manuscripta math. 68, 101 - 111 (1990)

manuscripta
mathematica

QUASISYMMETRY AND UNIONS

Jussi Väisälä

1. Introduction

1.1. Let X and Y be metric spaces with distance written as $|a-b|$, and let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism. An injective map $f : X \to Y$ is called *η-quasisymmetric*, abbreviated η-QS, if $|a-x| \le t|b-x|$ implies

$$|f(a)-f(x)| \le \eta(t)|f(b)-f(x)|$$

for every triple of points $a, b, x \in X$ and for every $t \ge 0$. If f is η–QS in a neighborhood of each point in X, it is called *locally* η–QS. The basic theory of QS maps is given in [TV] and [Vä$_3$].

An increasing map $f : R^1 \to R^1$ of the real axis is η–QS if and only if it is ρ–QS in the classical sense of Beurling and Ahlfors; here η and ρ depend only on each other. For homeomorphisms between open sets in the euclidean n–space R^n, $n \ge 2$, local quasisymmetry is equivalent to quasiconformality. The following lemma gives a precise form of this statement. For proofs, see [Vä$_3$, 2.3, 2.4] and [AVV, 5.23]. We let $B(x,r)$ denote the open ball with center x and radius r and abbreviate quasiconformal as QC.

1.2. LEMMA. *Suppose that $n \ge 2$, that G and G' are open sets in R^n and that $f : G \to G'$ is a homeomorphism. If f is locally η-QS, then f is K-QC with $K = \eta(1)^{n-1}$. If f is K-QC and $B(x,ar) \subset G$ for some $a > 1$, then $f|B(x,r)$ is η-QS with η depending only on K and a.*

This paper deals with the following problem: Suppose that $X = E_1 \cup \ldots \cup E_s$ and that $f : X \to Y$ is an injective map such that each restriction $f|E_j$ is locally η-QS. Is f locally η_1–QS with some η_1? The following example shows that the answer can be negative in very simple cases: Define $f : R^1 \to R^1$ by $f(x) = x$

for $x \leq 0$ and $f(x) = x^2$ for $x \geq 0$. Then f is a homeomorphism, which is QS in $(-\infty, 0]$ and in $[0, \infty)$ but not in any neighborhood of the origin.

In 3.10 we show that the answer is positive in the case where X and Y are open sets in R^n, $n \geq 2$. As a special case we obtain a new removability theorem for quasiconformal maps (Theorem 3.3). The related result 3.12 states that a homeomorphism $f : \overline{G} \to \overline{G}'$ is QS if f is QC in G and QS on ∂G. This was suggested to the author by O. Martio.

The proofs are based on an improved version of F.W. Gehring's local maximum principle for QC maps [Ge, Theorem 2.1]. This is given in Section 2, and the main results in Section 3. Some open problems are stated in 3.16.

I thank Olli Martio for useful and enjoyable discussions and Pekka Alestalo for careful reading of the manuscript.

2. The local maximum principle

2.1. *Terminology.* In the rest of the paper we assume that $n \geq 2$. We consider open sets G, G' in R^n, which are not necessarily connected. A homeomorphism $f : G \to G'$ is called K–quasiconformal if f is K–quasiconformal in each component of G. If $F \subset \partial G$, the cluster set $\operatorname{clus}(f, F)$ of f at F is the set of all points $y \in \dot{R}^n = R^n \cup \{\infty\}$ such that there is a sequence of points $x_j \in G$ with $x_j \to x_0 \in F$ and $f(x_j) \to y$.

As in the introduction, $B(x, r)$ denotes the open ball with center x and radius r, and we abbreviate $B(r) = B(0, r)$. For spheres we write $S(x, r) = \partial B(x, r)$ and $S(r) = S(0, r)$. For $0 < r < s$ the open spherical annulus $B(s) \setminus \overline{B}(r)$ is written as $A(r, s)$.

A domain is an open connected set. We first give an elementary topological result:

2.2. LEMMA. *Suppose that D_1, D_2 are domains in a locally connected topological space X with $D_1 \cap D_2 \neq \emptyset$ and that U is a component of $D_1 \cap D_2$ with $D_1 \neq U \neq D_2$. Then ∂U meets both ∂D_1 and ∂D_2.*

Proof. It suffices to show that ∂U meets ∂D_1. Since the connected set D_2 meets U and $X \setminus U$, there is a point $x \in D_2 \cap \partial U$. We show that $x \in \partial D_1$.

Clearly $x \in \overline{U} \subset \overline{D}_1$. If $x \notin \partial D_1$, then $x \in D_1 \cap D_2$, and we can choose a domain V with $x \in V \subset D_1 \cap D_2$. Since $U \cap V \neq \emptyset$, the set $W = U \cup V$ is connected and hence $W \subset U$. This implies $x \in U$, a contradiction. □

2.3. THEOREM (local maximum principle). *Suppose that G and G' are open sets in R^n and that $f : G \to G'$ is a K-quasiconformal homeomorphism. Suppose also that $r > 0$, $0 < q < 1$, $b > 0$ and that $B(r) \setminus G$ contains two points x_1, x_2 with $|x_1| \leq qr$ and $|x_1 - x_2| \geq br$. If $R > 0$ and*

$$\operatorname{clus}(f, \partial G \cap B(r)) \subset \overline{B}(R),$$

then

$$f[G \cap B(qr)] \subset B(cR),$$

where c is a constant depending only on the data $v = (K, n, q, b)$.

Proof. We may assume that $G \cap B(qr) \neq \emptyset$. Then also the sets $F = \partial G \cap B(r)$ and $E = \text{clus}(f, F)$ are nonempty. Let $x' \in G \cap B(qr)$ and write $y' = f(x')$. It suffices to find a bound $|y'| \leq cR$, $c = c(v)$. We may assume that $|y'| > 2R$.

Let γ' be the ray $\{ty' : t \geq 1\}$ and let β' be the component of $\gamma' \cap G'$ containing y'. Then $\beta = f^{-1}\beta'$ is a half open arc starting at x' and clustering to ∂G. Since it cannot cluster to points in F, it has a subarc $\beta_0 \subset B(r)$ with diameter $d(\beta_0) = (1-q)r$.

Let D be the component of $G \cap B(r)$ containing β_0. Since $x_1 \in \overline{B}(qr) \setminus G$, ∂D meets $\overline{B}(qr)$. Fix a point $z_1 \in \partial D \cap \overline{B}(qr)$ and choose a number $b_1 = b_1(v)$ with $b_1 < b/2$ and $b_1 < 1 - q$. We first show that $\partial D \cap B(r)$ contains a point z_2 with $|z_1 - z_2| \geq b_1 r$.

Assume that this is false. Since the connected set $W = B(r) \setminus B(z_1, b_1 r)$ does not meet ∂D, we have either $W \subset D$ or $W \cap \overline{D} = \emptyset$. In the first case, x_1 and x_2 are in $\overline{B}(z_1, b_1 r)$, which implies $|x_1 - x_2| \leq 2b_1 r < br$, a contradiction. In the second case, $\overline{D} \subset \overline{B}(z_1, b_1 r) \subset B(r)$. Hence D is a component of G, and $\partial f D \subset E$. This implies $fD \subset B(R)$ and thus $fD \cap \beta' = \emptyset$, which is again a contradiction. The existence of z_2 is proved.

We next show that the set $V = D \cap f^{-1}B(2R)$ has a component V_0 with $d(V_0) \geq b_1 r/2$. Suppose that this is false. Since $z_1, z_2 \in \partial V$, there are components V_1, V_2 of V with distance $d(z_i, V_i) < b_1 r/4$. Then $V_1 \neq V_2$, since otherwise

$$|z_1 - z_2| \leq d(z_1, V_1) + d(V_1) + d(z_2, V_1) < b_1 r.$$

Since $V_1 \subset B(z_1, 3b_1 r/4)$, we have $\overline{V}_1 \subset B(r)$. If $x \in \partial V_1 \cap D$ and $|f(x)| < 2R$, there is a connected neighborhood $W \subset D$ of x with $fD \subset B(2R)$. Then $W \cup V_1$ is a connected set in V, and hence V_1 is not a component of V. It follows that

$$\partial V_1 \subset (B(r) \cap \partial D) \cup f^{-1}S(2R),$$

and thus

$$\partial f V_1 \subset E \cup S(2R).$$

The case $\partial f V_1 \subset E$ is impossible, because then $\partial V_1 \subset \partial D$ and thus $V_1 = D$. Hence $\partial f V_1$ meets $S(2R)$. Since the annulus $A(R, 2R)$ is connected, it is contained in fV_1. Since f is injective, this yields $fV_2 \subset B(R)$. Since fV_2 is a component of $fV = fD \cap B(2R)$, Lemma 2.2 implies $fV_2 = fD$. This gives $V_2 = D$, a contradiction. The existence of V_0 is proved.

As usual, we let $\Delta(A, B; C)$ denote the family of all paths joining the sets A and B in C. We next show that the family $\Gamma_1 = \Delta(\beta_0, V_0; B(r))$ is minorized [Vä$_1$, 6.3] by $\Gamma = \Delta(\beta_0, V, D)$. Let γ be a path in Γ_1 starting at β_0 and terminating at V_0. If γ lies in D, then $\gamma \in \Gamma$. If γ goes out of D, let γ_1 be the maximal subpath of γ in D, starting at β_0. Then γ_1 converges to $\partial D \cap B(r)$, and hence $f\gamma_1$ converges to E. It follows that a subpath of γ_1 is in Γ. By [Vä$_1$, 6.4] this implies $M(\Gamma) \geq M(\Gamma_1)$. Since the diameters of β_0 and V_0 are at least $b_1 r/2$, standard modulus estimates yield $M(\Gamma_1) \geq c_0 = c_0(n, b_1) > 0$; see e.g. [GM, 2.6] and [GV, Lemma 3.3].

On the other hand, $f\Gamma$ is minorized by the family associated to the ring $A(2R, |y'|)$, and hence

$$M(f\Gamma) \le \omega_{n-1}(\ln \frac{|y'|}{2R})^{1-n}.$$

Since $M(\Gamma) \le KM(f\Gamma)$, these inequalities give the desired estimate $|y'| \le cR$, $c = c(v)$. □

2.4. *Remarks.* If $\overline{G} \subset B(r)$, then $fG \subset B(R)$, and the theorem holds with $c = 1$. If $\overline{G} \not\subset B(r)$ and ∂G is connected and meets $B(qr)$, one can choose $b = 1-q$. Thus Gehring's result [Ge, Theorem 2.1] is a special case of 2.2.

With the aid of suitable inversions we obtain from 2.3 the following "local minimum principle":

2.5. THEOREM. *Suppose that G and G' are open sets in R^n and that $f : G \to G'$ is a K-quasiconformal homeomorphism. Suppose also that $0 \notin G \cup G'$, $r > 0$, $0 < q < 1$, $c_0 \ge 1$ and that $A(qr, c_0 r) \setminus G \ne \emptyset$. If $R > 0$ and*

$$\operatorname{clus}(f, \partial G \setminus \overline{B}(qr)) \cap B(R) = \emptyset,$$

then

$$f[G \setminus \overline{B}(r)] \cap B(R/c) = \emptyset,$$

where c depends only on $v = (K, n, q, c_0)$.

3. Main results

3.1. *Notation.* Let G and G' be open sets in R^n, $n \ge 2$, and let $f : G \to G'$ be a homeomorphism. If $\overline{B}(x, r) \subset G$, we let $L(x, f, r)$ and $l(x, f, r)$ denote the supremum and infimum of $|f(y) - f(x)|$ over $y \in S(x, r)$. The number

$$H(x, f) = \limsup_{r \to 0} \frac{L(x, f, r)}{l(x, f, r)}$$

is called the *metric dilatation* of f at x, often also the linear dilatation. By the metric definition of quasiconformality [Vä$_1$, 34.1], f is QC if and only if $H(x, f)$ is bounded over $x \in G$. If, in addition, $H(x, f) \le c$ almost everywhere in G, then f is c^{n-1}–QC.

The main result of this paper is Theorem 3.10. Its proof is based on the removability theorem 3.3, which is in fact a special case of 3.10.

We start with an easy distortion result for maps of spherical rings. A different proof appears in [MN, Lemma 2].

3.2. LEMMA. *Suppose that $r > 0$, $a > 1$ and that f is a K-QC map of the annulus $A = A(r/a, ar)$ onto a domain A' not containing the origin. If $x, y \in S(r)$, then $|f(x)| \le c|f(y)|$ with c depending only on (K, n, a).*

Proof. We may assume that $|f(x)| > |f(y)|$. Let k and k' denote the quasihyperbolic metrics [GP] of A and A', respectively. Integration along a circular arc in $S(r)$ gives

$$k(x,y) \leq \frac{\pi a}{a-1}.$$

Hence [GO, Theorem 3] implies that $k'(f(x),f(y))$ is bounded by a constant $b = b(K,n,a)$. On the other hand, since $d(f(y),\partial A') \leq |f(y)|$, [GP, (2.2)] yields

$$k'(f(x),f(y)) \geq \ln(1 + \frac{|f(x)-f(y)|}{|f(y)|}) \geq \ln\frac{|f(x)|}{|f(y)|}.$$

Hence the lemma is true with $c = e^b$. □

3.3. THEOREM. *Suppose that G, G' are open sets in R^n, that E is closed in G, and that $f : G \to G'$ is a homeomorphism such that $f|G \setminus E$ is K-QC and $f|E$ locally η-QS. Then f is K_1-QC with K_1 depending only on $v = (K,n,\eta)$.*

Proof. Let $x_0 \in E$. According to the metric definition of quasiconformality it suffices to find an upper bound $H(x_0,f) \leq c(v)$. We may assume that $x_0 = 0 = f(x_0)$. Choose $r > 0$ such that $B(2r) \subset G$ and such that $f|B(2r) \cap E$ is η–QS. It suffices to find an estimate

$$|f(x)| \leq c|f(y)| \tag{3.4}$$

for $x, y \in S(r)$ with $c = c(v)$.

If the annulus $A = A(r/2, 2r)$ does not meet E, (3.4) follows from 3.2. Hence we may assume that there is a point $z_1 \in E \cap A$. If $z \in E \cap B(2r)$, the quasisymmetry condition implies

$$|f(z)| \leq \eta(4)|f(z_1)|. \tag{3.5}$$

Write $G_1 = G \setminus E$. We want to apply the local maximum principle 2.3 with the substitution $(G, qr, r) \mapsto (G_1, r, 2r)$. Since 0 and z_1 are in $B(2r) \cap E$ with $|z_1| > r/2$, we can choose $b = 1/4$. By (3.5) we can choose $R = \eta(4)|f(z_1)|$. Hence 2.3 gives

$$f[B(r) \setminus E] \subset B(c_1\eta(4)|f(z_1)|)$$

with $c_1 = c_1(K,n) \geq 1$. Hence

$$|f(x)| \leq c_1\eta(4)|f(z_1)|$$

for all $x \in S(r)$.

Similarly we can apply the local minimum principle 2.5 with $(qr, r, c_0) \mapsto (r/2, r, 2)$. By quasisymmetry we have

$$|f(z)| \geq |f(z_1)|/\eta(4)$$

for all $z \in E \setminus B(r/2)$. Hence 2.5 yields

$$|f(y)| \geq |f(z_1)|/c_2\eta(4)$$

for all $y \in S(r)$ with $c_2 = c_2(K, n)$, and we obtain (3.4) with $c = c_1c_2\eta(4)^2$. □

3.6. *Remarks.* 1. We can replace the local η–quasisymmetry in 3.3 by the following weaker condition: There is $H > 0$ such that for every $x \in E$ there is $s > 0$ such that $|a - x| \leq 4|b - x| < s$ implies $|f(a) - f(x)| \leq H|f(b) - f(x)|$. The number 4 can be replaced by any number $\alpha > 1$, but K_1 will then depend on H and α.

2. S. Rickman [Ri, Lemma 2] proved in 1969 a removability result with a weaker condition on $f|E$. However, in that result E could not always be completely removed. I do not know whether the local quasisymmetry in 3.3 can be replaced by Rickman's condition.

3. One can replace the local η–quasisymmetry in 3.3 by a local η–quasimöbius condition. The result is then valid for maps in the extended space $\dot{R}^n = R^n \cup \{\infty\}$. This version is easily reduced to 3.3 by auxiliary Möbius maps; cf. the proof of 3.15.

3.7. LEMMA. *Suppose that G and G' are open sets in R^n, that $f : G \to G'$ is a locally* QS *homeomorphism and that $E \subset G$ with $f|E$ locally η –* QS. *If $x_0 \in E$ and x_0 is a point of outer density for E, then $H(x_0, f) \leq \eta(1)$.*

Proof. Write $H = \eta(1)$. We may assume that $x_0 = 0 = f(x_0)$. Choose $r_0 > 0$ such that $\overline{B}(r_0) \subset G$, $f|\overline{B}(r_0) \cap E$ is η–QS and $f|\overline{B}(r_0)$ is η_1–QS for some η_1. Since f is η–QS in $\overline{B}(r_0) \cap \overline{E}$, we may assume that $\overline{B}(r_0) \cap E$ is closed. Fix t with $0 < t < 1/4$ and set

$$F(t) = \{r : 0 < r \leq r_0,\ m_{n-1}(S(r) \setminus E) \leq t m_{n-1}(S(r))\}.$$

Here m_{n-1} is the $(n - 1)$-dimensional measure on $S(r)$. Since 0 is a point of density for E, $F(t)$ has the origin as a point of right–hand linear density. Writing $L(r) = L(0, f, r)$ and $l(r) = l(0, f, r)$ we want to estimate the ratio $L(r)/l(r)$.

Assume first that $r \in F(t)$. Choose $x_1 \in S(r) \cap E$ such that $|f(x_1)|$ is minimal. Then

$$|f(x_1)| \leq |f(x)| \leq H|f(x_1)| \tag{3.8}$$

for every $x \in S(r) \cap E$. Let $x \in S(r) \setminus E$ and let C be the cap of the form $S(r) \cap B(x, r')$ such that $m_{n-1}(C) = 2t m_{n-1}(S(r))$. We can write $r' = \alpha(t)r$ where $\alpha(t) \to 0$ as $t \to 0$. Since $r \in F(t)$, we can choose a point $y \in C \cap E$. Since $f|\overline{B}(r_0)$ is η_1–QS, (3.8) implies

$$\begin{aligned}|f(x)| &\leq |f(y)| + |f(x) - f(y)| \\ &\leq |f(y)| + \eta_1\left(\frac{|x - y|}{|y|}\right)|f(y)| \\ &\leq (1 + \eta_1(\alpha(t)))H|f(x_1)|,\end{aligned}$$

and similarly a lower bound

$$|f(x)| \geq (1 - \eta_1(\alpha(t)))|f(x_1)|.$$

For small t we have $\eta_1(\alpha(t)) < 1$, and then

$$\frac{L(r)}{l(t)} \leq \frac{1 + \eta_1(\alpha(t))}{1 - \eta_1(\alpha(t))} H \tag{3.9}$$

for every $r \in F(t)$.

Next assume that $0 < r \leq r_0$ and $r \notin F(t)$. Since the origin is a point of right–hand density for $F(t)$, we can find a number $r_1 \in F(t)$ such that

$$r/(1 + \beta(r)) < r_1 < r$$

where $\beta(r) \to 0$ as $r \to 0$. Suppose that $x \in S(r)$ and set $y = r_1 x/r$. Using again the η_1–quasisymmetry of $f|\overline{B}(r_0)$ we obtain

$$\begin{aligned} |f(x)| &\leq |f(y)| + |f(x) - f(y)| \\ &\leq |f(y)| + \eta_1(\frac{|x - y|}{|y|})|f(y)| \\ &\leq L(r_1)(1 + \eta_1(\beta(r))). \end{aligned}$$

Since $l(r) > l(r_1)$, this and (3.9) give

$$\frac{L(r)}{l(r)} \leq \frac{(1 + \eta_1(\beta(r)))(1 + \eta_1(\alpha(t)))}{1 - \eta_1(\alpha(t))} H$$

for every $r \leq r_0$. Letting first $r \to 0$ and then $t \to 0$ yields $H(0, f) \leq H$. □

3.10. THEOREM. *Suppose that G and G' are open sets in R^n and that $f : G \to G'$ is a homeomorphism. Suppose also that $G = E_1 \cup \ldots \cup E_s$ and that $f|E_j$ is locally $\eta -$ QS for each j. Then f is $K -$ QC with $K = \eta(1)^{n-1}$.*

Proof. We first show that f is K_1–QC with K_1 depending on η, n and s. If $s = 1$, this follows from 1.2. Proceeding inductively, assume that the assertion is true if the number of the sets E_j is less than s. Fix $x_0 \in G$. It suffices to show that f is K_1–QC in a neighborhood ot x_0. We may assume that $x_0 \in E_1$. Choose a neighborhood $U \subset G$ of x_0 such that $f|U \cap E_1$ is η–QS. By [TV, 2.25] we may assume that $U \cap E_1$ is closed in U. By the inductive hypothesis, f is K_2–QC in $U \setminus E_1$ with $K_2 = K_2(\eta, n, s - 1)$. Applying 3.2 with the substitution $G \mapsto U$, $E \mapsto E_1 \cap U$ we conclude that $f|U$ is K_1–QC with $K_1 = K_1(\eta, n, s)$. Hence $H(x, f)$ is bounded for $x \in G$.

In view of 1.2, 3.7 and the density theorem, $H(x, f) \leq \eta(1)$ for almost every $x \in G$. The theorem follows from the metric definition of quasiconformality, see 3.1. □

3.11. *Remarks.* 1. In the special case where $s = 2$ and E_1 is closed in G, 3.10 gives 3.3.

2. The hypothesis that $f|E_j$ be locally η−QS can be weakened in 3.10 as in 3.6.1 and 3.6.3.

3.12. THEOREM. *Suppose that G and G' are domains in R^n and that $f : \overline{G} \to \overline{G}'$ is a homeomorphism such that $f|G$ is K−QC and $f|\partial G$ η−QS. Then f is η_1−QS with η_1 depending only on $v = (K, \eta, n)$.*

Proof. In the theorem the closures and boundaries are taken in R^n. However, if G is unbounded, the set $\mathrm{clus}(f, \infty)$ cannot contain any finite point of $\overline{G}$. Hence we can extend f to a homeomorphism, also written as $f : \overline{G} \cup \{\infty\} \to \overline{G}' \cup \{\infty\}$, $f(\infty) = \infty$. In the rest of the proof all closures, boundaries and complements are taken in $\dot{R}^n$.

If $G = R^n$, then $G' = R^n$, and f is η_1−QS by Lemma 1.2. Thus we may assume that $G \neq R^n \neq G'$. By continuity and by [Vä$_5$, 2.9] it suffices to show that $f|G$ is weakly H−QS with $H = H(v)$. Let x_0, $a, b \in G$ with $|a - x_0| \leq |b - x_0| = r$. Writing $R' = |f(a) - f(x_0)|$, $r' = |f(b) - f(x_0)|$ we must show that

$$R' \leq Hr' \tag{3.13}$$

for some $H = H(v)$. We normalize $x_0 = 0 = f(x_0)$ by auxiliary translations. If $2r \leq d(0, \partial G)$, then (3.13) follows from Lemma 1.2. Hence we may assume that $2r > d(0, \partial G)$.

Suppose first that there are numbers r_1 and r_2 such that $r' \leq r_1 < r_2 \leq R'$ and such that G' contains the spherical annulus $A' = \{y : r_1 < |y| < r_2\}$. We shall show that

$$r_2 \leq \lambda r_1 \tag{3.14}$$

for some $\lambda = \lambda(v)$. We may assume that $\overline{A}' \subset G$. Set $A = f^{-1}A'$, and let C_0 and C_1 be the components of $\dot{R}^n \setminus A$ with $\infty \in C_1$. Since A separates $\{0, b\}$ from a, there are two possibilities:

Case 1. $\{0, b\} \subset C_0$, $a \in C_1$. Let Γ_A be the path family associated to the ring A. By the Teichmüller estimate [Vä$_1$, 11.9] we have $M(\Gamma_A) \geq c_n > 0$. Since

$$M(\Gamma_A) \leq KM(f\Gamma_A) = K\omega_{n-1}(\ln \frac{r_2}{r_1})^{1-n},$$

this yields (3.14).

Case 2. $\{0, b\} \subset C_1$, $a \in C_0$. Let C be the ∞−component on $\dot{R}^n \setminus G$. Then $B = \overline{G} \cap C$ is a component of ∂G. Since A separates B and a, A' separates fB and $f(a)$, which implies $fB \subset B(r_1)$. Since fB is bounded and $f^{-1}|fB$ QS, B is bounded, which means that G is bounded. The component C_0 meets ∂G, since otherwise fC_0 would be the ∞−component of A', which is impossible. Choose a point $x_1 \in C_0 \cap \partial G$. Let L be a line through x_1. Choose points $x_2, x_3 \in L \cap B$ such that $x_1 \in [x_2, x_3]$. Writing $y_j = f(x_j)$ we have

$$\frac{|y_1 - y_2|}{|y_3 - y_2|} \leq \eta(\frac{|x_1 - x_2|}{|x_3 - x_2|}) \leq \eta(1).$$

Since A' separates y_1 and y_2, we have $|y_1 - y_2| \geq r_2 - r_1$. Since $|y_3 - y_2| \leq 2r_1$, we obtain (3.14) with $\lambda = 1 + 2\eta(1)$.

Write $\delta = d(0, \partial G) < 2r$ and $\delta' = d(0, \partial G')$. If $r' < \delta'$, the well-known distortion theorem of Gehring gives

$$1/2 < r/\delta \leq \Theta_K^n(r'/\delta'),$$

where $\Theta_K^n : [0,1) \to [0,\infty)$ is a homeomorphism; see [Vä$_1$, 18.1]. Hence in every case there is $M = M(v) \geq 1$ such that $\delta' \leq Mr'$. We can thus choose a point $y_4 \in \partial G'$ with $|y_4| \leq Mr'$. If $R' \leq 3\lambda Mr'$, then (3.13) holds with $H = 3\lambda M$. Assume that $R' > 3\lambda Mr'$. The annulus

$$A_1' = \{y : 2Mr' < |y| < 3\lambda Mr'\}$$

is not contained in G', since otherwise (3.14) gives a contradiction. Choose a point $y_5 \in A_1' \cap \partial G'$.

We apply the local maximum principle 2.3 with

$$(G, f, qr, r) \mapsto (G', f^{-1} - x_4, 2Mr', 3\lambda Mr')$$

where $x_4 = f^{-1}(y_4)$. Since $y_4, y_5 \in \partial G$ and

$$|y_4| \leq Mr', \ |y_5| < 3\lambda Mr', \ |y_4 - y_5| \geq |y_5| - |y_4| \geq Mr',$$

the hypotheses of 2.3 are satisfied with $b = 1/3\lambda$. Writing

$$E = \overline{B}(3\lambda Mr') \cap \partial G, \ E = f^{-1}E'$$

we thus have

$$f^{-1}[B(2Mr') \cap G'] \subset B(x_4, c(v)d(E)).$$

Since $B(2Mr')$ contains 0 and $f(b)$, this implies

$$r = |b| \leq |b - x_4| + |x_4| \leq 2cd(E)$$

and

$$|a - x_4| \leq |a| + |x_4| \leq r + cd(E) \leq 3cd(E).$$

Writing $F = \overline{B}(e_4, 4cd(E) \cap \partial G)$, another application of the local maximum principle gives

$$|f(a) - y_4| \leq c_1(v)d(fF).$$

Since $E \subset F$, [TV, 2.5] yields

$$\frac{d(fF)}{d(E')} \leq 2\eta(\frac{d(F)}{d(E)}) \leq 2\eta(8c).$$

Combining these inequalities we obtain

$$R' = |f(a)| \leq |f(a) - y_4| + |y_4| \leq (12c_1\lambda\eta(8c) + 1)Mr' = Hr'. \quad \square$$

3.15. THEOREM. *Suppose that* G *and* G' *are domains in* $\dot{R}^n$ *and that* $f : \overline{G} \to \overline{G'}$ *is a homeomorphism such that* $f|G$ *is* K–QC *and* $f|\partial G$ θ*-quasimöbius. Then* f *is* θ_1*-quasimöbius with* θ_1 *depending only on* $v = (K, \theta, n)$.

Proof. We may assume that $\partial G \neq \emptyset$, since otherwise the result is well known [Vä$_4$, p. 232]. Fix a point $x_0 \in \partial G$ and choose Möbius maps α and β of $\dot{R}^n$ with $\alpha(x_0) = \infty = \beta(f(x_0))$. Then $g = \beta f \alpha^{-1}|\alpha\overline{G}$ is a homeomorphism $g : \alpha\overline{G} \to \beta\overline{G}'$, $g|\alpha G$ is K–QC and $g|\partial\alpha G$ θ–quasimöbius. Since $g(\infty) = \infty$, $g|\partial\alpha G$ is θ–QS. By 3.12 g is η_1–QS with η_1 depending only on v. By [Vä$_4$, 3.2] g is θ_1–quasimöbius with θ_1 depending only on v. □

3.16. *Open problems.* 1. Is 3.10 true for countable unions? The essential question is whether f is QC, since the estimate $H(x, f) \leq \eta(1)$ a.e. follows then as in 3.10.

If each $f|E_j$ has a K–QC extension to a neighborhood of E_j with a fixed K, the answer is positive by [Vä$_2$, Theorem 2]. Indeed, f is then K–QC.

2. In Theorem 3.10 we can replace the K–quasiconformality of f by the condition that f is locally η_1–QS with η_1 depending on η and n. Can η_1 be chosen to be independent of n?

3. Problem 2 is related to the following important question: Is a locally η–QS homeomorphism $f : R^n \to R^n$, $n \geq 2$, globally η_1–QS with η_1 depending only on η?

4. Are there any infinite–dimensional versions of the results of this paper? We give an elementary result, which is valid in all spaces of dimension at least two. In other respects it is considerably weaker than Theorem 3.10. The definition of the metric dilatation $H(x, f)$ in 3.1 can obviously be extended to the present situation.

3.17. THEOREM. *Suppose that* G *and* G' *are open sets in a normed vector space* X *with* dim $X \geq 2$ *and that* $f : G \to G'$ *is a homeomorphism. Suppose also that* $G = E_1 \cup \ldots \cup E_s$ *where each* E_j *is closed in* G *and each* $f|E_j$ *is locally* η – QS. *Then* $H(x, f) \leq \eta(1)^s$ *for every* $x \in G$.

Proof. Set $H = \eta(1)$. To prove that $H(x_0, f) \leq H^s$ we may assume that $x_0 = 0 = f(x_0)$. Since the sets E_j are closed in G, we may assume, replacing G by a neighborhood of 0, that $0 \in E_j$ for every j. Choose r_0 such that $\overline{B}(r_0) \subset G$ and such that $f|\overline{B}(r_0) \cap E_j$ is η–QS for all j. Let $0 < r \leq r_0$ and let $x, y \in S(r)$. Since dim $X \geq 2$, $S(r)$ is connected. Hence we can pick a finite sequence $F_1, \ldots, F_k$ of distinct sets E_j such that $x \in F_1$, $y \in F_k$ and $F_j \cap F_{j+1} \cap S(r) \neq \emptyset$ for all $1 \leq j \leq k - 1$. Choose points $x_j \in F_j \cap F_{j+1} \cap S(r)$. Then

$$|f(x)| \leq H|f(x_1)| \leq H^2|f(x_2)| \leq \ldots \leq H^k|f(y)|.$$

Thus $H(0, f) \leq H^k \leq H^s$. □

References

[AVV] G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen, Dimension-free quasiconformal distortion in n-space, Trans. Amer. Math. Soc. 297, 1986, 687–706

[Ge] F.W. Gehring, Extension of quasiisometric embeddings of Jordan curves, Complex Variables 5, 1986, 245–263

[GM] F.W. Gehring and O. Martio, Quasiextremal distance domains and extension of quasiconformal mappings, J. Analyse Math. 45, 1985, 181–206

[GO] F.W. Gehring and B.G. Osgood, Uniform domains and the quasi-hyperbolic metric, J. Analyse Math. 36, 1979, 50–74

[GP] F.W. Gehring and B.P. Palka, Quasiconformally homogeneous domains, J. Analyse Math. 30, 1976, 172–199

[MN] O. Martio and R. Näkki, Boundary Hölder continuity and quasiconformal mappings, preprint

[Ri] S. Rickman, Removability theorems for quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. A I Math. 449, 1969, 1–8

[TV] P. Tukia and J. Väisälä, Quasisymmetric embeddings of metric spaces, Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 97–114

[Vä$_1$] J. Väisälä, Lectures on n-dimensional quasiconformal mappings, Lecture Notes in Mathematics 229, Springer-Verlag, 1971

[Vä$_2$] J. Väisälä, Piecewise quasiconformal maps are quasiconformal, Ann. Acad. Sci. Fenn. Ser. A I Math. 1, 1975, 3–6

[Vä$_3$] J. Väisälä, Quasi-symmetric embeddings in euclidean spaces, Trans. Amer. Math. Soc. 264, 1981, 191–204

[Vä$_4$] J. Väisälä, Quasimöbius maps, J. Analyse Math. 44, 1984/85, 218–234.

[Vä$_5$] J. Väisälä, Quasiconformal maps of cylindrical domains, Acta Math. 162, 1989, 201–225

Matematiikan laitos
Helsingin yliopisto
Hallituskatu 15
SF-00100 Helsinki
Finland

(Received January 12, 1990)