r \cap \operatorname{Sing} u$ and is independent of r.

Suppose for contradiction, that there is an energy minimizing map $u : \mathbf{B}^4 \to \mathbf{S}^2$ and a positive $\epsilon < \frac{d_0}{8}$ so that one of the above conclusions fails for some sequence $r_i \to 0$ and any choice of homogeneous minimizers v_{r_i}. Passing to a subsequence, we may by 1.4, assume that

$$\|u_i - v\|_{H^1(\mathbf{B})} \to 0 \text{ where } u_i = u(r_i\cdot) \tag{1}$$

and that v is a homogeneous energy minimizing map from $\mathbf{B}^4_2$ to $\mathbf{S}^2$ with $\mathbf{S}^3 \cap \operatorname{Sing} v = \emptyset$.
From 1.5 and Theorem 3.1 we easily deduce that, for i sufficiently large,

$$(\overline{\mathbf{B}} \sim \mathbf{B}_{\frac{1}{2}}) \cap \operatorname{Sing} u_i \subset \{x : \operatorname{dist}(x, \operatorname{Sing} v) < \epsilon\};$$

hence,

$$\left(\overline{\mathbf{B}}_{r_i} \sim \mathbf{B}_{\frac{r_i}{2}}\right) \cap \operatorname{Sing} u \subset \{x : \operatorname{dist}(x, \operatorname{Sing} v) < r_i\epsilon\}. \tag{2}$$

On the compact set

$$\overline{\mathbf{B}} \sim \mathbf{B}_{\frac{1}{2}} \sim \{x : \operatorname{dist}(x, \operatorname{Sing} v) < \epsilon\},$$

the u_i converge, by 1.5, uniformly to v; hence by Theorem 3.1(3),

$$\deg u_i \mid \partial\mathbf{B}_s \cap \partial\mathbf{B}_{s\beta}(a) = \deg v \mid \partial\mathbf{B}_s \cap \partial\mathbf{B}_{s\beta}(a) = \pm 1$$

for all sufficiently large i, all $s \in \left[\frac{1}{2}, 1\right]$, and all $a \in \mathbf{S}^3 \cap \operatorname{Sing} v$. For all such i, s, and a, we find, as in the proof of 4.2, a point $b_i^a \in \operatorname{Sing} u_i$ with $|b_i^a| = s$ and $|a - \frac{b_i^a}{s}| < \epsilon$. Arguing as before, we may assume that $b_i^a \notin \operatorname{Sing}_0 u_i$, and so

$$\partial\mathbf{B}_s \cap \mathbf{B}_\epsilon(sa) \cap \operatorname{Sing}_1 u_i \neq \emptyset,$$

which, along with (1), (2), and (3) gives the desired contradiction with i sufficiently large and $v_{r_i} = v$. ∎

4.4 COROLLARY. *For any energy minimizing map u from an $\mathbf{R}^4$ domain Ω into $\mathbf{S}^2$, the set $\operatorname{Sing}_0 u$ is a discrete subset of Ω.*

PROOF: If not, then there would, by 4.1, be a sequence $b_i \in \operatorname{Sing}_0 u$ convergent to a point in $\Omega \cap \operatorname{Sing}_1 u$, which for notational convenience, we assume to be the origin. By Theorems 4.3 and 3.1(2), each $\frac{b_i}{|b_i|}$, for i sufficiently large, determines a point a_i, the nearest point, in the finite set $\mathbf{S}^3 \cap \operatorname{Sing} v_{r_i}$ where $r_i = 4|b_i|3$ and the normalized distance $|a_i - \frac{b_i}{|b_i|}|$ approaches 0 as $i \to \infty$. Since $\operatorname{Sing}_1 u$ is, by 1.3 and 1.6, closed, we may find for each i a nearest point c_i in $\operatorname{Sing}_1 u$ to b_i. Since both b_i and c_i are singular points, Theorem 4.3 implies that the normalized distances $r_i^{-1}|b_i - c_i|$ also approaches 0 as $i \to \infty$. Choose and fix $\delta = \delta\left(\frac{1}{10}\right)$ from Lemma 4.2 and observe that, for i sufficiently large,

$$r_i^{-1}|b_i - |b_i|a_i| < \beta(\delta) \text{ and } r_i^{-1}|b_i - c_i| < \gamma(\delta)$$

where $\beta = \beta(\delta)$ and $\gamma = \gamma(\delta)$ are as in Theorem 3.1. For such i, 3.1(4) implies that

$$(r_i\gamma)^{-2} \int_{\mathbf{B}_{r_i\gamma(b_i)}} |\nabla v_{r_i}|^2 dx < 4\pi^2 + \delta.$$

Since $\|u(r_i\cdot) - v_{r_i}\|_{H^1(\mathbf{B}_2)} \to 0$ as $i \to \infty$, we find that, for i sufficiently large,

$$(r_i\gamma)^{-2} \int_{\mathbf{B}_{r_i\gamma(b_i)}} |\nabla u|^2 dx < 4\pi^2 + \delta.$$

We now rescale by defining $w_i(x) = u(s_i x + c_i)$ where $s_i = |b_i - c_i|$. Then $d_i = s_i^{-1}(b_i - c_i) \in \mathbf{S}^3 \cap \operatorname{Sing} w_i$ and, by monotonicity 1.1,

$$4\pi^2 \le \Theta_u(b_i) = \Theta_{w_i}(0) \le 2^{-2} \int_{\mathbf{B}_2^4} |\nabla w_i|^2 dx$$

$$= (2s_i)^{-2} \int_{\mathbf{B}_{2s_i}(b_i)} |\nabla u|^2 dx \le (r_i\gamma)^{-2} \int_{\mathbf{B}_{r_i\gamma}(b_i)} |\nabla u|^2 dx < 4\pi^2 + \delta.$$

Thus we may apply Lemma 4.2 to w_i to infer that

$$\overline{\mathbf{B}} \cap \operatorname{Sing} w_i \subset \{x : \operatorname{dist}(x, p^{-1}\{0\}) < \frac{1}{10}\} \text{ and}$$

$$\overline{\mathbf{B}} \cap p^{-1}\{0\} \subset \{x : \operatorname{dist}(x, \operatorname{Sing}_1 w_i) < \frac{1}{10}\}$$

for some projection $p : \mathbf{R}^4 \to \mathbf{R}^3$. The first inclusion implies that $|p(d_i)| < \frac{1}{10}$. Let e_i be the point on the line $p^{-1}\{0\}$ with $|e_i| = \frac{1}{2}$ and $e_i \cdot d_i > 0$. Then the second inclusion implies that there is a point $x_i \in \mathbf{B}_{\frac{1}{10}}(e_i) \cap \operatorname{Sing}_1 w_i$. But then

$$|x_i - d_i| \le |x_i - e_i| + |e_i - d_i| < .1 + \sqrt{(.5)^2 + (.1)^2} < 1.$$

Scaling back, we see that $s_i x_i + c_i \in \operatorname{Sing}_1 u$ and

$$|b_i - (s_i x_i + c_i) = s_i |x_i - d_i| < s_i = |b_i - c_i|.$$

This contradicts that c_i is the nearest point to b_i in $\operatorname{Sing}_1 u$ and completes the proof. ∎

5. Structure of the singular set.

5.1 THEOREM. *For every $\epsilon > 0$, there is a positive number $\delta_0 = \delta_0(\epsilon)$ so that if $u : \mathbf{B}_2^4 \to \mathbf{S}^2$ is energy minimizing, $\left(\overline{\mathbf{B}}_1 \sim \mathbf{B}_{\frac{1}{2}}\right) \cap \operatorname{Sing} u \ne \emptyset$, and*

$$2^{-2} \int_{\mathbf{B}_2^4} |\nabla u|^2 dx < 4\pi^2 + \delta_0,$$

then,

$$\overline{\mathbf{B}} \cap \operatorname{Sing} u \subset \{x : \operatorname{dist}(x, L) < \epsilon\} \text{ and}$$

$$\overline{\mathbf{B}} \cap L \subset \{x : \operatorname{dist}(x, \operatorname{Sing}_1 u) < \epsilon\}$$

for some line L passing through 0, and

$$\overline{\mathbf{B}}_r(b) \cap \operatorname{Sing} u \subset \{x : \operatorname{dist}(x, L_b^r) < r\epsilon\} \text{ and}$$

$$\overline{\mathbf{B}}_r(b) \cap L_r^b \subset \{x : \operatorname{dist}(x, \operatorname{Sing}_1 u) < r\epsilon\}$$

for each point $b \in \mathbf{B} \cap \operatorname{Sing}_1 u$, $0 < r \le \frac{1}{2}$, and for some line L_r^b passing through b. Moreover,

$$\mathbf{B}_{\frac{1}{2}} \cap \operatorname{Sing}_1 u \subset \Gamma \subset \mathbf{B} \cap \operatorname{Sing}_1 u$$

for some single embedded Hölder continuous arc Γ.

Proof: Recalling 4.2 and 3.3, we can, for $\epsilon > 0$, choose a positive $\beta_0 = \beta_0(\epsilon)$ so that, for any rotation ω and projection p as before,

$$\int_{\mathbf{B}_1(b)} \left| \nabla\omega \circ \left(\frac{p}{|p|}\right) \right|^2 dx < 4\pi^2 + \frac{1}{2}\delta(\epsilon)$$

whenever $\operatorname{dist}(b, p^{-1}\{0\}) < \beta_0$. From Lemma 4.2, we find that, with $\delta_0 = \inf\{\frac{1}{2}\delta(\epsilon), \delta(\beta_0(\epsilon))\}$ we have, for all $b \in \mathbf{B} \cap \operatorname{Sing}_1 u$,

$$\int_{\mathbf{B}_1(b)} |\nabla u|^2 dx < 4\pi^2 + \delta(\epsilon),$$

hence,

$$4\pi^2 \le \Theta_u(b) \le (2r)^{-2} \int_{\mathbf{B}_{2r}(b)} |\nabla u|^2 dx \le \int_{\mathbf{B}_1(b)} |\nabla u|^2 dx < 4\pi^2 + \delta(\epsilon)$$

for any positive $r \le \frac{1}{2}$. We wish to apply Lemma 4.3 to the mapping $w_r^b = u(b + r(\cdot))\,|_{\mathbf{B}_2}$ to obtain a projection p_r^b so that

$$\overline{\mathbf{B}} \cap \operatorname{Sing} w_r^b \subset \{x : \operatorname{dist}(x, (p_r^b)^{-1}\{0\} < \epsilon\}, \text{ and}$$
$$\overline{\mathbf{B}} \cap (p_r^b)^{-1}\{0\} \subset \{x : \operatorname{dist}(x, \operatorname{Sing}_1 u) < \epsilon\},$$

and thus obtain the estimates with $L_r^b = (p_r^b)^{-1}\{0\} + b$. For this, we still need to verify the hypothesis

$$\left(\overline{\mathbf{B}}_1 \sim \mathbf{B}_{\frac{1}{2}}\right) \cap \operatorname{Sing} w_r^b \ne \emptyset, \text{ that is,}$$

(1) $$(\overline{\mathbf{B}}_r(b) \sim \mathbf{B}_{\frac{r}{2}}(b)) \cap \operatorname{Sing} u \ne \emptyset$$

for all positive $r \le \frac{1}{2}$. This is clearly true for $r = \frac{1}{2}$. But then applying 4.3 with $r = \frac{1}{2}$ shows that (1) remains true for $2\epsilon \le r \le \frac{1}{2}$. Applying 4.3 with $r = 2\epsilon$ then gives (1) for $2\epsilon \cdot 4\epsilon \le r \le 2\epsilon$. Continuing, we see that (1) holds true for all $0 < r \le \frac{1}{2}$, and we may indeed apply 4.3 to each w_b^r.

We have now verified precisely the conditions necessary to use Reifenberg's topological disk theorem [**R**], [**M**, 10.5.1]. In the notation of [**M**, 10.5.1] we need only fix ϵ smaller than the ϵ_0 associated with 1 dimensional sets in $\mathbf{R}^4$. We also observe that Reifenberg's argument actually guarantees that the resulting topological disk is Hölder continuous with the Hölder exponent depending on ϵ. ∎

5.2 **Theorem.** *Suppose u is an energy minimizing map of an $\mathbf{R}^4$ domain Ω into $\mathbf{S}^2$. Then each point $b \in \operatorname{Sing} u$ has an open neighborhood N so that $(N \sim \{b\}) \cap \operatorname{Sing} u$ consists of an even number of disjoint embedded Hölder continuous arcs $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ joining $\{b\}$ to a point of ∂N. Moreover, $k \le N_0$, and $\operatorname{dist}(\Gamma_i \cap \partial\mathbf{B}_r(a), \Gamma_j \cap \partial\mathbf{B}_r(a)) \ge \frac{1}{2}d_0 r$ for $1 \le i < j \le k$ and r sufficiently small.*

Proof: In case $b \in \operatorname{Sing}_0 u$, b is an isolated singularity. We now assume that $b \in \operatorname{Sing}_1 u$. After a suitable translation and scaling, we may, using Corollary 4.4, assume that $b = 0$, that $\Omega = \mathbf{B}$, and that $\operatorname{Sing}_0 u = \emptyset$.

First we choose $\delta_0 = \delta_0(2^{-4}d_0)$ as in Lemma 5.1. Then we choose $\beta = \beta\left(\frac{1}{2}\delta_0\right)$ and $\gamma = \gamma\left(\frac{1}{2}\delta_0\right)$ as in Theorem 3.1(4). And finally we choose $R = R(\epsilon)$ as in Theorem 4.3 corresponding to

$$\epsilon = \inf\{2^{-2}\gamma\delta_0,\ 2^{-6}\beta d_0\} \le \inf\{2^{-2}\gamma\delta_0,\ 2^{-8}\gamma d_0,\ 2^{-16}d_0^2\}.$$

For each $r \in (0, R]$ choose an approximating homogeneous minimizer v_r as in Theorem 4.3. For each point $a \in \mathbf{S}^3 \cap \operatorname{Sing} v_r$ there is by 4.3 at least one point $b_r^a \in \partial\mathbf{B}_{\frac{3r}{4}} \cap \mathbf{B}_{r\epsilon}\left(\frac{3a}{4}\right) \cap \operatorname{Sing} u$. From Theorems 4.3 and 3.1 we then obtain the energy estimate

$$(2r\gamma)^{-2} \int_{\mathbf{B}_{2r\gamma}(b_r^a)} |\nabla u|^2 dx \le (2r\gamma)^{-2} \int_{\mathbf{B}_{2r\gamma}(b_r^a)} |\nabla v_r|^2 dx + \frac{1}{2}\delta_0 < 4\pi^2 + \delta_0.$$

We note also that $(\overline{\mathbf{B}}_{r\gamma}(a) \sim \mathbf{B}_{\frac{r\gamma}{2}}(a)) \cap \operatorname{Sing} u \ne \emptyset$ by Theorem 4.3 and the fact that ϵ is chosen suitably smaller than γ. We can now apply Theorem 5.1 to the mapping $u_a^r(x) = u(b_a + r\gamma x) \mid \mathbf{B}_2$ to conclude that

$$\overline{\mathbf{B}}_{\frac{r\gamma}{2}}(b_a) \cap \operatorname{Sing} u \subset \Gamma_r^a \subset \mathbf{B}_{r\gamma}(b_a) \cap \operatorname{Sing} u \subset \{x : \operatorname{dist}(x, L_r^a) < 2^{-4}d_0 r\gamma\}$$

for some embedded Hölder continuous arc Γ_r^a and some line L_r^a passing through b_a. Noting that $|b_a - a| < r\epsilon < 2^{-8}\gamma d_0 r\gamma^2$, we readily see that the direction of L_r^a is very close to radial and that

$$\left(\overline{\mathbf{B}}_{\frac{3}{4}r+\frac{1}{4}r\gamma} \sim \overline{\mathbf{B}}_{\frac{3}{4}r-\frac{1}{4}r\gamma}\right) \cap \{x : \operatorname{dist}(x, \operatorname{Sing} v_r) < \epsilon r\} \subset \bigcup_{a \in \operatorname{Sing} v_r} \overline{\mathbf{B}}_{\frac{r\gamma}{2}}(b_a),$$

hence,

$$(\overline{\mathbf{B}}_s \sim \mathbf{B}_{\lambda s}) \cap \operatorname{Sing} u \subset \bigcup_{a \in \operatorname{Sing} v_r} \Gamma_r^a \subset \operatorname{Sing} u$$

where $s = \frac{3}{4}r + \frac{1}{4}r\gamma$ and $\lambda = 1 - \frac{2\gamma}{(3-\gamma)}$. By Theorem 4.3 each arc Γ_r^a intersects both the outer sphere $\partial\mathbf{B}_s$ and the inner sphere $\partial\mathbf{B}_{\lambda s}$. Reasoning as in the beginning of the proof of Theorem 4.3, we see from the above inclusions that Γ_r^a overlaps $\Gamma_{\lambda r}^{\overline{a}}$ where $\overline{a}$ is the nearest point to a in $\mathbf{S}^3 \cap \operatorname{Sing} v_{\lambda r}$. Moreover, it is clear that $\Gamma_r^a \cup \Gamma_{\lambda r}^{a}$ is also an arc.

Starting with $r = R$, we repeat the above argument with $r = R,\ \lambda R,\ \lambda^2 R, \ldots$. We then obtain the desired arcs $\Gamma_1, G_2, \ldots, \Gamma_k$ by forming, for each $a \in \mathbf{S}^3 \cap \operatorname{Sing} v_R$, a union of a chain of overlapping arcs starting with Γ_R^a. The remaining conclusions follow from Theorem 4.3. ∎

5.3 COROLLARY. *Suppose u is an energy minimizing map of a bounded $\mathbf{R}^4$ domain Ω into $\mathbf{S}^2$ and both $\partial\Omega$ and $u \mid \partial\Omega$ are C^1 smooth. Then $\operatorname{Sing} u$ is the union of a finite set and a finite family of Hölder continuous embedded closed curves with only finitely many crossings.*

PROOF: Combine 4.4, 5.2, and boundary regularity **[SU2]**, **[HL1]**. ∎

6. Additional Remarks on Homogeneous Minimizers..

6.1 THEOREM. *The family $\mathcal{V}_{\rm reg}$ of all homogeneous energy minimizing $v : \mathbf{B}^4 \to \mathbf{S}^2$ such that $v|\mathbf{S}^3$ is smooth is compact in H^1.*

PROOF: By (1.8), (1.4) and (1.9), any sequence in $\mathcal{V}_{\rm reg}$ contains a subsequence v_i that is strongly convergent to a homogeneous energy minimizing map $v : \mathbf{B}^4 \to \mathbf{S}^2$. If $v|\mathbf{S}^3$ did have a singular point a then we could, by 3.1, choose an $\epsilon > 0$ so that $v|\mathbf{S}^3 \cap \partial\mathbf{B}_\epsilon(a)$ is smooth of degree 1. By (1.5) this would also be true for the approximating map v_i. But then elementary topology implies that $v_i|\mathbf{S}^3 \cap \partial\mathbf{B}_\epsilon(a)$ would have a singularity, contradicting that $v_i \in \mathcal{V}_{\rm reg}$. Thus $\operatorname{Sing}(v|\mathbf{S}^3) = \emptyset$ and $v \in \mathcal{V}_{\rm reg}$. ∎

6.2 COROLLARY. *The family $\{v|\mathbf{S}^3 : v \in \mathcal{V}_{\rm reg}\}$ is compact in C^k for all k. In particular,*

$$\sup\{\textit{Hopf invariant } (v|\mathbf{S}^3) : v \in \mathcal{V}_{\rm reg}\} < \infty.$$

6.3 THEOREM. *Suppose g is a smooth metric on $\mathbf{S}^2$, then there exists at least one homogeneous energy minimizing map $v : \mathbf{B}^3 \to (\mathbf{S}^2, g)$ whose restriction to $\mathbf{S}^2$ has degree 1.*

PROOF: Let g_t be a smooth curve of metrics with g_0 being the standard round metric on $\mathbf{S}^2$ and $g_1 = g$. Also let

$$\tau = \sup\{t : \text{there exists a homogeneous energy minimizing map } v : \mathbf{B}^3 \to (\mathbf{S}^2, g) \text{ with } \deg(v|\mathbf{S}^2) = 1\}.$$

Thus $\tau \ge 0$. Suppose $0 \le t_i < \tau$, $t_i \uparrow \tau$, and $v_i : \mathbf{B}^3 \to (\mathbf{S}^2, g_{t_i})$ are homogeneous energy minimizing maps with $\deg(v_i|\mathbf{S}^2) = 1$. Arguing as in [**HKL** §3] we find a universal energy bound and that a subsequence of v_i converges strongly to a homogeneous energy minimizing map $v_\tau : \mathbf{B}^3 \to (\mathbf{S}^2, g_\tau)$. By [**SU1**, Th.2], $v_\tau|\mathbf{S}^3$ is regular and the convergence is uniform away from the origin. Thus $\deg(v_\tau|\mathbf{S}^2) = 1$.

If $\tau < 1$ then we could choose $\tau < s_i < 1$, $s_i \downarrow \tau$, and find, for each i, a (not necessarily homogeneous) energy minimizing map $u_i : \mathbf{B}^3 \to (\mathbf{S}^2, g_{t_i})$ such that $u_i|\mathbf{S}^2 = v_\tau|\mathbf{S}^2$. Then a subsequence of the u_i converge strongly in H^1 to an energy minimizing map $u_\tau : \mathbf{B}^3 \to (\mathbf{S}^2, g_\tau)$ with $u_\tau|\mathbf{S}^2 = v_\tau|\mathbf{S}^2$. Thus they have the same energy on the unit ball,

$$\mathcal{E}(u_\tau, g_\tau) = \mathcal{E}(v_\tau, g_\tau).$$

(In fact $u_\tau \equiv v_\tau$ because a homogeneous energy minimizer is a unique for its boundary data.) Since $\deg(v_\tau|\mathbf{S}^2) \ne 0$ there exist at least one point $a_1 \in \operatorname{Sing} u_i$. Moreover, by 1.5, $a_i \to 0$ as $i \to \infty$. Since $s_i > \tau$ the degree m_i of any tangent map w_i of u_i at a_i has absolute value at least (2). Since the restriction of w_i to $\mathbf{S}^2$ is necessarily conformal, we see that the energy density

$$\Theta_{u_i}(a_i) = 2|m_i| \operatorname{Area}(\mathbf{S}^2, g_{s_i}).$$

Similarly $\Theta_{v_\tau}(0) = 2\,\text{Area}(\mathbf{S}^2, g_\tau)$ because $\deg(v_\tau|\mathbf{S}^2) = 1$. Monotonicity (1.1) implies now the contradiction

$$\begin{aligned}\mathcal{E}(u_\tau, g_\tau) &= \lim_{i\to\infty} \mathcal{E}(u_i, g_i) \geq \liminf_{i\to\infty} \Theta_{u_i}(a_i)\\ &= \liminf_{i\to\infty} 2|m_i|\,\text{Area}(\mathbf{S}^2, g_{s_i}) \geq 4\,\text{Area}(\mathbf{S}^2, g_\tau) = 2\mathcal{E}(u_\tau, g_\tau).\end{aligned}$$

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