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Regularity of Ideals and their Radicals

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Abstract. In this paper we compare the regularity reg I of a homogeneous ideal $I \subset K[x_1, ..., x_n]$ with that of its radical. We prove that reg $I \geq$ reg $\sqrt{I}$ if R/I is a Buchsbaum R–module or if I is a monomial ideal. We also prove the same result when $\sqrt{I}$ defines a non–singular curve in P^3 under some additional hypotheses

Introduction. Let $R = K[x_1, .., x_n]$ be the graded polynomial ring. Recall that a graded R-module, N is said to be k-regular if $[H^i_M(N)]_j = 0$ for all $i, j \in Z$ such that $i+j \geq k+1$. Here $M = (x_1, .., x_n)$ denotes the maximal ideal of R. The regularity of N denoted by reg N is defined to be the smallest integer k for which N is k-regular. Recently there has been interest in finding upper bounds on the regularity of a homogeneous ideal I of R. The upper bounds depend on whether I is a radical ideal. For instance, if I is a saturated ideal defining a reduced, one-dimensional subscheme X of P^{n-1} then the regularity of I is at most linear in the degree of X ([GLP]). On the other hand, for a general one-dimensional subscheme X of degree d and arithmetic genus g, the best possible upper bound on the regularity of the saturated ideal defining X is given by $d(d-1)/2+1-g$ ([B],[G]). In view of this it seems reasonable to us to compare the regularity of an ideal I with that of its radical $\sqrt{I}$. In particlar, we address the following question:

QUESTION. *Let I be a homogeneous ideal of R. Is it always true that reg $(\sqrt{I}) \leq$ reg (I)?*

We have not yet been able to answer this question in its complete generality. In this paper, we shall answer the question in the affirmative in the following cases:

(1) $\frac{R}{\sqrt{I}}$ is a Buchsbaum R-module (Prop. 1.1).
(2) I is a monomial ideal (Thm 3.4).
(3) In some cases when $\sqrt{I}$ defines a curve in P^3 (Section 4).

1. Buchsbaum case. For details on the definition and properties of Buchsbaum modules, we refer to [S-V,1].

PROPOSITION 1.1. *Let $I \subset R$ be a homogeneous ideal such that R/I is a Buchsbaum R-module of depth at least 1. Let $J \subset I$ be another homogeneous ideal such that* $\dim(R/I) = \dim(R/J)$. *Then* $reg(J) \geq reg(I)$.

PROOF: By induction on $\dim(R/I)$. Since K is an infinite field, there exists a linear form h in R which is generic for I and J ([B-St],1.5).

(1) If $\dim(R/I) = 1$ then $(J,h) \subset (I,h)$ and $\dim R/(I,h) = \dim R/(J,h) = 0$. If $\mathrm{reg}(J,h) = m$, then $[(J,h)]_m = [R]_m$ ([B-St],1.7). Therefore $[(I,h)]_m = [R]_m$ and hence (I,h) is also m-regular. Since the depth of $R/I \geq 1$, I is a saturated ideal. So by ([B-St],1.8) I is also m-regular. Hence $\mathrm{reg}\,(I) \leq \mathrm{reg}\,(J,h)$. But $\mathrm{reg}\,(J,h) \leq \mathrm{reg}\,(J)$.

(2) If $\dim(R/I) \geq 1$ then $R/(I,h) : M$ is also a Buchsbaum module of depth greater than or equal to one. Therefore by the induction hypothesis $\mathrm{reg}\,(J,h) \geq \mathrm{reg}\,((I,h) : M)$. Now by ([S-V 2], Lemma 2) $\mathrm{reg}\,((I,h) : M) = \mathrm{reg}\,(I)$. Hence $\mathrm{reg}\,(J) \geq \mathrm{reg}\,(J,h) \geq \mathrm{reg}\,(I)$. ∎

COROLLARY 1.2. *Let X contained in P^n be an arithmetically Buchsbaum scheme. Let Y be another subscheme of P^n such that $\dim Y = \dim X$ and $X \subset Y$. Then* $reg\,(\mathcal{I}_X) \geq reg\,(\mathcal{I}_Y)$.

2. Gröbner basis algorithm. Before we prove our result for monomial ideals we wish to recall some facts about the algorithmic construction of free resolutions using the Gröbner basis algorithm. This construction is originally due to [Spe], [Sch] and [Z].Further details of this construction can also be found in [B] and [M-M]. Here we shall only define the terms and state the facts we shall be using in our proof.

Let $M = \bigoplus_{i=1}^{l} R(-m_i)$ be a free module over the polynomial ring. Let $e_1,..,e_l$ be a canonical basis for M. We first define a total ordering on the monomials of R which satisfy the following two properties :

(1) If the total degree of X^α is less than the total degree of X^β then $X^\alpha < X^\beta$.

(2) If $X^\alpha < X^\beta$ then for any non-zero monomial X^γ, $X^\alpha X^\gamma < X^\beta X^\gamma$.

There are many such orders, for our purpose we fix one such order. We can extend this order to 'monomials' in M, that is elements of M of the form $X^\alpha e_i$, by defining an order on the e_i, say

$$e_1 < e_2 < .... < e_l.$$

Define $X^\alpha e_i < X^\beta e_j$ if either $i < j$ or $i = j$ and $X^\alpha < X^\beta$. Now, given any $h \in M$ define in(h) to be the largest monomial occurring in h with non-zero coefficient. Let T be an ordered finite subset of M. Given any $h \in M$ we define rem (h) mod T recursively as follows:

(1) Let $k = h$.

(2) If $k = 0$ or there is no $f \in T$ such that in(k) is a multiple of in(f) then rem (h) mod $T = 0$.

(3) Let f be the smallest element in T such that in$(f)aX^\alpha =$ in(k). Let $l = k - aX^\alpha f$.

(4) Set $k = l$ and proceed to Step 2.

$T \subset M$ is called a Gröbner basis for a submodule P of M if $T \subset P$ and for any $h \in P$, rem (h) mod $T = 0$.

Given $X^\alpha = x_1^{\alpha_1}..x_n^{\alpha_n}$ and $X^\beta = x_1^{\beta_1}..x_n^{\beta_n}$ we define $X^\alpha \vee X^\beta = X^\gamma$ where $\gamma_i = \max\{\alpha_i, \beta_i\}$. We also denote γ as $\alpha \vee \beta$. Given $h, k \in M$ we define hSk as follows:

(1) Let $h = aX^\alpha e_i +$ terms of lower order with respect to the total order $<$. Similarly let the highest term in k be $bX^\beta e_j$. Here a and b are in K.

(2) If $h = 0$ or $k = 0$ or $i \neq j$ then $hSk = 0$.

(3) If $i = j$, define δ and γ by $\alpha \vee \beta = \alpha + \gamma = \beta + \delta$. Define $hSk = X^\gamma h - \frac{a}{b} X^\delta k$.

FACT 1. *If $T \subset M$ is a finite subset such that for any $f, g \in T$, rem (fSg) mod $T = 0$, then T is a Gröbner basis for the submodule generated by T.*

Let $T = \{f_1, ..., f_r\}$ be a Gröbner basis for a submodule P of M. Then for each pair

$1 \leq i < j \leq r$, we have an expression of the form $f_iSf_j = \sum_{k=1}^r h_k^{ij} f_k$, which in turn gives a relation among the f_i of the form $\sum_{k=1}^r g_k^{ij} f_k = 0$. Let $M' = \bigoplus_{i=1}^{r} R(-m_i)$ where $m_i = \deg(f_i)$. We define a map ϕ from M' to P by sending the i^{th} basis element d_i to f_i. Then ϕ is onto and further, for each pair $1 \leq i < j \leq r$, $s_{ij} = \sum_{k=1}^r g_k^{ij} d_k \in \mathrm{Ker}(\phi)$.

FACT 2. *With notation as above Ker(ϕ) is generated by* $T = \{s_{ij} | 1 \leq i < j \leq r\}$.

We can make the map ϕ, minimal by the following process: If d_r (say) occurs with an invertible coefficient a in a syzygy $s \in T$, then we set $M' = \bigoplus_{i=1}^{r-1} R'(-m_i)$ and define a new map, also called ϕ, by sending the i^{th} basis element d_i to f_i. This map to P is also surjective and further $\mathrm{Ker}(\phi)$ is generated by

$$T' = \{\sum_{k=1}^{r-1} h_i d_i + \frac{h_r}{a}(-t + ad_r) | \; \forall s = \sum_{1}^{r} h_k d_k \in T\}.$$

We can continue this process until none of the d_i occur with an invertible coefficient in any of the syzygies. At this stage ϕ is minimal, that is, $\mathrm{Ker}(\phi) \subset (x_1, ..., x_n).M'$.

3. Regularity of monomial ideals. Let $I \subset R$ be a monomial ideal, that is, I has a set of generators of the form $\{X^{\alpha_i}\}_1^r$. We assume that this is a minimal set of generators for I. Let $\Gamma = N^n$. Let $\alpha_i = (\alpha_{i1}, ..., \alpha_{in})$ and $k_j = \max_i\{\alpha_{ij}\}$ for each $j = 1, .., n$. Let $a = \sum k_j$ and $\Delta = N^a$. Let $R' = K[x_{11}, x_{12}, .., x_{1k_1}, x_{21}, ..., x_{2k_2}, .., x_{nk_n}]$ and $J \subset R'$ be a monomial ideal with a set of generators given by $\{X^{\beta_i}\}_1^r$ where $\beta_i = (\beta_{i11}, .., \beta_{i1k_1}, \beta_{i21}, ...\beta_{ink_n}) \in \Delta$ and

$$\beta_{ijl} = \begin{cases} 1 & \text{if } \alpha_{ij} \geq l \\ 0 & \text{otherwise.} \end{cases}$$

Hence J is a squarefree monomial ideal. Our main object in this section is to show that $\mathrm{reg}\,(J) = \mathrm{reg}\,(I) \geq \mathrm{reg}\,(\sqrt{I})$. Define $\phi : R' \to R$ by

$$\phi(x_{i1}) = x_i \text{ and } \phi(x_{ij}) = 1 \text{ if } j > 1.$$

Further let ψ from R' to R be defined by

$$\psi(x_{ij}) = x_i \text{ for all } j.$$

We first note that $\phi(J) = \sqrt{I}$ and $\psi(J) = I$. Further let

$$S = \{(\alpha_{11}, .., \alpha_{nk_n}) \in \Delta | \alpha_{ij} \leq 1 \text{ and for each } i,\ \alpha_{ij} = 1 \implies \alpha_{ik} = 1 \text{ for } k < j\}$$

and

$$T = \{(\beta_1, .., \beta_n) \in \Gamma | \beta_i \leq k_i\}.$$

Now ϕ and ψ can be considered as maps from S to T in the obvious way. Note that ψ is a 1-1, onto map from S to T.

DEFINITION 3.1. *We say that $\{M', P', M, P\}$ is ϕ-compatible if the following conditions hold:*

1. $M' = \bigoplus_1^l R'(-m_i)$ *with a canonical basis $e'_1, .., e'_l$. Each e'_i has a degree in Δ denoted by $\deg_\Delta(e'_i)$. For each i, $\deg_\Delta(e'_i) \in S$ and $m_i =$ total degree of e'_i.*
2. $M = \bigoplus_1^l R(-n_i)$ *with a canonical basis $e_1, .., e_l$. $\deg_\Gamma(e_i) = \phi(\deg_\Delta(e'_i))$. $n_i =$ total degree of e_i. Note that $n_i \leq m_i$.*
3. *P' is a Δ-homogeneous submodule of M'. There exists a set of Δ-homogeneus generators $\{h'_i\}_1^r$ for P' with $\deg_\Delta(h'_i) \in S$.*
4. *If $\phi : M' \to M$ is defined by*

$$\phi(\sum f_i e'_i) = \sum \phi(f_i) e_i.$$

then $P = \phi(P')$, and P is generated by $\{h_i = \phi(h'_i)\}_1^r$.

With the above setting we would first like to remark that the h_i are Γ-homogeneous. We can define the concept of ψ-compatibility in an analogous way. The main difference in this case is that the total degree of e'_i equals the total degree of e_i.

LEMMA 3.2. *Let $\{M', P', M, P\}$ be ϕ-compatible. Then we can construct a commutative diagram as follows:*

$$\begin{array}{ccccc} M'_1 & \xrightarrow{\sigma} & P' & \longrightarrow & 0 \\ \phi\downarrow & & \phi\downarrow & & \\ M_1 & \xrightarrow{\tau} & P & \longrightarrow & 0 \end{array}$$

where σ is minimal and $\{M_1', Ker(\sigma), M_1, Ker(\tau)\}$ is ϕ-compatible.

PROOF: Let $\{h_j'\}$ be the Δ-homogeneous generators for P' assured by definition 3.1. Since h_j' is Δ-homogeneous, it is of the form

$$h_j' = \sum_1^l a_{ji} X^{\alpha_{ji}} e_i'$$

where

$$\alpha_{ji} \vee \deg{}_\Delta(e_i') = \deg{}_\Delta(h_j') \text{ for all } i = 1, .., l \text{ and } a_{ji} \in K.$$

Let $h_i = \phi(h_i')$. Let us order the basis elements of M' and M consistently, say

$$e_1' < e_2' < \ldots < e_l' \text{ and } e_1 < e_2 \ldots < e_l.$$

Now $\phi(\text{in}(h_j')) = \text{in}(\phi(h_j')) = \text{in}(h_j)$. Hence from the definition of $h_i S h_j$ it is clear that $\phi(h_i' S h_j') = h_i S h_j$. Also, $h_i' S h_j'$ are Δ-homogeneous and $\deg{}_\Delta(h_i' S h_j') \in S$. Further, for any two Δ-homogeneous elements g' and h' in M', $\text{in}(g')$ divides $\text{in}(h')$ implies that $\text{in}(\phi(g'))$ divides $\text{in}(\phi(h'))$. Hence, we can construct a Gröbner basis $\{h_i'\}_1^s$ for P' such that $\{\phi(h_i') = h_i)\}_1^s$ forms a Gröbner basis for P. Define $M_1' = \bigoplus_1^s R'(-m_i)$ with a canonical basis $d_1', .., d_s'$. Define σ from M_1' by sending d_i' to h_i'. Define $\deg{}_\Delta d_i' = \deg{}_\Delta h_i'$. By the observations above we can ensure that if $t' = \sum_1^s g_i' d_i'$ is a generator for $\text{Ker}(\sigma)$ provided by the algorithm, as described in Section 2, then $t = \sum \phi(g_i') d_i$ is the corresponding generator for $\text{Ker}(\tau)$, with τ and d_i defined analagous to σ and d_i'. Also note that t' is Δ-homogeneous. Now, we can use the method described in Section 2 to make the map σ minimal. In this process, if we eliminate d_i' because it occurs with an invertible coefficient in a syzygy t' then d_i also occurs with an invertible coefficient in $\phi(t')$. The new generators obtained remain Δ-homogeneous with their degree in S. Hence, eventually we can make σ minimal with $\{M_1', \text{Ker}(\sigma), M_1, \text{Ker}(\tau)\}$ being ϕ-compatible. ∎

LEMMA 3.3. *Let $\{M', P', M, P\}$ be ψ-compatible. Then we can construct a commutative diagram as follows :*

$$\begin{array}{ccccc} M_1' & \xrightarrow{\sigma} & P' & \longrightarrow & 0 \\ \psi\downarrow & & \psi\downarrow & & \\ M_1 & \xrightarrow{\nu} & P & \longrightarrow & 0 \end{array}$$

where both σ and ν are minimal. Further, $\{M_1', Ker(\sigma), M_1, Ker(\nu)\}$ is ψ-compatible.

PROOF: The proof follows the same lines as that of the previos lemma. The main observation is that $\psi(\text{in}(h_i')) = \text{in}(\psi(h_i'))$ and therefore $\psi(h_i' s h_j') = \psi(h_i') S \psi(h_j')$ and also $\deg_\Delta(h_i' S h_j') \in S$. The rest of the proof is identical to the proof of Lemma 3.2 except for the additional observation that a generator $\psi(h_i')$ for P occurs in a syzygy $\psi(t')$ with an invertible coefficient, if and only if, the coefficient of h_i' in t' is invertible. Hence the map ν is also minimal. ∎

THEOREM 3.4. *Let $I \subset R$ be a monomial ideal. Then reg $(I) \geq$ reg $(\sqrt{I})$.*

PROOF: Let $\{X^{\alpha_i}\}_1^r$ be a minimal set of generators for I. We construct $J \subset R'$ as indicated before. As already noted $\psi(J) = I$ and $\phi(J) = \sqrt{I}$. We can apply Lemma 3.2 to $\{R', J, R, I\}$ to get:

$$\begin{array}{ccccc} M_0' & \xrightarrow{\sigma_0} & J & \longrightarrow & 0 \\ \phi\downarrow & & \phi\downarrow & & \\ M_0 & \xrightarrow{\tau_0} & \sqrt{I} & \longrightarrow & 0 \end{array}$$

where $M_0' = \bigoplus_i R'(-m_{0,i})$ and $M_0 = \bigoplus_i R(-n_{0,i})$ and $m_{0,i} \geq n_{0,i}$. Here the map σ is minimal. But this construction can be iterated to give the following :

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M_k' & \xrightarrow{\sigma_k} & \cdots & \longrightarrow & M_0' & \xrightarrow{\sigma_0} & J & \longrightarrow & 0 \\ & & \phi\downarrow & & & & \phi\downarrow & & \phi\downarrow & & \\ 0 & \longrightarrow & M_k & \xrightarrow{\tau_k} & \cdots & \longrightarrow & M_0 & \xrightarrow{\tau_0} & \sqrt{I} & \longrightarrow & 0 \end{array}$$

where $M_j' = \bigoplus_i R'(-m_{j,i})$ and $M_j = \bigoplus_i R(-n_{j,i})$. Here the top row is a minimal resolution of J and the bottom row is a resolution of $\sqrt{I}$. Hence by [B-St]

$$\text{reg }(J) = \max_{i,j}(m_{j,i} - j) \geq \max_{i,j}(n_{j,i} - j) \geq \text{ reg }(\sqrt{I}).$$

Similarly, using Lemma 3.2 we get :

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M_k' & \xrightarrow{\sigma_k} & \cdots & \longrightarrow & M_0' & \xrightarrow{\sigma_0} & J & \longrightarrow & 0 \\ & & \psi\downarrow & & & & \psi\downarrow & & & & \\ 0 & \longrightarrow & N_k & \xrightarrow{\nu_k} & \cdots & \longrightarrow & N_0 & \xrightarrow{\nu_0} & I & \longrightarrow & 0 \end{array}$$

where both the top and bottom rows are minimal resolutions. Here $N_j = \bigoplus_i R(-m_{j,i})$. Hence

$$\text{reg}\,(J) = \max_{i,j}(m_{j,i} - j) = \text{reg}\,(I). \quad \blacksquare$$

REMARK. *The fact that reg $(J) =$ reg (I) has also been proved in [D-E-P].*

4. Non-reduced structures on space curves. From now on let X contained in P^3 be a non-singular, non-degenerate curve over the complex numbers of degree d and genus g. Let $\mathcal{I}$ be the ideal sheaf of X in P^3.

The conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf of rank two on X of degree $-4d - 2g + 2$. We first recall the following results:

THEOREM ([H-S]). *Let N_X be the normal bundle of X and let $\mathcal{L}$ be an invertible subsheaf of N_X. If $g \geq 1$ then deg $\mathcal{L} \leq 3d + 2g - 5$.*

THEOREM ([E-V]). *Let X contained in P^3 be a smooth rational curve of degree d. Then*

$$\frac{\mathcal{I}}{\mathcal{I}^2} \simeq O_{P^1}(-2d + 1 - a) \oplus O_{P^1}(-2d + 1 + a)$$

where $0 \leq a \leq d - 4$.

PROPOSITION 4.1. *Let X contained in P^3 be a non-singular curve of degree d and genus g such that $d \geq g$. Then $\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}}$ is $(5n + 3)$-regular.*

PROOF: Let $\mathcal{M}$ be an invertible subsheaf of $\frac{\mathcal{I}}{\mathcal{I}^2}$ of maximal degree. Then by ([Gu],p.81) deg $\mathcal{M} \geq -2d - 2g + 1$. Let $\mathcal{L}$ be the quotient sheaf. Then by the results mentioned above deg $\mathcal{L} \geq -3d - 2g + 5$. So $\frac{\mathcal{I}}{\mathcal{I}^2}$ is an extension of locally free sheaves of degree $\geq -3d - 2g + 5$.

Now $\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \simeq \text{Sym}^n \frac{\mathcal{I}}{\mathcal{I}^2}$. Hence there is a filtration of $\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}}$ as follows:

$$\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} = F_0 \supset F_1 \supset \ldots. \supset F_{n+1} = 0$$

where $F_i/F_{i+1} \simeq \mathcal{M}^{\otimes i} \otimes \mathcal{L}^{\otimes(n-i)}$ for $i = 0, ..n$ ([Ha],2,Ex.4.12). Now by the hypothesis that $d \geq g$ we have that deg $\mathcal{M}^{\otimes i} \otimes \mathcal{L}^{\otimes(n-i)} \otimes O_{P^3}(5n+2) \geq 2g-1$. Hence $H^1(P^3, \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}}(5n+2)) = 0$. Hence it is $(5n+3)$-regular. ∎

REMARK 4.2. *By using the Riemann-Roch theorem on* $\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}}$ *and induction on* n *it can be shown that*

$$\chi(\mathcal{I}^n(m)) = \binom{m+3}{3} - \frac{md(n+1)}{2} + (2d+g)[\frac{n(n-1)}{2} + \frac{(n-1)(2n-1)n}{6}] + (g-1)\frac{n(n+1)}{2} - \frac{n(n-1)}{2} - \frac{n(n-1)(2n-1)}{6}.$$

THEOREM 4.3. *Let* X *be a non-singular curve of degree* d *and genus* g, *such that* $d \geq g$ *and* $d \geq 20$. *Let* $\mathcal{I}$ *be the ideal sheaf of* X *in* P^3. *Let* $\tilde{X}$ *be a non-reduced structure on* X *whose ideal sheaf in* P^3, $\mathcal{J}$ *is such that* $\mathcal{I}^{n+1} \subset \mathcal{J} \subset \mathcal{I}^n$ *for some* n. *Then* reg $\mathcal{J} \geq$ reg $\mathcal{I}^n \geq$ reg $\mathcal{I}$.

PROOF: From the sequence

$$0 \to \frac{\mathcal{J}}{\mathcal{I}^{n+1}} \to \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \to \frac{\mathcal{I}^n}{\mathcal{J}} \to 0$$

and Prop. 4.2 we get that $\frac{\mathcal{I}^n}{\mathcal{J}}$ is also $(5n+3)$-regular. Therefore from the sequence:

$$0 \to \mathcal{J} \to \mathcal{I}^n \to \frac{\mathcal{I}^n}{\mathcal{J}} \to 0$$

we get that $H^1(P^3, \mathcal{I}^n(m))$ is a quotient of $H^1(P^3, \mathcal{J}(m))$ for all $m \geq 5n+2$. Now we can check by a direct computation with the formula for $\chi(\mathcal{I}^n(m))$ in Remark 4.3 that $\chi(\mathcal{I}^n(5n+2)) < 0$. Hence $H^1(P^3, \mathcal{I}^n(5n+2)) \neq 0$. Further, for any m $H^2(\mathcal{I}^n(m)) \neq 0 \implies H^2(P^3, \mathcal{J}(m)) \neq 0$. Hence reg $\mathcal{J} \geq$ reg $\mathcal{I}^n$. ∎

REMARK. *The restriction that* $d \geq 20$ *can be dropped if* $g \geq 1$ *by replacing* $(5n+3)$ *by* $(3n+1)$ *in the calculations above.*

Recall that a line L is said to be an m-secant line to X if it intersects X at finitely many points and $H^0(P^3, \frac{O_{P^3}}{\mathcal{I}+\mathcal{I}_L}) \geq m$. It is well known that if a curve X has an m-secant line then reg $\mathcal{I} \geq m$. Now if $\tilde{X}$ is any non-reduced structure on X then the multiplicity of L along $\tilde{X}$ is at least m. Hence reg $\mathcal{I}_{\tilde{X}} \geq m$. Now we recall a result of D'Almeida:

THEOREM ([D]). *Let X be an irreducible non-singular curve in P^3. If X lies on a surface of degree ≤ 3, then $H^1(P^3, \mathcal{I}(n)) \neq 0$ implies that X has a $(n+2)$-secant.*

Remark: Though the statement of the result in [D] is different from the above, it is clear from the proof that the above statement is proved there.

COROLLARY 4.6. *Let X be a non-singular curve in P^3 lying on a surface of degree ≤ 3. Then for any non-reduced structure $\tilde{X}$ on X, reg $\mathcal{I} \leq$ reg $\mathcal{I}_{\tilde{X}}$.*

PROOF: If $H^1(P^3, \mathcal{I}(n)) \neq 0$ then by the above theorem X has an $(n+2)$-secant line. So by our previous observation reg $\mathcal{I}_{\tilde{X}} \geq n+2$. On the other hand, $H^2(P^3, \mathcal{I}(n)) \neq 0$ always implies that $H^2(P^3, \mathcal{I}_{\tilde{X}}(n)) \neq 0$. Hence reg $\mathcal{I} \leq$ reg $\mathcal{I}_{\tilde{X}}$. ∎

Remark: The above Corollary implies that any curve of type (m, n) on a smooth quadric surface in P^3 cannot be a set-theoretic complete intersection of surfaces of degrees e and f if $e + f - 1 \leq m$.

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