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The Tangent Surface of a Rational Algebraic Space Curve

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Introduction

Let C be a rational algebraic curve in projective 3-space over the complex numbers and let $\varphi : \mathbb{P}^1 \to C \subset \mathbb{P}^3$ be its normalization. Assume C is not contained in a plane.

The union of the tangent lines to the points of C is called the *embedded* tangent surface $T_C \subset \mathbb{P}^3$. Its normalization $\tilde{T}_C$ is a geometrically ruled surface on $\mathbb{P}^1$ (cf. [2], prop. 3), hence the projectivization of a rank 2 bundle $\mathcal{E}$. We call $\tilde{T}_C \cong \mathbb{P}(\mathcal{E})$ the *abstract* tangent surface of C.

Thus two rank 2 bundles on $\mathbb{P}^1$ are associated with C: the normal bundle $\mathcal{N}$ of C in $\mathbb{P}^3$ and $\mathcal{E}$. By a theorem of Grothendieck both are direct sums of line bundles $\mathcal{N} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ and $\mathcal{E} = \mathcal{O}(c) \oplus \mathcal{O}(d)$. If C is a smooth curve of degree n then $a + b = 4n - 2$. Hence it suffices to determine $|a - b|$. This is done in [2]...[5] where also the variety of all smooth curves of degree n with fixed a and b is investigated. The geometrical meaning of $|a - b|$ remains open.

Our note deals with the calculation of c and d. The sum $c + d$ is not an invariant of the tangent surface; but the difference $e := |c - d|$ determines $\tilde{T}_C$ up to isomorphism since $\tilde{T}_C$ is isomorphic to a *Hirzebruch Sigma surface*

$$\Sigma_e := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e)).$$

We prove that e vanishes for smooth curves, but not necessarily for cuspidal curves. It can be computed from the normalization map φ (prop. 2). The values which occur for curves of degree n and fixed number of cusps are determined in proposition 1 and 3.

Proposition 6 gives an idea of a geometrical meaning of the invariant e: Each hyperplane $H \subset \mathbb{P}^3$ cuts out a section s of the tangent surface. The self intersection number of s is determined by the number of points (counted without muliplicity) in $H \cap C$. As a consequence we get that e is large when there is a hyperplane in $\mathbb{P}^3$ that meets C in few points.

Finally we give examples that e is not determined by the numerical invariants of the curve. All results can easily be extended to curves in $\mathbb{P}^n$.

The abstract tangent surface is constructed via the Gauss map $\Phi : \mathbb{P}^1 \to G_2^4$ which maps $p \in \mathbb{P}^1$ to the tangent line to C in $\varphi(p)$. Φ corresponds to a surjective morphism of sheaves on $\mathbb{P}^1$

$$\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \to \mathcal{E} \to 0 \tag{1}$$

where $\mathcal{E}$ is the pullback of the universal (rank 2) bundle on G_2^4 and deg $\mathcal{E}$ = deg T_C. Equivalently (1) can be viewed as a mapping from the ruled surface $\mathbb{P}(\mathcal{E})$ to $\mathbb{P}^3$ that maps each fibre to a line in $\mathbb{P}^3$ (cf. lemma V.2.4 in [7]). So $\mathbb{P}(\mathcal{E})$ is the abstract tangent surface and we have to compute the splitting index of $\mathcal{E}$.

Moreover morphism (1) gives the link to the normal bundle since by formula IV.18 in [8] the kernel of (1) is isomorphic to $\mathcal{N}^\vee \otimes \varphi^*(\mathcal{O}(1))$. Especially

$$\deg \mathcal{N} = \deg T_C + 2n.$$

Some terminology: For $p \in \mathbb{P}^1$ the set

$$R_p = \{\mathrm{ord}_p \varphi^* s : s \in H^0(\mathbb{P}^3, \mathcal{O}(1))\}$$

contains four integers. For $0 < i \leq 3$ the *numerical invariant* α_{ip} is defined by

$$R_p = \{0, 1 + \alpha_{1p}, 2 + \alpha_{1p} + \alpha_{2p}, 3 + \alpha_{1p} + \alpha_{2p} + \alpha_{3p}\}.$$

The points $p \in \mathbb{P}^1$ with $\alpha_{1p} > 0$ are the ramification points of φ and their images in $\mathbb{P}^3$ are called the *cusps* of C. Remember the Plücker formulas (cf. [9] and [1]):

$$\deg T_C = 2n - 2 - \sum_{p \in \mathbb{P}^1} \alpha_{1p} \tag{2}$$

$$4 \deg C = 3 \sum \alpha_{1p} + 2 \sum \alpha_{2p} + \sum \alpha_{3p} - 12 \tag{3}$$

Cuspidal Rational Curves

Assume that C is of degree n. Then $\varphi^*\mathcal{O}(1) = \mathcal{O}(n)$ and for $0 \leq i \leq 3$

$$s_i := \varphi^* X_i \in H^0(\mathbb{P}^1, \mathcal{O}(n))$$

is a homogeneous polynomial of degree n in two variables (say T_0 and T_1). We may form the partial derivative

$$\partial s_i / \partial T_j \in H^0(\mathbb{P}^1, \mathcal{O}(n-1)).$$

To construct the Gauss map Φ consider the morphism

$$b : \mathcal{O} \oplus \ldots \oplus \mathcal{O} \to \mathcal{O}(n-1) \oplus \mathcal{O}(n-1)$$

given by the matrix

$$M = \begin{pmatrix} \partial s_0/\partial T_0 & \ldots & \partial s_3/\partial T_0 \\ \partial s_0/\partial T_1 & \ldots & \partial s_3/\partial T_1 \end{pmatrix}.$$

Using the Euler relation

$$ns = T_0 \partial s/\partial T_0 + T_1 \partial s/\partial T_1$$

we see that the first numerical invariants of φ appear in the following way:

$$\alpha_{1p} = \min\{\mathrm{ord}_p |M_{ij}| : 0 \le i < j \le 3\} \tag{4}$$

where the M_{ij} denote the 2 by 2 minors of M. In the unramified points of φ the morphism b is surjective and defines a map from $\{p \in \mathbb{P}^1 : \alpha_{1p} = 0\}$ to G_2^4 which in fact parametrizes the tangents to C. The Gauss map Φ is the unique extension of this map to all points of $\mathbb{P}^1$.

Let $\mathcal{E} := \mathrm{Im}(b)$. Then $\mathcal{E}$ is a coherent subsheaf of $\mathcal{O}(n-1) \oplus \mathcal{O}(n-1)$ and thereby is locally free of rank two either. So the morphism

$$b : \mathcal{O} \oplus \ldots \oplus \mathcal{O} \to \mathcal{E} \to 0$$

describes Φ and we get the abstract tangent surface as

$$\tilde{T}_C = \mathbb{P}(\mathcal{E}).$$

The degree of $\mathcal{E}$ is equal to the degree of the embedded tangent surface as calculated in the Plücker formula (2). As an immediate consequence we get:

Proposition 1 *Let $\tilde{T}_C = \Sigma_e$ be the abstract tangent surface of a (possibly singular) rational curve C in $\mathbb{P}^3$. Then*

$$0 \le e \le \sum_{p \in \mathbb{P}^1} \alpha_{1p} \text{ and } e \equiv \sum_{p \in \mathbb{P}^1} \alpha_{1p} \bmod 2$$

If C has no cusp then $e = 0$.

If the four homogeneous polynomials s_i are known the invariant can be calculated by linear algebra:

Proposition 2 *Let $W \subset \mathbb{P}^1$ be the set of ramification points of φ. Let $s := \varphi^* H \in H^0(\mathbb{P}^1, \mathcal{O}(n))$ for a hyperplane $H \subset \mathbb{P}^3$ that contains no cusp of C.*

(i) For $m \in \mathbf{N}$

$$H^0(\mathbb{P}^1, \mathcal{E}(-m)) \cong \{(f,g) \in H^0(\mathbb{P}^1, \mathcal{O}(n-m-1) \oplus \mathcal{O}(n-m-1)) : \mathrm{ord}_p(-f\frac{\partial s}{\partial T_1} + g\frac{\partial s}{\partial T_0}) \ge \alpha_{1p} \text{ for all } p \in W\}$$

(ii) Let $k \in \mathbf{N}$ be the smallest number such that there exist homogeneous polynomials f and g of degree k and

$$\mathrm{ord}_p(-f\frac{\partial s}{\partial T_1} + g\frac{\partial s}{\partial T_0}) \ge \alpha_{1p} \tag{5}$$

for all $p \in W$. Then

$$\mathcal{E} = \mathcal{O}(n-1+k-\sum_{p \in W} \alpha_{1p}) \oplus \mathcal{O}(n-1-k)$$

and

$$e = \sum_{p \in W} \alpha_{1p} - 2k$$

Proof: For simplicity assume $H = \{X_0 = 0\}$, $s = s_0$. Consider the diagram

$$\begin{array}{ccc} \mathcal{O} \oplus \ldots \oplus \mathcal{O} & \stackrel{M}{\rightarrow} & \mathcal{O}(n-1) \oplus \mathcal{O}(n-1) \\ & \searrow & \downarrow E \\ & & \mathcal{O}(n) \oplus \mathcal{O}(2n-2) \end{array}$$

where E is defined by the matrix

$$E = \begin{pmatrix} T_0 & T_1 \\ -\partial s/\partial T_1 & \partial s/\partial T_0 \end{pmatrix}$$

Let $U := \{p \in \mathbb{P}^1 : s(p) \neq 0\}$. Then $E_{|U}$ is an isomorphism since for all $p \in U$

$$\det E(p) = T_1 \partial s/\partial T_1 + T_0 \partial s/\partial T_0 = ns(p) \neq 0$$

So we have to determine the image of $E \circ M$:

$$EM = \begin{pmatrix} ns_0 & ns_1 & ns_2 & ns_3 \\ 0 & |M_{01}| & |M_{02}| & |M_{03}| \end{pmatrix}$$

By equation (4) we know that the divisor of

$$f := \gcd\{|M_{0i}| : 0 < i \leq 3\}$$

is

$$D(f) = \sum_{p \in W} \alpha_{1p} p$$

So it is obvious that for $V \subset \mathbb{P}^1$ and $(g, h) \in (\mathcal{O}(n) \oplus \mathcal{O}(2n-2))(V)$

$$(g, h) \in \operatorname{Im}(E \circ M)$$

if and only if for all $p \in W$

$$\operatorname{ord}_p h \geq \alpha_{1p}.$$

Since $E_{|U}$ is an isomorphism and b is surjective outside of $W \subset U$ this proves (i). Part (ii) is a direct consequence of (i). □

The number k in part two of the proposition can be found by simple linear algebra if the ramification points of φ are known. Thus we can compute the invariant of the abstract tangent surface and we may ask which values of e occur. The general Plücker formula (3) implies

$$\sum_{p \in W} \alpha_{1p} \leq \frac{4}{3}(n-3)$$

and as proved in [1] this bound is sharp. Nevertheless the invariant e cannot take all values:

Proposition 3 *(i) If Σ_e is the abstract tangent surface of a rational curve C in $\mathbb{P}^3$ not contained in a plane. Then*

$$e \leq \min(\sum_{p \in W} \alpha_{1p}, n-3)$$

(ii) *If $0 \leq e \leq k \leq n-3$ and $e \equiv k \bmod 2$ then there is a rational curve C of degree n in $\mathbb{P}^3$ such that*

$$\sum_{p \in W} \alpha_{1p} = k \text{ and } \tilde{T}_C \cong \Sigma_e$$

Proof: $\mathcal{E}$ is a subbundle of $\mathcal{O}(n-1) \oplus \mathcal{O}(n-1)$ and therefore $e \leq n-1$.

Case $e = n-1$: Then $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(n-1)$. The projection

$$\pi : \mathcal{O} \oplus \mathcal{O}(n-1) \to \mathcal{O} \to 0$$

defines a section of $\tilde{T}_C$ (cf. [7] proposition V.2.6.) that is mapped to a curve of degree 0, i.e. a point on the embedded tangent surface. So all tangents of C meet a common point which contradicts the finiteness of the normalisation map.

Case $e = n-2$: Then $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(n-2)$ or $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(n-1)$ The first case is excluded by the same argument as above. The second case means that the section of $\tilde{T}_C$ that corresponds to the projection to $\mathcal{O}(1)$ is mapped to a curve of degree 1, i.e. all tangents of C meet a common line. But then the dual curve C^* is contained in a plane and $C = C^{**}$ is a plane curve either. This proves (i).

All possible values of e can be generated by curves that have only one cusp. The following example proves part (ii). □

Example 4 *Let $0 \leq e \leq k \leq n-3$ and $e \equiv k \bmod 2$; $m := (k-e)/2$. Let the curve $\varphi : \mathbb{P}^1 \to C \subset \mathbb{P}^3$ be defined by*

$$\varphi(t_0 : t_1) = (t_0^n + t_0^{n-m} t_1^m : t_0^{n-k-1} t_1^{k+1} : t_0^1 t_1^{n-1} : t_1^n)$$

Then $\alpha_{1(1:0)} = k$, $\alpha_{1p} = 0$ for all other points and

$$\tilde{T}_C \cong \Sigma_e$$

Proof: We just calculate the tangent surface:

$s := T_0^n + T_0^{n-m} T_1^m$ suffices the assumptions of proposition 2. So we have to find homgeneous polynomials f, g of minimal degree such that

$$\text{ord}_{(1:0)}(-f(nT_0^{n-1} + (n-m)T_0^{n-m-1}T_1^m) + gmT_0^{n-m}T_1^{m-1}) \geq k.$$

Since $2m \leq k$ the degrees of f and g are minimal if we combine the zero polynomial taking $f = t^{m-1}$ and $g = n + (n-m)t^m$. So the invariant is $k - 2m = e$ as predicted. □

The polynomials f and g we chose above not only satisfied equation (5) but even

$$-f\frac{\partial s}{\partial T_1} + g\frac{\partial s}{\partial T_0} = 0 \tag{6}$$

In general polynomials f and g of minimal degree that fulfill this condition are

$$f = \frac{\partial s}{\partial T_0}\frac{1}{h} \text{ and } g = \frac{\partial s}{\partial T_1}\frac{1}{h} \tag{7}$$

where $h = \gcd(\partial s/\partial T_0, \partial s/\partial T_1)$ and $\deg f = \deg g = n - 1 - \deg h$. We may interpret the number $n - \deg h$ as the number of pairwise distinct zeros of s or geometrically as the intersection points of the curve C and the hyperplane $H \subset \mathbb{P}^3$ corresponding to s counted without multipicities (but counting the branches of C at each intersection point) since we have

Remark 5 *For each homogeneous polynomial $s \in \mathbb{C}[T_0, T_1], p \in \mathbb{P}^1$ and $k > 1$*

$$ord_p s = k + 1 \Leftrightarrow \min(ord_p \frac{\partial s}{\partial T_0}, ord_p \frac{\partial s}{\partial T_1}) = k$$

For a hyperplane $H \subset \mathbb{P}^3$ let $\sharp(H \cap \mathbb{P}^1)$ denote the number of pairwise distinct zeros of the hyperplane section $s = \varphi^* H$. We get the following bound on e:

Proposition 6 *Let*

$$k := \min\{\sharp(H \cap \mathbb{P}^1) : H \subset \mathbb{P}^3 \text{ hyperplane that contains no cusp of } C\}.$$

Then

$$\begin{aligned} 2k - 2 - \sum_{p \in W} \alpha_{1p} > 0 \quad &\Rightarrow \quad e \le 2k - 2 - \sum_{p \in W} \alpha_{1p} \\ 2k - 2 - \sum_{p \in W} \alpha_{1p} \le 0 \quad &\Rightarrow \quad e = -2k + 2 + \sum_{p \in W} \alpha_{1p} \end{aligned}$$

Proof: Assume that $\mathcal{E} = \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a \le b$ and $a + b = 2n - 2 - \sum \alpha_{1p}$. Choose a hyperplane H such that $\sharp(H \cap \mathbb{P}^1) = k$ and $s := \varphi^* H$. Let f, g and h be as in formula (7). Then f and g satisfy (6) and are of degree $k - 1$. Hence $H^0(\mathbb{P}^1, \mathcal{E} \otimes \mathcal{O}(k - n)) \neq 0$. We get $b \ge n - k$ and

$$e = 2b - (a + b) \ge -2k + 2 + \sum \alpha_{1p}$$

To complete the proof consider the surjective morphism

$$b_1 : \mathcal{O}(n-1) \oplus \mathcal{O}(n-1) \to \mathcal{O}(2n - 2 - \deg h) \to 0$$

defined by the matrix $(-\frac{1}{h}\partial s/\partial T_1, \frac{1}{h}\partial s/\partial T_0)$. Since $\mathcal{E} = \mathcal{O}(a) \oplus \mathcal{O}(b)$ the morphism $b_2 := b_{1|\mathcal{E}}$ is also described by homogeneous polynomials (u, v). Because $h(p) \neq 0$ for all $p \in W$ proposition 2 implies

$$w := \gcd(u, v) = \prod_{p \in W} (p_1 t_0 - p_0 t_1)^{\alpha_{1p}}.$$

So factoring out w we get

$$b_3 : \mathcal{E} \to \mathcal{O}(2n - 2 - \deg h - \sum \alpha_{1p}) \to 0$$

But b_3 can only be surjective if either

$$b \le 2n - 2 - \sum \alpha_{1p} - \deg h$$

or

$$a = 2n - 2 - \sum \alpha_{1p} - \deg h.$$

□

When starting the work on this paper we thought that e might be determined by the distribution of the numerical invariants. So we conclude with two example curves having the same invariants but different tangent surfaces.

Example 7 *Consider the rational curves:* $\varphi_1 : \mathbb{P}^1 \to C_1 \subset \mathbb{P}^3$ *and* $\varphi_2 : \mathbb{P}^1 \to C_2 \subset \mathbb{P}^3$ *defined by*

$$\begin{aligned} \varphi_1(t_0 : t_1) &= (t_0^6 + 6t_0^5 t_1 + 15t_0^4 t_1^2 + t_0 t_1^5 : t_0^3 t_1^3 : t_0^2 t_1^4 : t_1^6) \\ \varphi_2(t_0 : t_1) &= (t_0^3 + 6t_0^5 t_1 + t_0 t_1^5 : t_0^3 t_1^3 : t_0^2 t_1^4 + t_0 t_1^5 : t_1^6) \end{aligned}$$

Then for all $p \in \mathbb{P}^1$ *and* $0 < i \leq 3$

$$\alpha_{ip}(\varphi_1) = \alpha_{ip}(\varphi_2)$$

but $\tilde{T}_{C_1} \cong \Sigma_2$ *and* $\tilde{T}_{C_2} \cong \Sigma_0$.

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