manuscripta math. 67, 99 - 112 (1990)

manuscripta mathematica

A PRIORI BOUNDS AND NECESSARY CONDITIONS FOR SOLVABILITY OF PRESCRIBED CURVATURE EQUATIONS

Neil S. Trudinger

We prove an estimate for the magnitude of solutions of the prescribed higher order mean curvature equations and examine the necessity of our conditions. Our results include well known sharp estimates for the mean and Gauss curvature and our previous estimate for scalar curvature as special cases.

In the article [12], we formulated and proved in the scalar curvature case $m = 2$, an estimate for the magnitude of classical solutions of prescribed curvature equations of the form

$$H_m[u] = \psi , \quad m = 1,\dots,n , \tag{1}$$

in domains Ω in Euclidean n space $\mathbb{R}^n$, where $H_m[u]$ denotes the m mean curvature function of the graph of the function $u \in C^2(\Omega)$ and ψ is a given non-negative function on Ω. In this paper we establish such a result for the remaining cases and deduce from it, with aid of our first derivative estimates in [13] and the recent second derivative estimates of Caffarelli, Nirenberg and Spruck [3], Ivochkina [6,7], sharp existence theorems for the classical Dirichlet problem for equation (1), analogous to those of Serrin [11] for the mean curvature case, $m = 1$, and Trudinger and Urbas [15] for the Gauss curvature case, $m = n$. By relaxing our hypotheses, we also infer by approximation, existence theorems for weak (or viscosity) solutions, as introduced in [13].

We recall from [12], that if $\kappa = (\kappa_1,\dots,\kappa_n)$ denotes the principal curvatures of the graph of u, S, then H_m is given by

(2) $$H_m[u] = H_m[S] = H_m(\kappa) = \Sigma\kappa_{i_1}\ldots\kappa_{i_m} ,$$

where the sum is taken over all increasing m–tuples, $(i_1,\ldots,i_m) \subset (1,\ldots,n)$. Adapting the terminology of [3], we call a function $u \in C^2(\Omega)$, H_m admissible (or simply m admissible) if the principal curvatures $\kappa = (\kappa_1,\ldots,\kappa_n)$ of its graph S lie in the closure $\overline{K}_m$ of the convex cone $K_m = K_{m,n}$ in $\mathbb{R}^n$, given by

(3) $$\begin{aligned} K_{m,n} &= \left\{\kappa \in \mathbb{R}^n \;\middle|\; H_j(\kappa) > 0,\ j = 1,\ldots,m\right\} \\ &= \left\{\kappa \in \mathbb{R}^n \;\middle|\; H_m(\kappa+\eta) \geq H_m(\kappa) > 0 \ \ \forall\, \eta \in K^+\right\}, \end{aligned}$$

where $K^+ = K_n$ is the positive cone in $\mathbb{R}^n$; (see [3],[6],[8]). Clearly the operator H_m is degenerate elliptic with respect to admissible functions, but also elliptic if their graph curvatures lie in K_m. The cone K_m may also be characterized as that component of the positivity set of the function H_m containing K^+[3]. We shall also refer to the boundary $\partial\Omega \in C^2$ as m *admissible* if its principal curvatures $\kappa' = (\kappa_1',\ldots,\kappa_{n-1}')$ lie in $\overline{K}_{m,n-1}$. Similarly to [12], we shall assume that Ω is bounded, with boundary $\partial\Omega \in C^2$ and that ψ is a bounded, non–negative integrable function on Ω satisfying

(4) $$m\int_E \psi \;\leq\; (1-\chi)\int_{\partial E} H_{m-1}[\partial E] ,$$

for all subdomains $E \subset \Omega$ with (m–1) admissible boundary $\partial E \in C^2$, and for some positive constant χ. When $m = 1$, we set $H_o = H_{m-1} \equiv 1$, $K_o = K_{o,n} = \mathbb{R}^n$. We shall also assume here that $\partial\Omega$ itself is (m–1) admissible, whence (4) also holds for $E = \partial\Omega$.

Theorem 1. *Let* $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *be an admissible solution of the differential inequality*

(5) $$H_m[u] \leq \psi$$

in Ω *. Then we have the estimate*

(6) $$\inf_{\partial\Omega} u - C \le u \le \sup_{\partial\Omega} \psi$$

in Ω, *where* C *is a positive constant depending only on* n, m, χ, $\sup_{\Omega} \psi$ *and* Ω.

Proof of Theorem 1. The upper bound in (6) is an immediate consequence of the assumed m admissibility of u . To get the lower bound we shall employ, as in [12], a method based on Moser iteration but instead of using the full strength of the special Sobolev type inequality, [12], Lemma 2, we make do with a relatively simpler Poincaré type inequality. The details of the present proof will be technically more intricate than in [12]. We adopt similar notation to [12], so that if $a = [a_{ij}]$ is an $n \times n$ matrix, we let

(7) $$A_m(a) = [a]_m$$

denote the sum of its $m \times m$ principal minors, (with $A_0 = 1$), and set

(8) $$A_m^{ij}(a) = \frac{\partial A_m}{\partial a_{ij}}(a).$$

For graphs S of functions $u \in C^2(\Omega)$, we compute curvature with respect to the downwards directed normal,

(9) $$(\nu, \nu_{n+1}) = (\frac{Du}{v}, \frac{-1}{v}), \quad v = \sqrt{1+|Du|^2}.$$

The principal curvatures are then the eigenvalues of the Jacobian matrix $D\nu$, so that we have the formulae,

(10) $$H_m[u] = H_m[S] = [D\nu]_m = A_m(D\nu).$$

For boundaries $\partial E \in C^2$ of domains $E \subset \mathbb{R}^2$, we let γ denote the unit outer normal to ∂E with $\kappa_1, \ldots, \kappa_{n-1}$ the principal curvatures of ∂E given by the eigenvalues of $D\gamma$, excluding zero, and

(11) $$H_m[\partial E] = [D\gamma]_m \cdot$$

If g is any continuously differentiable vector field on Ω , we can write $[Dg]_m$ in the divergence form

(12) $$A_m[D_g] = \frac{1}{m} D_i[A_m^{ij} g_j],$$

while, as proved in [12],

(13) $$A_m^{ij}(Dg) g_i g_j = |g|^{m+1} [D\tilde{g}]_{m-1},$$

whenever $g \neq 0$, $\tilde{g} = g/|g|$. When we substitute $g = \nu$ in (13), we obtain

(14) $$A_m^{ij} \nu_i D_j u = \left[\frac{|Du|}{v}\right]^m C_m[u]$$

where

(15) $$C_m[u] = \begin{cases} |Du|^{2-m} [D_{ij}u - \gamma_i \gamma_k D_{kj} u]_{m-1} & \text{if } |Du| \neq 0 \\ 0 & \text{if } Du = 0 \end{cases}$$

and $\gamma = Du/|Du|$. Since H_m is degenerate elliptic with respect to u, we have $C_m[u] \geq 0$.

We now proceed as in [12] by replacing u by $u - \inf_{\partial\Omega} u$, so that $u \geq 0$ on $\partial\Omega$ and denote $\Omega_o = \{u < 0\} \subset \Omega M$. Integrating (14) over Ω_o and using (12), we then obtain

(16) $$\int_{\Omega_o} f'(-u) \left[\frac{|Du|}{v}\right]^m C_m[u] = m \int_{\Omega_o} f(-u)\psi$$

for any $f \in C^1(\mathbb{R})$, $f' \geq 0$, $f(0) = 0$. Choosing initially the function $f(t) = t$, we conclude, precisely as in [12], the preliminary estimate

(17) $$\begin{aligned} \int_{\Omega_o} C_m[u] &\leq \frac{m}{\chi} \int_{\Omega_o} [D\gamma]_{m-1} \\ &\leq \frac{m}{\chi(m-1)} \int_{\partial\Omega_o} H_{m-2}[\partial\Omega_o] \end{aligned}$$

where $[D\gamma]_{m-1}$ is defined to vanish when $Du = 0$. (Note that by Sard's theorem, there is no loss of generality in assuming $\partial\Omega_o \in C^2$ is a non-degenerate level surface of u). The last inequality in (17) follows by application of [12], Lemma 3 to the approximations

$$\gamma_\epsilon = \frac{Du}{v_\epsilon}, \quad v_\epsilon = \sqrt{\epsilon^2 + |Du|^2}, \quad \epsilon > 0,$$

noting that the m admissibility of u implies the non-negativity of $[D\gamma_\epsilon]_{m-1}$ for $0 \le \epsilon \le 1$. Next, for any $\delta \in (0,1)$, we may estimate from (16),

$$\int_{|Du|>\delta} f'(-u)C_m[u] \le m\, 2^{m/2}\delta^{-m} \int_{\Omega_0} f(-u)\psi. \tag{18}$$

But we may also estimate

$$\begin{aligned}\int_{|Du|\le\delta} f'(-u)C_m[u] &\le \delta \int_{\Omega_0} f'(-u)[D\gamma]_{m-1} \\ &\le \frac{\delta}{m-1}\int_{\Omega_0} f''(-u)C_{m-1}[u] \\ &\le \frac{\delta}{m-1}\int_{\Omega_0} C_{m-1}[-f'(-u)]\end{aligned} \tag{19}$$

provided $f \in C^2(\mathbb{R}^+)$, $f'(0) = f(0) = 0$. To proceed further we use the following Poincaré type inequality.

LEMMA 2 *For any* $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *with* $v = 0$ *on* $\partial\Omega$, $C_m[v] \ge 0$ *in* Ω, *we have*

$$\int_\Omega C_{m-1}[v] \le \frac{(m-1)R}{(n-m+1)} \int_\Omega C_m[v] \tag{20}$$

where $R = \frac{1}{2}\mathrm{diam}\,\Omega$ *and* $1 < m \le n-1$.

The proof of Lemma 2 if provided at the end of that of Theorem 1. We shall also need the follwing Sobolev type inequality which arises on combination of Lemma 2 and the usual Sobolev inequality, ([5], Theorem 7.10),

$$\|v\|_{\frac{n}{n-1}} \le \frac{c_n R^{m-1}}{\binom{n-1}{m-1}} \int_\Omega C_m[v] \tag{21}$$

where the constant c_n can be taken to be the isoperimetric constant $(n\omega_n)^{-1}$.

Returning to the proof of Theorem 1, we combine (19) and (20) to estimate

$$(22)\qquad \int_{|Du|\le\delta} f'(-u)C_m[u] \le \frac{\delta R}{(n-m+1)}\int_{\Omega_0} f''(-u)C_m[u]$$

provided $m < n$. Selecting the same function f as in [12], namely

$$(23)\qquad f(t) = (1+t)^\beta - \beta t - 1, \quad \beta > 1 \quad ,$$

we then obtain from (18) and (22), for $w = 1-u$,

$$(24)\qquad \beta\int_{\Omega_0} (w^{\beta-1}-1)C_m[u] \le C\left\{\delta^{-m}\int_{\Omega_0} \psi\omega^\beta + \beta(\beta-1)\delta\int_{\Omega_0} \omega^{\beta-2}C_m[u]\right\}$$

where C depends on n , m and R . Choosing

$$(25)\qquad \delta = \frac{1}{2C(\beta-1)} ,$$

we then deduce from (24) and (17),

$$(26)\qquad \begin{aligned}\int_\Omega C_m(-f(-u)) &= \beta\int_\Omega (\omega^{\beta-1}-1)C_m[u] \\ &\le C\beta^m\int_\Omega (1+\psi)\omega^\beta\end{aligned}$$

where now C depends on n , m and Ω_0 . Applying inequality (21), we obtain

$$(27)\qquad \|f(-u)\|_{\frac{n}{n-1}} \le C\beta^m\int_{\Omega_0} (1+\psi)\omega^\beta$$

and consequently, for any $\beta > 1$,

$$(28)\qquad \|w\|_{\frac{n\beta}{n-1}} \le (C\beta)^{m/\beta}\|w\|_\beta ,$$

where now the constant C depends on n , m , $\sup\psi$ and Ω_0 . Successive iteration of (28), from $\beta = n/(n-1)$, then yields our desired estimate

$$(29)\qquad \begin{aligned}\sup_\Omega w &\le C\|w\|_{\frac{n}{n-1}} \\ &\le C ,\end{aligned}$$

by virtue of (17) and (21). This completes the proof of Theorem 1, (at least in the case $\Omega_0 = \Omega$), except for Lemma 1 which we now treat.

Calculus on hypersurfaces

Lemma 1 follows from and extension to higher order curvatures of the basic integration formula for hypersurfaces ([10],[5],Lemma 16.1). Let $\mathcal{U}$ be an open set in $\mathbb{R}^n$ and suppose we have a C^2 hypersurface S in $\mathcal{U}$ which is represented as the level set of a function $\phi \in C^2(\mathcal{U})$ so that we can take $|D\phi| \neq 0$, $\phi = 0$ on S. The tangential gradient operator on S is defined by

$$\delta g = Dg - (\gamma \cdot Dg)\gamma \tag{30}$$

for any $g \in C^1(\mathcal{U})$, where

$$\gamma = \frac{D\phi}{|D\phi|}$$

is the unit normal to S (in the direction of increasing ϕ). It follows that the matrix $\delta\gamma$ is symmetric on S with eigenvalues $\kappa_1,..,\kappa_{n-1},0$ where $\kappa_1,...,\kappa_{n-1}$ are principal curvatures of S (with respect to γ), so that we have the following formulae for the higher order curvatures of S,

$$H_m[S] = [\delta\gamma] = A_m(\delta\gamma) \quad . \tag{31}$$

LEMMA 3 *Letting* dA *denote the area element in* S *and* $H_m^{ij}[S] = A_m^{ij}[\delta\gamma]$, *we have*

$$\int_S H_m^{ij}\delta_j g \, dA = m \int_S gH_m\gamma_i \, dA \tag{32}$$

for all $g \in C_0^1(\mathcal{U})$.

Proof We first establish the divergence formula,

$$\delta_j H_m^{ij} = H_{m-1}^{kl}(\delta_l\gamma_j)(\delta_k\gamma_j)\gamma_i \, . \tag{33}$$

To prove (33), we make use of the recursion formula,

$$A_m^{ij} = A_{m-1}\delta^{ij} - a_{jk}A_{m-1}^{ik} \tag{34}$$

so that, with $a = \delta\gamma$, we have

$$\delta_j H_m^{ij} = H_{m-1}^{kl}\delta_i\delta_k\gamma_l - H_{m-1}^{jk}\delta_j\delta_i\gamma_k - \delta_j(H_{m-1}^{jk})\delta_i\gamma_k \,. \tag{35}$$

If we assume that (33) is valid when m is replaced by $m-1$, then the last term in (35) vanishes and using the commutator formula [10],

$$\delta_i\delta_j - \delta_j\delta_i = (\gamma_i\delta_j\gamma_k - \gamma_j\delta_k\gamma_k)D_k \,, \tag{36}$$

we thus obtain

$$\begin{aligned} \delta_j H_m^{ij} &= H_{m-1}^{jk}(\delta_i\delta_j\gamma_k - \delta_j\delta_i\gamma_k) \\ &= H_{m-1}^{ij}(\gamma_i\delta_j\gamma_l - \gamma_j\delta_i\gamma_l)\delta_l\gamma_k \\ &= H_{m-1}^{jk}\delta_l\gamma_l\delta_j\gamma_k \end{aligned} \tag{37}$$

whence (33) follows for all $m \geq 1$ by induction. To derive (32), we now integrate (33) over S, thereby obtaining from [5], Lemma 16.1, (which corresponds to the case $m = 1$),

$$\begin{aligned} \int_S H_m^{ij}\delta_j g &= -\int_S (\delta_j H_m^{ij})g\,dA + \int_S HH_m^{ij}\gamma_j g\,dA \\ &= \int_S (HH_{m-1} - H_{m-1}^{ij}\delta_i\gamma_k\delta_j\gamma_k)g\gamma_i\,dA \\ &= m\int_S H_m g\gamma_i\,dA\,. \end{aligned} \tag{38}$$

To derive Lemma 2, it suffices to assume that S is compact and select

$$g = x_i - y_i \tag{39}$$

for some fixed $y \in \mathcal{U}$, to obtain

$$\int_S H_m^{ij}(\delta^{ij} - \gamma_i\gamma_j)dA = \int_S H_m\gamma\cdot(x-y)dA$$

and hence

$$(40) \qquad (n-m)\int_S H_{m-1}dA = m\int_S H_m \gamma\cdot(x-y)dA$$
$$\leq mR\int_S |H_m|\, dA$$

by appropriate choice of y. Finally by taking S to be a C^2 level set of the function v in Lemma 2, we conclude by the co–area formula [4],

$$(41) \qquad (n-m+1)\int_\Omega C_{m-1}[v] \quad = (m-1)\int_\Omega C_m[v]\frac{Dv\cdot(x-y)}{|Dv|}$$
$$\leq (m-1)R\int_\Omega |C_m[v]| \quad ,$$

whence Lemma 2 follows. Note that by Sard's Theorem [4], almost all levels sets of v are C^2 provided $v \in C^n(\Omega)$ so strictly speaking, we should pass to our condition $v \in C^2(\Omega)$ by approximation.

The proof of Theorem 1 is thus complete in the case when $\Omega_0 = \Omega$, or equivalently when $\partial\Omega$ is a level set of u. Note that our condition $H_{m-1}(\partial\Omega) \geq 0$ becomes redundant in this case. To complete the proof in general we still need to estimate the term $\int_{\partial\Omega_0} H_{m-2}[\partial\Omega_0]$ occuring in the estimate (17), when Ω is replaced by Ω_0. It is convenient for us to defer this step until after our existence considerations. We observe here that Lemma 2 guarantees our condition (3) is non void in that it will certainly be satisfied if $\sup\psi$ is sufficiently small.

We are also indebted to Robert Bartnik for providing an earlier derivation of the key inequality (40) from the first variation formula for the integral

$$\int_S H_{m-1}dA$$

rather than our integration formula, Lemma 3. For a compactly supported variation η, Bartnik's formula asserts

$$\delta\int_S H_{m-1}dA \; = \; m\int_S H_m(\eta,\gamma)dA \; ,$$

whence (40) follows again, with the choice $\eta = g$.

Application to the Dirichlet problem

Recently Ivochkina [7] has succeeded in obtaining global second derivative estimates for solutions of the prescribed m curvature equation (1) under geometrically natural conditions, thereby extending earlier work of herself [6] and Caffarelli, Nirenberg and Spruck [3] for the case of uniformly convex domains and constant boundary values. The incorporation of our Theorem 1 here and our gradient estimates in [13] with these second derivative estimates yields corresponding existence theorems for the classical Dirichlet problem for equation (1). Accordingly let us now assume $\partial\Omega \in C^{3,1}$, $g \in C^{3,1}(\overline{\Omega})$, $\psi \in C^{1,1}(\overline{\Omega})$, $\psi > 0$ in $\overline{\Omega}$, together with (3) and the geometric condition:

$$H_m[\partial\Omega] \geq \psi \text{ on } \partial\Omega, \tag{43}$$

with $\partial\Omega$ assumed m admissible, the latter being redundant if $\partial\Omega$ is connected.

THEOREM 4. *Under the above hypotheses, there exists a unique, admissible, classical solution of the Dirichlet problem,*

$$H_m[u] = \psi \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega. \tag{44}$$

Theorem 4 follows by combination of the above mentioned solution and derivative estimates with the second derivative Hölder estimates of Krylov [9] and the method of continuity as described, for example in [5]. By virtue of the Schauder theory [5] the solution $u \in C^{3,\alpha}(\overline{\Omega})$ for any $\alpha < 1$. When the boundary values g are constant we need only assume $\partial\Omega \in C^{2,1}$, $g \in C^{2,1}(\overline{\Omega})$ with resultant solution $u \in C^{3,\alpha}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ for any $\alpha < 1$, [7]. When we further reduce the above smoothness and positivity hypotheses, we obtain the existence of weak solutions in the viscosity sense [13]. These results may be achieved by direct approximation from Theorem 4, rather than through the uniformly elliptic regularization approach of [13]. To formulate such results, let us first recall from [13], that a function $u \in C^0(\Omega)$ is called a viscosity solution of equation (1) if:

(i) for any $\varphi \in C^2(\Omega)$ and local maximum x_0 of $u - \varphi$, we have

$$H_m[\varphi](x_0) \geq \psi(x_0);$$

(ii) for any admissible $\varphi \in C^2(\Omega)$ and local minimum of $u - \varphi$, we have

$$H_m[\varphi](x_0) \leq \psi(x_0) .$$

We now assume only $\partial\Omega \in C^2$, $g \in C^0(\partial\Omega)$, $\psi^{1/m} \in C^{0,1}(\overline{\Omega}) \cap C^{1,1}(\Omega)$, $\psi > 0$ in Ω, together with (3) and (43). Then, also utilizing the interior gradient bounds of Korevaar [8] and the stability of viscosity solutions under uniform convergence [13], we get a weak existence theorem.

THEOREM 5. *Under the above hypotheses, there exists a unique viscosity solution* $u \in C^0(\overline{\Omega})$ *of the Dirichlet problem* (44), *which is locally uniformly Lipschitz continuous in* Ω. *If* $g \in C^{1,1}(\overline{\Omega})$, *then* $u \in C^{0,1}(\overline{\Omega})$.

The uniqueness in Theorem 5 follows since viscosity solutions can be approximated by classical solutions using Theorem 4; (see [13], Corrigendum). Note that all the above theorems extend to embrace equations of the form

$$H_m[u] = \psi(x,u) , \tag{45}$$

provided ψ is monotone increasing with respect to u, and in (3) and (43), $\psi(x)$ is replaced by $\psi(x,\inf g)$, $\psi(x,g(x))$ respectively.

We observe also that only the already proven case of Theorem 1 when u is constant on $\partial\Omega$ is necessary to derive Theorems 4 and 5 as the solutions so obtained in these cases will provide lower bounds for solutions in the general case.

On the necessity of condition (3)

Let $u \in C^2(\overline{\Omega})$ satisfy equation (1) in Ω. By the Reilly formula ([12], Lemma 3), inequality (3) is satisfied whenever $E = \{u < t\}$ for any $t \in \mathbb{R}$, and

$$1-\chi = \sup_{\Omega} \left[1 - \frac{|Du|^m}{v^m}\right] . \tag{46}$$

To get a similar inequality for other sets E, we first observe that we can find a further admissible function $\tilde{u} \in C^2(\overline{\Omega})$ with

$$H_m[\tilde{u}] = \tilde{\psi} > \psi \tag{47}$$

in $\overline{\Omega}$. To see this, we assume (without loss of generality) that $\mathrm{dist}(0,\Omega) \geq 1$ and set

(48) $$\tilde{u} = u + \eta ,$$

where η is a uniformly convex function given by

(48) $$\eta(x) = \frac{1}{A} \exp \tfrac{1}{2}A(|x|^2 - d^2) , \quad d = \sup_{x\in\Omega} |x| ,$$

and A is a positive constant to be determined. Writing

(50) $$H_m^{1/m}[u] = F(Du,D^2u) ,$$

for admissible $u \in C^2(\Omega)$, we then have

(51) $$\begin{aligned} H^{1/m}[\tilde{u}] - H^{1/m}[u] &= F(Du+D\eta,D^2u+D^2\eta) - F(Du+D\eta,D^2u) \\ &\quad + F(Du+D\eta,D^2u) - F(Du,D^2u) \\ &\geq F(Du+D\eta,D^2\eta) - c_1|D\eta| \\ &\geq c_0(\det D^2\eta)^{1/n} - c_1|D\eta| \end{aligned}$$

using the concavity and homogenity of F with respect to D^2u , where c_0 and c_1 are positive constants depending only on n , m , $|Du|_0$ and $|D^2u|_0$. By fixing the constant A appropriately in terms of c_0 , c_1 , n and d , we infer (41). Consequently, if $\partial E \in C^{2,1} \cap K_m$, we can, (by Theorem 4), solve the classical Dirichlet problem

(52) $$H_m[\bar{u}] = \bar{\psi} \text{ in } E, \bar{u} = 0 \text{ on } \partial E$$

for any $0 < \bar{\psi} \leq \psi$ in E , with $\bar{\psi} \in C^{1,1}(\bar{E})$, sufficiently small on ∂E , since the function $\tilde{u}$ will provide a lower barrier. Consequently by approximation, we obtain inequality (3) for any $E \subset \Omega$ with $\partial E \in C^2 \cap K_m$ and for some fixed χ depending on n , m , $|Du|_0$, $|D^2u|_0$.

The above considerations also facilitate the completion of the proof of Theorem 1. For we observe that the inequality

$$(53) \qquad m\int_{\Omega}\left[D\left(\frac{Du}{v}\right)\right]_m \le \int_{\partial\Omega} H_{m-1}[\partial\Omega]$$

continues to hold when the function v is replaced by $v_\epsilon = \sqrt{\epsilon^2+|Du|^2}$ for any $\epsilon > 0$. Sending $\epsilon \to 0$, we may then deduce, back in (17),

$$(54) \qquad (m-1)\int_{\Omega_0}[D\gamma]_{m-1} \le \int_{\Omega_0}[D\gamma]_{m-1} \le \int_{\partial\Omega} H_{m-2}[\partial\Omega]$$

and this provides the missing estimate for $\int_{\Omega_0} H_{m-2}[\partial\Omega_0]$ in inequality (17) in the proof of Theorem 1, when $\Omega_0 \subseteq \Omega$, $\Omega_0 \neq \Omega$. It is easy to check that any further dependence on Ω_0 in the remainder of the proof can be replaced by the corresponding dependence on Ω . Indeed our proof yields a constant C depending on $\operatorname{diam}\Omega$ and $\int_{\Omega} H_{m-2}[\partial\Omega]$ (which is not as good as the proof in [12] where C depends only on the perimeter of Ω in the case $m = 2$).

Condition (3) will be further examined, in conjunction with our treatment of isoperimetric inequalities in [14] and its essential necessity will result as a by-product of that work.

REFERENCES

1. Bakel'man, I. Ya., *Geometric methods for solving elliptic equations*, Moscow: Izdat. Nauka 1965 [Russian]

2. Burago, Yu. D. and Zalgaller, V.A., *Geometric inequalities*, Springer- Verlag, Berlin,(1988)

3. Caffarelli,L., Nirenberg,L. and Spruck,J., *Nonlinear second-order elliptic equations* V., *The Dirichlet problem for Weingarten surfaces*, Comm. Pure Appl. Math. **41**, 47–70(1988)

4. Federer,H., *Geometric Measure Theory*, Springer–Verlag, Berlin–Heidelberg–New York, 1969

5. Gilbarg,D., and Trudinger,N.S., *Elliptic differential equations of second order*, Second edition, Springer–Verlag, Berlin, (1983)

6. Ivochkina, N.M., *Solution of the Dirichlet problem for the equation of curvature of order m*, Mat. Sbornik **180**, 867–887(1989)

7. Ivochkina, N.M., *Solution of the Dirichlet problem for the equation of curvature of order m*, J. Algebra and Analysis, **6** (1989), to appear

8. Korevaar, N.J., *A priori interior gradient bounds for solutions to elliptic Weingarten equations*, Ann. Inst. Henri Poincaré, Analyse Non Lineaire **4**, 405–421 (1987)

9. Krylov, N.V., *Nonlinear elliptic and parabolic equations of the second order*, Reidel, Dordrecht, 1987

10. Miranda, M., *Una maggiorazione integrale per le curvature delle ipersuperfici minimale*, Rend. Sem. Mat. Univ. Padova **38**, 91–107 (1967)

11. Serrin, J., *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A. **264**, 413–496(1969)

12. Trudinger, N.S., *A priori bounds for graphs with prescribed curvature*, In: Festschrift for Jürgen Moser, Academic Press, 1989

13. Trudinger, N.S., *The Dirichlet problem for the prescribed curvature equations*, Arch. Rat. Mech. Anal. (to appear)

14. Trudinger, N.S., *Isoperimetric inequalities for quermassintegralen*, (in preparation)

15. Trudinger, N.S. and Urbas, J.I.E., *The Dirichlet problem for the equation of prescribed Gauss curvature*, Bull. Aust. Math. Soc. 28, 217–231 (1983)

Neil S. Trudinger
Centre for Mathematical Analysis
Australien National University
P.O. Box 4, Canberra, A.C.T. 2607, Australia

(Received December 18, 1989)