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Estimates for the Eigenvalues of Hill's Equation and Applications for the Eigenvalues of the Laplacian on Toroidal Surfaces

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In this paper we give upper and lower bounds for each eigenvalue λ_n of Hill's differential equation. We apply the results to toroidal surfaces of revolution in order to get estimates for the eigenvalues of the Laplacian in terms of curvature expressions; they are sharp for the flat torus. As an example, we investigate the standard torus in $\mathbb{R}^3$; here, the bounds depend on the radii only.

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1 Introduction and Results

The problem of the eigenvalues of the second-order differential equation with periodic coefficients

$$(py')' + (\lambda s - q)\,y = 0\,, \tag{1}$$

called Hill's Equation (1877), has been extensively investigated for a long time. We came across this problem in a particular situation, namely while trying to find geometrically meaningful bounds for the eigenvalues of the standard circular torus. Eventually, we realized that our results could be formulated for Hill's Equation in general.

In (1), we suppose that $\lambda \in \mathbb{R}$ and that p, q, s are real continuous periodic functions, all with the same period L (> 0); moreover we assume that p and s are positive and that p' is continuous. For the general theory see [CL], [E], [MW]. Of special interest are solutions y of (1) satisfying the periodic boundary conditions

$$y(0) = y(L), \quad y'(0) = y'(L) \tag{2}$$

(*periodic eigenvalue problem*). The parameters λ with periodic solutions are the *eigenvalues* of the differential equation, and the corresponding solutions the *eigenfunctions*.

The eigenvalues form an infinite sequence of the type

$$\lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \ldots < \lambda_{2n-1} \le \lambda_{2n} < \ldots \tag{3}$$

([E], S. 27; here, we do not include the semi–periodic problem).

In certain transformations of equation (1), there appear terms which may be expressed by the so-called "Schwarz' Differential" $\mathcal{S}(f)$ of a function f; for its definition see 2.3.

The main results of this paper are two theorems which give upper and lower bounds for the eigenvalues of the periodic eigenvalue problem (1); the proofs are obtained (in section 2) in a surprisingly simple way from two classical theorems, namely Sturm's Comparison Theorem and a theorem of O. Haupt (1914) on the zeros of the eigenfunctions of (1).

1.1 Theorem: *Assume that the function p in (1) is twice differentiable, and let P_* be a primitive of $1/p$. By means of the coefficient functions in* (1), *we define the "estimating function"*

$$\mathcal{B}_n := \frac{p}{s}\left(\left(\frac{2\pi}{L}\right)^2 n^2 + \frac{q}{p} - \frac{1}{2}\mathcal{S}(P_*)\right).$$

Then we have the following bounds for the eigenvalues of (1) *(where for abbreviation* $\lambda_{-1} := \lambda_0$*):*

$$\min_{x\in[0,L]} \mathcal{B}_n(x) \le \lambda_{2n-1} \le \lambda_{2n} \le \max_{x\in[0,L]} \mathcal{B}_n(x).$$

Periodic re-parametrization of the differential equation does not change the eigenvalues but the boundary function $\mathcal{B}_n$ and, therefore, the upper and lower bounds for the eigenvalues. This observation leads to

1.2 Theorem: *Let the assumptions of Theorem 1.1 hold. Let* $\mathcal{G} := \{g \in C^2 \mid g > 0,\ g$ *periodic with period* $L\}$, $M_g := \frac{1}{L}\int_0^L g(t)dt$ *for* $g \in \mathcal{G}$ *and let* G *be a primitive of* g. *We define the extended estimating function*

$$\mathcal{B}_n(g) := \frac{p}{s}\left(\left(\frac{2\pi}{L}\right)^2 n^2 \left(\frac{g}{M_g}\right)^2 + \frac{q}{p} - \frac{1}{2}\mathcal{S}(P_*) + \frac{1}{2}\mathcal{S}(G)\right). \tag{3}$$

Then

$$\sup_{g\in\mathcal{G}} \min_{x\in[0,L]} \mathcal{B}_n(g)(x) \le \lambda_{2n-1} \le \lambda_{2n} \le \inf_{g\in\mathcal{G}} \max_{x\in[0,L]} \mathcal{B}_n(g)(x), \tag{i}$$

and for the first eigenvalue we get the equation

$$\sup_{g\in\mathcal{G}} \min_{x\in[0,L]} \mathcal{B}_0(g)(x) = \lambda_0 = \inf_{g\in\mathcal{G}} \max_{x\in[0,L]} \mathcal{B}_0(g)(x). \tag{ii}$$

With the special choice $g = \sqrt{s/p}$, we get the estimating function

$$\mathcal{B}_n(\sqrt{s/p}) = \left(\frac{2\pi}{L\,M_g}\right)^2 n^2 + \frac{q}{s} + \frac{1}{4}\frac{p}{s}\left(\left(\frac{(ps)'}{ps}\right)' - \frac{1}{4}\left(\frac{(ps)'}{ps}\right)^2 + \frac{(ps)'}{ps}\frac{p'}{p}\right),$$

which can be used to obtain explicit bounds for the spectral gaps, which are of the size expected by Weyl's asymptotic formula:

1.3 Corollary: *The gaps between two successive eigenvalues are bounded by*

$$\lambda_{2n} - \lambda_{2n-1} \le \max_{x\in[0,L]} \mathcal{B}_0\left(\sqrt{s/p}\right)(x) - \min_{x\in[0,L]} \mathcal{B}_0\left(\sqrt{s/p}\right)(x) =: \mathcal{A}\left(\frac{q}{s}, \sqrt{s/p}\right),$$

and

$$\lambda_{2n+1} - \lambda_{2n} \le \left(\frac{2\pi}{L\,M_g}\right)^2 (2n+1) + \mathcal{A}\left(\frac{q}{s}, \sqrt{s/p}\right).$$

Note the significantly different behavior of the two bounds for large n.

As a special case of 1.3 we obtain a sufficient condition for the coexistence of two linearly independent periodic solutions (*coexistence problem*, cf. [MW], p. 90ff):

1.4 Corollary: *For the class of differential equations* (1) *with* $ps = 1$ *and* $q/s =$ const. $= b \in \mathbb{R}^3$, *all eigenvalues (for* $n \ge 1$*) are double eigenvalues:*

$$\lambda_{2n-1} = \lambda_{2n} \equiv \mathcal{B}_n(s) = (2\pi)^2 n^2 \left(\int_0^L s(t)dt\right)^{-2} + b.$$

1.5 Remark: For special choices of g, the estimating function becomes particularly simple, e.g. with $g = 1/p$ or $g = p$, we obtain in 1.2 the terms: $\mathcal{S}(P_*) = \mathcal{S}(G)$ or $-\frac{1}{2}\mathcal{S}(P_*) + \frac{1}{2}\mathcal{S}(G) = \left(\frac{p'}{p}\right)'$, respectively.

Which choice of g is especially convenient depends on the coefficient functions and on the number of the eigenvalue under consideration. With $g = 1/pw^2$ for $w \in \mathcal{G}$ the left equality of Theorem 1.2 (ii) yields the same expression for λ_0 as is obtained in [PW], p. 38, by the maximum principle, cf. the proof of Theorem 1.2.

In §3, these results are applied to the eigenvalues of the Laplacian on surfaces of revolution homeomorphic to the torus (cf. [Bk 1], §7), and in this way we obtain estimates containing curvature expressions of the surface. In order to be able to formulate the result 1.6, we briefly sketch the necessary notions. By the action of the rotation group $O(2)$ on the surface, the eigenspace E_λ of an eigenvalue λ splits up into the isotypical components E_λ^k with respect to the irreducible representations of $O(2)$, which are characterized by their winding number k. The eigenvalues λ with $E_\lambda^k \neq \{0\}$ will be called *k-eigenvalues*. The search for eigenfunctions in E_λ^k leads to a differential equation of Hill type with eigenvalue λ.

The metric of such surfaces of revolution is given by $(\mathbf{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r(t)^2 \end{pmatrix}$, with a L-periodic function r (cf. §3).

1.6 Theorem: *On a surface of revolution homeomorphic to the torus the following estimates for the k-eigenvalues λ_n^k of the Laplacian are valid:*

$$\sup_{g\in\mathcal{G}} \min_{x\in[0,L]} \mathcal{B}_n^k(g)(x) \leq \lambda_{2n-1}^k \leq \lambda_{2n}^k \leq \inf_{g\in\mathcal{G}} \max_{x\in[0,L]} \mathcal{B}_n^k(g)(x)$$

with

$$\mathcal{B}_n^k(g) := \left(\frac{2\pi}{L}\right)^2 n^2 \left(\frac{g}{M_g}\right)^2 + \frac{k^2}{r^2} - \frac{1}{2}K - \frac{1}{4}B^2 + \frac{1}{2}\mathcal{S}(G),$$

where K denotes the Gauss curvature and B the geodesic curvature of the parallels. The bounds are sharp for the flat torus and $g = 1$.

For concrete estimates, one may try different choices of g, adapted to the metric of the surface. For instance, the choice $g = r^\alpha$ with $\alpha \in \mathbb{R}$ yields

$$\mathcal{B}_n^k(r^\alpha) = \left(\frac{2\pi}{L}\right)^2 n^2 \left(\frac{r^\alpha}{M_{r^\alpha}}\right)^2 + \frac{k^2}{r^2} - \beta K - \beta^2 B^2$$

with $\beta = (\alpha + 1)/2$, cf. 3.3. This shows quite well the asymptotic behavior of the eigenvalues on toroidal surfaces:

$$\lambda_{2n-1}^k, \lambda_{2n}^k \sim \mathcal{O}(n^2) + \mathcal{O}(k^2).$$

In §4, we consider the imbedded standard torus T (with radius S and meridian radius R) in $\mathbb{R}^3$ and we derive some estimates for its eigenvalues. For the first eigenvalue of each k-spectrum ($k \geq 0, n = 0$) we obtain the bounds

$$\frac{k^2}{(S+R)^2} \leq \lambda_0^k \leq \frac{k^2}{S^2 - R^2}$$

(see 4.1 and 4.2), and for the higher eigenvalues ($n \geq 0$) we get the estimates (cf. 4.3)

$$\frac{n^2}{R^2} + \frac{k^2}{(S+R)^2} - \frac{1}{2}\frac{1}{R(S+R)} \leq \lambda_{2n-1}^k \leq \lambda_{2n}^k \leq \frac{n^2}{R^2} + \frac{k^2}{(S-R)^2} + \frac{1}{2}\frac{1}{R(S-R)} .$$

These bounds show the asymptotic behaviour of the eigenvalues as $R \to 0$, i.e. if the torus degenerates to a circle (with fixed radius S): then

$$\lim_{R\to 0} \lambda_0^k = \frac{k^2}{S^2} , \qquad \text{and for } \nu \geq 1 : \quad \lim_{R\to 0} \lambda_\nu^k = \infty .$$

The first eigenvalue of each k-spectrum remains finite and tends to the corresponding eigenvalue of the circle of radius S; all the other eigenvalues tend to ∞.

Remark: In [Bk 1], 16.4, 5, we investigated the eigenvalues of surfaces consisting in a cylindrical tube with half-spheres on both ends. Here, only the invariant eigenvalues λ_n^0 remain finite (for $R \to 0$, the length H of the cylinder staying fixed) and tend to the eigenvalues of a vibrating string with fixed ends of length H, whereas all k-eigenvalues ($k \geq 1$) tend to infinity.

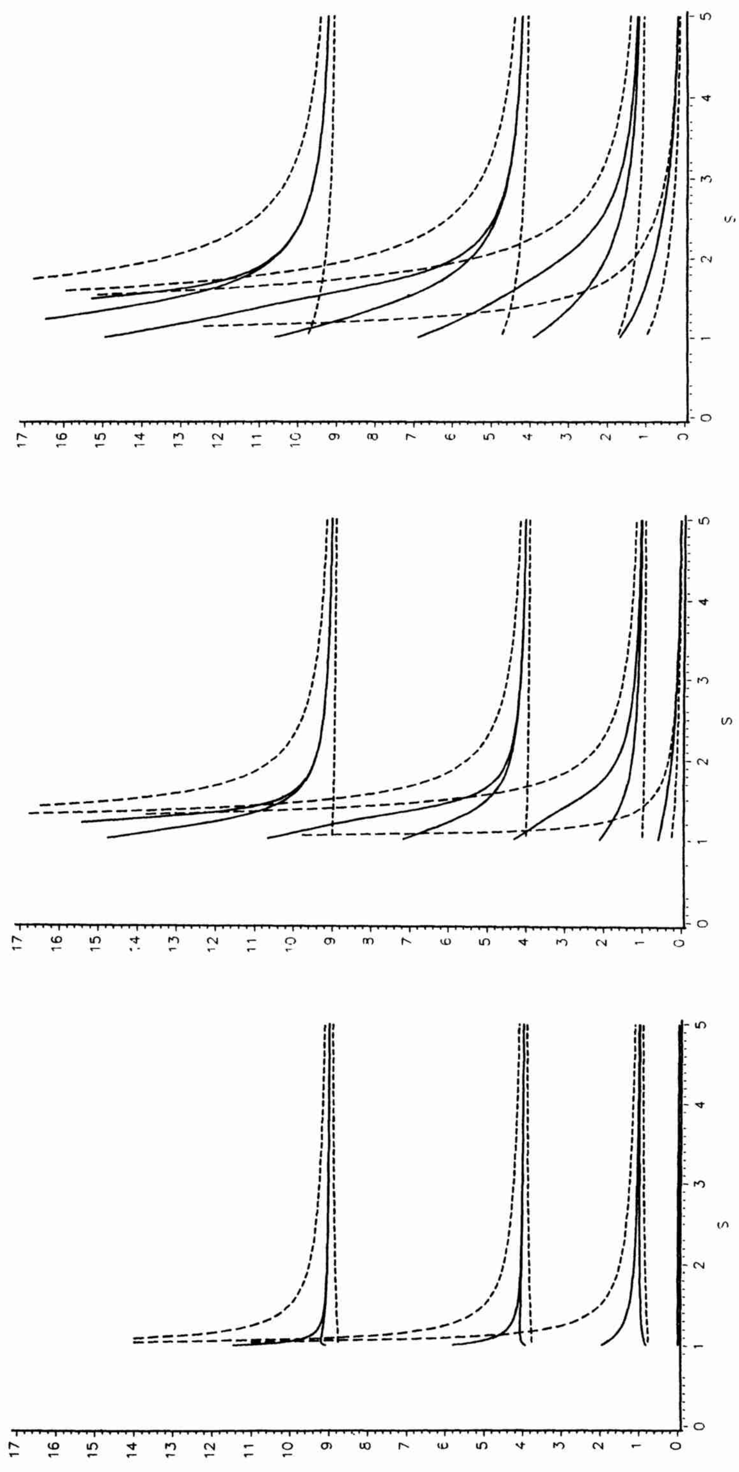
For the example of the standard torus with radius $R = 1$ we computed the eigenvalues (using Fortran NAG D02KAF and [Ho]) and compared them with the new estimates. The figures on the next page show the first seven k-eigenvalues for $k = 0, 1, 2$ and the corresponding bounds as functions of S for $1 < S \leq 5$. As was to be expected, the quality of the bounds increases with S, n and k. For $S \to 1$ the torus degenerates and so do our upper bounds. For small S ($1 < S < 2$ say), Rayleighs' principle yields better bounds for λ_0^k and λ_ν^k, since they remain finite for $S \to 1$; for example one gets $\lambda_0^k \leq \frac{2}{3} k^2$ and $\lambda_{2n}^k \leq (n + \frac{1}{2})^2 + (2n+1) k^2$ in the limit $S \to 1$.

The same effect can be seen when considering other known upper bounds. For instance, Yang and Yau have shown that for the first eigenvalue $\lambda_1(T)$ of the complete spectrum, $\lambda_1(T) \leq 16\pi / \text{vol}\, T = \frac{4}{\pi R S}$ (cf. [C], p. 94). Thus, for $S/R \to 1$ (more general whenever the torus becomes "narrow"), our upper bounds are worse.

In [Bk 1], some estimates for the eigenvalues on toroidal surfaces have been derived, too, but only as subordinate matter. The estimates of this paper are generally better, with the same restrictions for $S/R \to 1$ as described above. In this context, we want to note that the upper bound given in [Bk 1], 15.8 (ii) for imbedded tori is only valid for the cases $\frac{4}{3} r_{\max} \leq L$; in the other cases the formula has to be alterated. This restriction is missing there.

Comparing our lower bound for the first invariant eigenvalue λ_1^0, (4.3), to the upper bound of Yang and Yau, one can conclude that the 1-eigenvalue λ_0^1 equals the first eigenvalue of the complete spectrum, viz. $\lambda_0^1 = \lambda_1$, at least for $S/R > 1.579$. In tendency this is always the case when $S \gg R$ or when the volume of the surface becomes large compared with its distance from the axis of revolution. (When comparing with known estimates for the eigenvalues always keep in mind that usually the eigenvalues are simply sorted with respect to their growth, disregarding the classification by the winding number k.)

Thus the question as to the cases, in which our estimates improve known ones, cannot be answered in a general way. The answer depends on the choice of the function g in Theorems 1.2 and 1.6, it depends on the shape of the manifold and also on the type k



The k-eigenvalues and their bounds (dashed lines) as functions of the radius S for $k = 0, 1, 2$ and $n = 0, \ldots, 3$

of the spectrum and on the number n of the eigenvalue under consideration. The known estimates, e.g. those due to Cheeger, Yang-Yau, Rayleigh and others, see [C], are either upper bounds or lower bounds and pertain often only to the first eigenvalue and are independent of the winding number k.

The aim of this paper is to present a method (to be developed in §2) which follows from the Theorems of Haupt and Sturm in a simple way, which immediately yields two-sided estimates for each eigenvalue and which is tailor made to handle the problem of eigenvalues of the torus and its periodic differential equation. Such global assertions on *intervals* for each eigenvalue were known only in special cases, to the best of our knowledge (cf. [MW], p. 77). Also, the significance of our estimates originates rather from the bounds for the higher eigenvalues than from those for the first eigenvalues. In fact, our estimates virtually are bounds for the length of the intervals of instability $]\lambda_{2n-1}, \lambda_{2n}[$ of Hill's equation (cf. [E], p. 19ff, [MW], p. 12ff, [Ho]).

2 Proof of Theorems 1.1 and 1.2

We consider the differential equation (1) with boundary conditions (2) and the sequence (3) of its eigenvalues, as described in the introduction. For the sake of convenience we recall the mentioned result about the zeros of the eigenfunctions ψ_ν with eigenvalue λ_ν, cf. [CL], p. 214, [E], p.39 or [MW], p.11, or the original papers [Ha], [H 1], [H 2]:

2.1 Theorem (O. Haupt, 1914): *(i)* ψ_0 *has no zeros in* $[0, L]$,
(ii) ψ_{2n-1} *and* ψ_{2n} *have exactly* $2n$ *zeros in* $[0, L)$.

Equally well-known and much older is the Comparison Theorem, which we restate in terms of our given equation; for abbreviation put $Q := \lambda s - q$ (cf. e.g.[CL], p. 208):

2.2 Theorem (Sturm): *Suppose* ψ_1 *is a real solution on* $(0, L)$ *of* $(py')' + Q_1 y = 0$ *and* ψ_2 *a real solution on* $(0, L)$ *of* $(py')' + Q_2 y = 0$. *Let* $Q_2(x) > Q_1(x)$ *for* $x \in (0, L)$. *If* $x_1, x_2 \in (0, L)$ *are two successive zeros of* ψ_1, *then* ψ_2 *must vanish at some point of* (x_1, x_2).

2.3 The **Schwarz' Differential** or **Schwarz' Derivative** is an invariant which appears when transforming Hills' equation into standard form (cf. e.g. [Ca], p. 116ff, [K], p. 120). For $f' \neq 0$ it is defined as

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2;$$

it obeys the chain rule

$$\mathcal{S}(f \circ g) = g'^2 \mathcal{S}(f) \circ g + \mathcal{S}(g),$$

hence $\mathcal{S}$ is invariant under (real) Möbius transformations (homographies), i.e. for $h(x) = \frac{ax+b}{cx+d}$ with $ad - bc \neq 0$, we have $\mathcal{S}(h \circ g) = \mathcal{S}(g)$. From the identity

$$0 = \mathcal{S}(f \circ f^{-1}) = f'^2 \mathcal{S}(f^{-1}) \circ f + \mathcal{S}(f)$$

follows the expression for the Schwarz' Derivative of the inverse function:

$$\mathcal{S}(f^{-1}) \circ f = -\frac{1}{f'^2}\mathcal{S}(f).$$

2.4 For the **proof of Theorem 1.1**, we put $y = p^{-1/2}v$ and transform equation (1) into the standard form

$$v'' + \mathcal{V}v = 0 \tag{4}$$

with

$$\mathcal{V} := \frac{1}{4}\frac{p'^2}{p^2} - \frac{1}{2}\frac{p''}{p} + (\lambda s - q)\frac{1}{p} = \frac{1}{2}\mathcal{S}(P_*) + (\lambda s - q)\frac{1}{p} = \frac{s}{p}(\lambda - \mathcal{B}_0)$$

(P_* and $\mathcal{B}_0$ as in 1.1). Then v and y have the same period and the same zeros.

First, let $n \geq 1$ and let v be a periodic solution of (4) with eigenvalue $\lambda = \lambda_{2n}$ or $\lambda = \lambda_{2n-1}$. Then, by 2.1, we know that v has exactly $2n$ zeros in $[0, L)$; if we extend v on the interval $[0, \nu L)$, $\nu \in \mathbb{N}$, by periodic continuation, then v has exactly $2\nu n$ zeros. In the same interval, we consider the function

$$S(x) = \sin\Big((2\nu n - 1)\frac{\pi x}{\nu L}\Big) ;$$

this function has $2\nu n - 2$ zeros in $(0, \nu L)$ and satisfies the differential equation

$$w'' + \Big((2n - \frac{1}{\nu})\frac{\pi}{L}\Big)^2 w = 0 . \tag{5}$$

If we assume

$$\Big((2n - \frac{1}{\nu})\frac{\pi}{L}\Big)^2 > \mathcal{V} \qquad \text{on } (0, \nu L),$$

then each solution of (5) vanishes at least once between any two successive zeros of v, by 2.2, and consequently S has at least $2\nu n - 1$ zeros in $(0, \nu L)$, a contradiction. Hence there is an $x_0 \in [0, L]$ such that

$$\mathcal{V}(x_0) \geq \Big((2n - \frac{1}{\nu})\frac{\pi}{L}\Big)^2 .$$

Hence by definition of $\mathcal{V}$,

$$\lambda \frac{s}{p}\Big|_{x=x_0} \geq \bigg(\Big((2n - \frac{1}{\nu})\frac{\pi}{L}\Big)^2 + \frac{s}{p}\mathcal{B}_0\bigg)\Big|_{x=x_0} ,$$

consequently

$$\lambda \geq \min_{x\in[0,L]} \Big(\frac{p}{s}(2n - \frac{1}{\nu})\frac{\pi}{L}\Big)^2 + \mathcal{B}_0\Big) ,$$

and this is true for all $\nu \in \mathbb{N}$. Letting $\nu \longrightarrow \infty$ yields

$$\lambda \geq \min_{x\in[0,L]} \Big(\frac{p}{s}\Big(\frac{2\pi}{L}\Big)^2 n^2 + \mathcal{B}_0\Big)(x) = \min_{x\in[0,L]} \mathcal{B}_n(x) ; \tag{6a}$$

this holds for both $\lambda = \lambda_{2n-1}$ and $\lambda = \lambda_{2n}$.

The upper bound

$$\lambda_{2n-1} \leq \lambda_{2n} \leq \max \mathcal{B}_n \tag{6b}$$

in 1.1 follows similarly using $\sin\Big((2\nu n + 1)\frac{\pi x}{\nu L}\Big)$ and the assumption $\mathcal{V} > \Big((2n + \frac{1}{\nu})\frac{\pi}{L}\Big)^2$.

The assertion for $n = 0$ can be proved by 2.1 alone: We know that the eigenfunction v_0 which belongs to the first eigenvalue λ_0 has no zero in $[0, L]$; let $v_0 > 0$. If we assume that $\mathcal{V} > 0$ (resp. $\mathcal{V} < 0$), then $v_0'' = -\mathcal{V}v_0 < 0$ (resp. > 0), hence v_0' is strictly decreasing (resp. increasing) and therefore v_0 is not periodic – a contradiction. Thus there is an $x_0 \in [0, L]$ such that $0 = \mathcal{V}(x_0) = (\frac{s}{p}(\lambda_0 - \mathcal{B}_0))(x_0)$, i.e. $\lambda_0 = \mathcal{B}_0(x_0)$. Therefore

$$\min_{x\in[0,L]} \mathcal{B}_0(x) \le \lambda_0 \le \max_{x\in[0,L]} \mathcal{B}_0(x), \tag{7}$$

which proves Theorem 1.1. □

2.5 Proof of Theorem 1.2: Let $g \in \mathcal{G}$, M_g and $\mathcal{B}_n(g)$ be given as in Theorem 1.2. Clearly $\mathcal{B}_n(g) = \mathcal{B}_n(cg)$ for all $c \in \mathbb{R}, c > 0$. Without loss of generality, we therefore can choose g such that $M_g = 1$. Let

$$G(x) := \int_0^x g(t)dt,$$

thus $g = G'$, and let F be the inverse function of G. Then

$$F(0) = 0, \quad F(L) = L, \quad F(x + L) = F(x) + L, \tag{8}$$

since the same assertions obviously hold for G. The function F is strictly increasing and $F: \mathbb{R} \to \mathbb{R}$ is bijective.

Now, let y be a periodic solution of Hill's equation (1) with eigenvalue λ. We consider the re-parametrization $z = y \circ F$. Since y solves (1),

$$\left(py'' + p'y' + (\lambda s - q)y\right) \circ F = 0,$$

and therefore the following differential equation for z holds:

$$(p \circ F)(z'' - z'\frac{F''}{F'})\frac{1}{F'^2} + (p' \circ F)\frac{z'}{F'} + \left(\lambda \cdot (s \circ F) - (q \circ F)\right) z = 0. \tag{9}$$

Because of the periodicity of y and the properties (8) of F, the function z is also of period L, and so are the functions $\tilde{p} = \frac{p \circ F}{F'}$, $\tilde{s} = F' \cdot (s \circ F)$ and $\tilde{q} = F' \cdot (q \circ F)$. Since $F' > 0$, we have $\tilde{p}, \tilde{s} > 0$. Thus the periodic boundary value problem

$$(\tilde{p}z')' + (\lambda\tilde{s} - \tilde{q})z = 0 \tag{10}$$

has exactly the same eigenvalues as (1) and fulfills the same conditions regarding the coefficient functions. Therefore, Theorem 1.1 is applicable and gives estimates for the eigenvalues of (10). In the expression

$$\tilde{\mathcal{B}}_n = \frac{\tilde{p}}{\tilde{s}}\left(\left(\frac{2\pi}{L}\right)^2 n^2 + \frac{\tilde{q}}{\tilde{p}} - \frac{1}{2}\mathcal{S}(\widetilde{P_*})\right)$$

observe that $\widetilde{P_*} = P_* \circ F$ and therefore

$$-\mathcal{S}(\widetilde{P_*}) = F'^2\left(\left(-\mathcal{S}(P_*) + \mathcal{S}(F^{-1})\right) \circ F\right),$$

using the formulas of 2.3. By elementary calculations, we then obtain

$$\tilde{\mathcal{B}}_n = \frac{p}{s}\left(\left(\frac{2\pi}{L}\right)^2 n^2 \frac{1}{F'^2 \circ F^{-1}} + \frac{q}{p} - \frac{1}{2}\mathcal{S}(P_*) + \frac{1}{2}\mathcal{S}(F^{-1})\right) \circ F.$$

We now make the simple but crucial observation that the extrema of this function $\widetilde{\mathcal{B}}_n$ are invariant under re-parametrization; this allows us to combine $\widetilde{\mathcal{B}}_n$ with $G = F^{-1}$. Therefore

$$\widetilde{\mathcal{B}}_n \circ G = \mathcal{B}_n(g)$$

(note that $1/(F'^2 \circ F^{-1}) = g^2$). By 1.1, applied to $\widetilde{\mathcal{B}}_n \circ G$, we get

$$\min_{x\in[0,L]} \mathcal{B}_n(g)(x) \leq \lambda_{2n-1} \leq \lambda_{2n} \leq \max_{x\in[0,L]} \mathcal{B}_n(g)(x) . \tag{11}$$

This is assertion (i).

In the particular case $n = 0$, we choose $g = 1/pw^2$ with $w \in \mathcal{G}$; then $\frac{g'}{g} = -\frac{p'}{p} - 2\frac{w'}{w}$ and

$$\mathcal{B}_0\Big(\frac{1}{pw^2}\Big) = \frac{1}{sw}\Big(-(pw')' + qw\Big) .$$

Assertion (i) for $n = 0$ now gives

$$\min_{x\in[0,L]} \frac{1}{sw}\Big(-(pw')' + qw\Big)(x) \leq \lambda_0 \leq \max_{x\in[0,L]} \frac{1}{sw}\Big(-(pw')' + qw\Big)(x) . \tag{12}$$

The left-hand inequality has already been shown in [PW], p. 38, with the help of the maximum principle. If we choose w to be an eigenfunction y_0 of λ_0 (assume $y_0 > 0$), then $w = y_0$ solves the differential equation (1):

$$\Big((py_0')' - qy_0\Big) = -\lambda_0 s y_0 ,$$

and therefore $\mathcal{B}_0\Big(1/p\,y_0^2\Big) \equiv \lambda_0$. This means equality in (11) for $n = 0$ and proves the second assertion of Theorem 1.2. □

2.6 Corollary 1.3 follows by straightforward calculations.
For Corollary 1.4, note that for $ps = 1$, $q/s = const.$ and with $g = \sqrt{s/p}$ we get $\max \mathcal{B}_n(g) = \min \mathcal{B}_n(g)$.

2.7 Remark. Eastham also gives an upper bound for λ_{2n} ([E], p. 42, 3.3.2):

$$\lambda_{2n} \leq \frac{\sup p(x)}{\inf s(x)} \Big(\frac{2n\pi}{L}\Big)^2$$

and equality occurs only when $q \equiv 0$, $p(x) = const.$, $s(x) = const.$ However for this result, one has to assume that the complex Fourier coefficients c_j of q vanish for $0 \leq j \leq 2n$, a condition which often is not satisfied in applications, for instance in the standard torus example, which will be investigated in section 4. Our estimates do not require these restrictions.

3 Applications to toroidal surfaces of revolution

Our motivation for the preceding investigations was the study of the eigenvalues of the Laplacian on the standard torus imbedded in $\mathbb{R}^3$ (and endowed with the induced metric) and moreover on all surfaces of revolution diffeomorphic to it. On the flat tori the eigenvalues are explicitly known (see e.g. [C], p. 28, or [BGM], [Br], [SW]); in other cases one only has estimates for the eigenvalues, especially for the first one, e.g. the bounds given by Cheeger, Yang and Yau, Rayleigh's quotient and others, cf. [C].

A *surface of revolution diffeomorphic to the torus* is the standard torus in $\mathbb{R}^3$

$$T = \{((2+\sin\tau)\cos\varphi,\ (2+\sin\tau)\sin\varphi,\ \cos\tau) \mid \tau \in [0,2\pi],\ \varphi \in [0,2\pi]\},$$

but endowed with any Riemannian metric which is invariant under the group $O(2)$ of the rotations about the $z-$axis and the reflections at planes containing the rotation axis. These Riemannian metrics may differ from the ordinary metric induced by $\mathbb{R}^3$ on the standard torus; it may happen that such surfaces cannot be embedded into $\mathbb{R}^3$ as surfaces of revolution in the usual sense (with a generating curve). This notion even includes some of the flat tori; in fact, a flat torus is a surface of revolution in the sense of our definition if and only if it may be defined by a rectangular lattice. For details, see [Bk 1], [Bk 2].

The meridians of T are minimal geodesics, they are all of the same length L; the arc length parameter is denoted by $t \in [0,L]$. The parallels, the orbits under $O(2)$, are orthogonal to the meridians and are parametrized by the angle φ. Using (t,φ) as local coordinates, the metric on T is of the form

$$(g_{ij})\,|_{(t,\varphi)} = \begin{pmatrix} 1 & 0 \\ 0 & r(t)^2 \end{pmatrix}$$

with a twice differentiable function $r\colon [0,L] \longrightarrow \mathbb{R}_+$, which we may define on $\mathbb{R}$ by periodic extension.

This definition is based on a specific action of the group $O(2)$ as transformation group on the surface; the set of fixed points of any reflection consists of two disjoint meridians. There are other Riemannian manifolds diffeomorphic to the torus which admit different actions of $O(2)$ as a group of isometries, for instance the equilateral torus, which is not covered by our definition.

Since the Laplacian Δ commutes with isometries, one obtains an action of $O(2)$ on the eigenspaces E_λ of Δ and eventually a decomposition of each E_λ into the isotypic components with respect to the irreducible representations of $O(2)$. These irreducible representations are one- or two-dimensional; the twodimensional representations are characterized by the *winding number* $k \in \mathbb{N}\setminus\{0\}$.

Among the one-dimensional representations, only the trivial one has to be considered; it will be assigned the winding number $k = 0$. Given a decomposition of E_λ into $O(2)$-invariant irreducible subspaces, the *isotypic component* E_λ^k is defined to be the sum of the irreducible components with winding number k. The eigenvalues λ with $E_\lambda^k \neq \{0\}$ will be called *k-eigenvalues*; we order them as an increasing sequence, the so-called *k-spectrum*: $0 \leq \lambda_0^k < \lambda_1^k \leq \lambda_2^k < \cdots < \lambda_{2n-1}^k \leq \lambda_{2n}^k < \cdots$, where $k = 0,1,2,\cdots$. The elements of E_λ^0 are the eigenfunctions which are invariant under the action of $O(2)$, and $0 = \lambda_0^0 < \lambda_1^0 \leq \lambda_2^0 \cdots$ is the *invariant spectrum*. For $k \geq 1$, λ_0^k is positive. In the numbering of the eigenvalues, we follow [E], [MW], contrary to [Bk 1] and consistent with sections 1, 2.

The elements of E_λ^k are called *k-eigenfunctions*, and we call them *pure eigenfunctions* if they are contained in an irreducible subspace of E_λ^k. In the above defined coordinates the pure k-eigenfunctions are of the form

$$F(t,\varphi) = f(t)\,\cos k(\varphi - \varphi_0) \tag{13}$$

with $f \in C^2[0,L]$ (cf. [Bk 1], 9.3, or [Bk 2]).

The fact that the variables are separated in $F(t,\varphi)$ implies that the partial differential equation for the pure eigenfunctions becomes an ordinary differential equation for f: in the local coordinates t, φ, the equation

$$\Delta F - \lambda F \;=\; -\frac{\partial^2 F}{\partial t^2} - \frac{r'}{r}\frac{\partial F}{\partial t} - \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} - \lambda F \;=\; 0$$

leads to

$$f'' + \frac{r'}{r} f' + \left(\lambda - \frac{k^2}{r^2}\right) f = 0\,. \tag{14}$$

Observe that the existence of a basis consisting of separated eigenfunctions *follows* from the action of $O(2)$, we do not need to make an *ansatz* with separated solutions.

The eigenvalues of (14) are exactly the k-eigenvalues λ_ν^k. Clearly, $\dim E_\lambda^0 \leq 2$ and $\dim E_\lambda^k \in \{0,2,4\}$ for $k \geq 1$. If F is a pure eigenfunction as in (13) with eigenvalue λ_{2n-1}^k or λ_{2n}^k, the system of its nodal lines evidently consists of $2k$ meridians ($k \geq 0$) and a number of parallels. By Haupt's Theorem 2.1, this number is $2n$, the number of zeros of f in the interval $[0,L)$. Therefore

3.1 Theorem: *On surfaces of revolution diffeomorphic to the torus, the eigenfunctions with eigenvalue λ_0^k have exactly $2k$ sectors as nodal domains, and the pure eigenfunctions with eigenvalue λ_{2n-1}^k or λ_{2n}^k ($n \geq 1$) have exactly $4kn$ nodal domains, bordered by parallels and equidistant meridians.*

3.2 Proof of Theorem 1.6: We identify (14) with Hill's equation by multiplying with r and putting $p = s = r$, $q = \frac{k^2}{r}$. Thus the estimating function $\mathcal{B}_n(g)$ depends on k and becomes

$$\mathcal{B}_n^k(g) = \left(\frac{2\pi}{L}\right)^2 n^2 \left(\frac{g}{M_g}\right)^2 + \frac{k^2}{r^2} - \frac{1}{2}K - \frac{1}{4}B^2 + \frac{1}{2}\mathcal{S}(G)\,, \tag{15}$$

where $K = -r''/r$ and $B = r'/r$ are the Gauss curvature and the geodesic curvature of the parallels, respectively. Therefore Theorem 1.6 is a direct reformulation of Theorem 1.2. □

3.3 Corollary of Theorem 1.6: *With $g = r^\alpha$, $\alpha \in \mathbb{R}$, the estimating function $\mathcal{B}_n^k$ in Theorem 1.6 becomes*

$$\mathcal{B}_n^k(r^\alpha) = \left(\frac{2\pi}{L}\right)^2 n^2 \left(\frac{r^\alpha}{M_{r^\alpha}}\right)^2 + \frac{k^2}{r^2} - \beta K - \beta^2 B^2\,,$$

where $\beta = (\alpha+1)/2$.

Proof: With $g = r^\alpha$, we obtain $-\frac{1}{2}K - \frac{1}{4}B^2 + \frac{1}{2}\mathcal{S}(G) = -\beta K - \beta^2 B^2$. □

3.4 Special cases of the corollary:

Put $r_{\min} := \min r(t)$, $r_{\max} := \max r(t)$, $r_0 := r_{\max}/r_{\min} \geq 1$, and define $K_{\min}, K_{\max}$ similarly.

(1) If $\alpha = -1$ then $g = 1/r$ and $\beta = 0$. Therefore

$$\left(\frac{2\pi}{L}\right)^2 n^2 \frac{1}{r_0^2} + \frac{k^2}{r_{\max}^2} \le \lambda_{2n-1}^k \le \lambda_{2n}^k \le \left(\frac{2\pi}{L}\right)^2 n^2 r_0^2 + \frac{k^2}{r_{\min}^2} .$$

This gives very simple bounds especially for the first k-eigenvalue λ_0^k.

(2) If $\alpha = 0$ then $g \equiv 1$ and $M_g = 1$. Thus

$$\left(\frac{2\pi}{L}\right)^2 n^2 + \min\left(\frac{k^2}{r^2} - \frac{1}{2}K - \frac{1}{4}B^2\right) \le \lambda_{2n-1}^k \le \lambda_{2n}^k \le \left(\frac{2\pi}{L}\right)^2 n^2 + \max\left(\frac{k^2}{r^2} - \frac{1}{2}K - \frac{1}{4}B^2\right).$$

(3) The choice $\alpha = 1$, i.e. $g = r$, gives

$$\left(\frac{2\pi}{L}\right)^2 n^2 \frac{1}{r_0^2} + \min\left(\frac{k^2}{r^2} - K - B^2\right) \le \lambda_{2n-1}^k \le \lambda_{2n}^k \le \left(\frac{2\pi}{L}\right)^2 n^2 r_0^2 + \max\left(\frac{k^2}{r^2} - K - B^2\right).$$

Remark: If one wants to use the same function g for all n then the choice $g \equiv 1$ is optimal for large n.

3.5 With the choice $\beta = \pm k$ in 3.3, we get

$$B_n^k(r^{\mp 2k-1}) = \left(\frac{2\pi}{L}\right)^2 n^2 \left(\frac{g}{M_g}\right)^2 + \frac{k^2}{r^2} \mp kK - k^2 B^2 .$$

For $n = 0$, this gives the same bounds as can be obtained by Barta's Theorem, see the proof of 14.7 in [Bk 1]; these estimates for λ_0^k are of the same type as the result 14.1 in [Bk 1] on spherical surfaces of revolution.

4 The imbedded standard tori

Now, we consider the class of tori generated by rotating a circle of radius R around an rotation axis so that its center traces out a circle of radius S $(S > R)$, and endowed with the standard metric, i.e. the metric induced from $\mathbb{R}^3$. In our description of surfaces of revolution, their metrics are determined by

$$r(t) = S + R \sin \tfrac{1}{R} t , \tag{16}$$

and

$$r_{\max} = S+R, \; r_{\min} = S-R, \; K_{\max} = \frac{1}{R(S+R)}, \; K_{\min} = -\frac{1}{R(S-R)}, \; L = 2\pi R. \tag{17}$$

In this case where r is explicitly known, we can use specific functions g to get good bounds for the λ_ν^k from the results of section 3.

First, we consider the case $n = 0$. For $k = 0$, $\lambda_0^0 = 0$; hence assume $k > 0$. We start from Corollary 3.3: with $g = r^\alpha$, $\beta = (\alpha+1)/2$, the expression $B_0^k(g)$ becomes

$$B_0^k(r^\alpha)(t) = \left(k^2 - \beta^2 - \beta \tfrac{S}{R} \sin \tfrac{1}{R} + (\beta^2 - \beta)\left(\sin \tfrac{1}{R} t\right)^2\right) \Big/ \left(S + R \sin \tfrac{1}{R} t\right)^2 .$$

This can be written as a quadratic polynomial in $X(t) := 1\Big/\left(S + R\sin\tfrac{1}{R}t\right)$:

$$B_0^k(r^\alpha) = aX^2 + bX + c$$

with $a = k^2+\beta^2(\frac{S^2}{R^2}-1) > 0$, $b = \frac{S}{R^2}(\beta-2\beta^2)$, $c = \frac{1}{R^2}(\beta^2-\beta)$. Maximal values of $\mathcal{B}_0^k(r^\alpha)$ can occur only when $X' = 0$ (since at $2aX + b = 0$ the parabola has its minimum), i.e. at $t_1 = \frac{\pi}{2}R \in [0, 2\pi R]$ and $t_2 = \frac{3\pi}{2}R \in [0, 2\pi R]$. At these points,

$$\mathcal{B}_0^k(r^\alpha(t_{1,2})) = \frac{k^2}{(S \pm R)^2} \mp \frac{\beta}{R}\frac{1}{(S \pm R)}\,.$$

In order to optimize the upper bounds of λ_0^k which can be obtained in this way, we have to determine β such that the maximum of these two values becomes minimal. This is the case precisely for

$$\beta = \beta_o = -\frac{2R^2}{S^2-R^2}\,k^2\,.$$

With this value of β, Corollary 3.3 gives

4.1
$$\lambda_0^k \leq \frac{k^2}{S^2-R^2}\,,$$

which is a fairly good upper bound for the first eigenvalue of each k-spectrum, λ_0^k, for large S.

Bounds for general λ_ν^k, $(\nu \geq 0, k \geq 0)$ may be obtained Corollary 3.4. With $g = 1/r$, 3.4 (1) gives

4.2
$$\frac{1}{R^2}\frac{(S-R)^2}{(S+R)^2}n^2 + \frac{k^2}{(S+R)^2} \leq \lambda_{2n-1}^k \leq \lambda_{2n}^k \leq \frac{1}{R^2}\frac{(S+R)^2}{(S-R)^2}n^2 + \frac{k^2}{(S-R)^2}\,.$$

Further estimates can be derived from Corollary 3.4 (2), i.e. with $g \equiv 1$: The term $\mathcal{A} := \frac{k^2}{r^2} - \frac{1}{2}K - \frac{1}{4}B^2$ has its extrema at the points $t_1 = \frac{\pi}{2}R$ and $t_2 = \frac{3\pi}{2}R$, where the geodesic curvature B of the parallels vanishes. Using (17), one thus obtains

4.3
$$\frac{n^2}{R^2} + \frac{k^2}{(S+R)^2} - \frac{1}{2}\frac{1}{R(S+R)} \leq \lambda_{2n-1}^k \leq \lambda_{2n}^k \leq \frac{n^2}{R^2} + \frac{k^2}{(S-R)^2} + \frac{1}{2}\frac{1}{R(S-R)}\,.$$

For $n \geq 1, k \geq 0$, these bounds are better than those of 4.2, but for $n = 0$, $k \geq 1$, the result 4.2 is better than 4.3. Both estimates improve those which can be obtained from 3.4 (3).

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