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REPRESENTATION OF A p-HARMONIC FUNCTION NEAR A CRITICAL POINT IN THE PLANE

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A representation theorem is given for a p-harmonic function $\varphi(1 < p < \infty)$ in the plane, near a zero z_0 of grad φ. The proof uses "stream functions" and the hodograph transformation. The stream function of a p-harmonic function is p'-harmonic, where $\frac{1}{p} + \frac{1}{p'} = 1$. In principle, all properties of φ near z_0 can be found from the representation. Some consequences are derived here, e.g. the optimal Hölder continuity of grad φ.

1. Introduction

Let φ be a p-harmonic function, i.e. a weak solution of the equation

$$\operatorname{div}(|\nabla\varphi|^{p-2}\nabla\varphi) = 0,$$

in a domain $\Omega \subset R^2$. This equation is the Euler equation for the functional $\int_\Omega |\nabla\varphi|^p dx$. In this work, it is assumed that $1 < p < \infty$. It is well known that there is an $\alpha > 0$ such that $\varphi \in C^{1,\alpha}$ on any compact subset of Ω. It is also known that the set E of critical points (i.e. where $\nabla\varphi = 0$) consists of isolated points only, unless φ is a constant. Further, with the use of elliptic regularity theory it can be proved that $\varphi \in C^\infty$, in fact φ is real analytic, in $\Omega \setminus E$. However, the equation degenerates on E to some extent (for $p \neq 2$) and the argument does not work there. In fact, it is known from examples that φ need not be in C^2 near a point in E.

It has therefore been an open problem for some time to determine more precisely the structure of φ near a critical point. We derive here a ***representation theorem*** for φ, valid in a full neighbourhood of a critical point z_0. In principle, all properties of φ near z_0 are determined by that theorem (Th. 4).

The first step towards the representation is the observation (Th. 1) that φ has a conjugate, or "stream" function ψ, which is p'-harmonic, where $\frac{1}{p} + \frac{1}{p'} = 1$. Accordingly, the representation includes φ and ψ.

Some consequences of Theorem 4 are proved, and the following ones should be mentioned here:

a) determination of the *exact* Hölder exponent of $\nabla\varphi$ near z_0 as a function of p and the order N of the critical point z_0 (Theorem 5). By varying N, this gives at once the best possible α above.

b) singular expansions of φ and ψ near z_0 (Theorem 6).

The derivation of the representation theorem is based on the hodograph method in a convenient form, which uses a stream function ψ (Theorem 3). Accordingly, the representation has the form of a linear superposition

$$\begin{pmatrix} z \\ \varphi(z) \\ \psi(z) \end{pmatrix} = \sum_{m=N+1}^{\infty} A_m \begin{pmatrix} Z_m(\varsigma') \\ \Phi_m(\varsigma') \\ \Psi_m(\varsigma') \end{pmatrix}$$

where $\varsigma' = \sqrt[N]{\varphi_x + i\varphi_y}$, and where each triple

$$\begin{pmatrix} Z_m(\varsigma') \\ \Phi_m(\varsigma') \\ \Psi_m(\varsigma') \end{pmatrix},$$

considered separately, generates a (p,p')-harmonic pair (φ_m, ψ_m) of a simple nature (namely quasi-radial), at least locally for $\varsigma' \neq 0$.

2. Some background material

Consider a partial differential equation of the form

$$\operatorname{div}\left(\frac{F(|\nabla\varphi|)}{|\nabla\varphi|}\nabla\varphi\right) = 0,$$

where $F(t)$ is strictly increasing. Let φ be smooth, with $\nabla\varphi \neq 0$, and satisfy the differential equation in a simply connected domain $\Omega \subset \boldsymbol{R}^2$. It is then elementary to construct a "stream function" ψ, attached to φ, which satisfies in Ω the reciprocal equation

$$\operatorname{div}\left(\frac{F^{-1}(|\nabla\psi|)}{|\nabla\psi|}\nabla\psi\right) = 0.$$

In particular, the choice $F(t) = t^{p-1}$, i.e. the p-harmonic equation, gives $F^{-1}(s) = s^{p'-1}$, where $p' = \frac{p}{p-1}$. Thus, ψ is p'-harmonic, where $\frac{1}{p} + \frac{1}{p'} = 1$. Details concerning this are found in [4], pp. 80-81. Before proceeding, it is necessary to state the precise solution concept to be used in the sequel.

We now say that φ is *p-harmonic* in $\Omega \subset R^n$ if $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi \cdot \nabla\eta \, dx = 0$$

for all $\eta \in C^1_0(\Omega)$. Let $1 < p < \infty$, and let φ be p-harmonic in $\Omega \subset R^n$. Then φ is locally in $C^{1,\alpha}$. See [10] or [16]. Further, φ is real analytic away from the zeros of $\nabla\varphi$; see [15], p. 208. If $n = 2$, then it is known that the zeros of $\nabla\varphi$ are isolated. For this we refer to [8], p. 8. We also refer to [1] where a complete proof is given. Finally, this fact is also discussed in [5].

The above background motivates our

Theorem 1. *Let $1 < p < \infty$ and let $\varphi \not\equiv$ constant be p-harmonic in a simply connected domain $\Omega \subset R^2$. Then there exists a p'-harmonic function $\psi \in C^1(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, such that*

$$\begin{cases} \psi_x = -|\nabla\varphi|^{p-2}\varphi_y \\ \psi_y = |\nabla\varphi|^{p-2}\varphi_x. \end{cases} \tag{3}$$

Both φ and ψ have locally Hölder continuous gradients. The zeros of $\nabla\varphi$ and $\nabla\psi$ are isolated in Ω. Streamlines of φ are level lines of ψ, and conversely.

The short proof is found in [4], p. 82.

Remark. The use of stream functions is, of course, classical. In this case it apparently appeared first in [3]. There is a physical interpretation of the p-Laplace equation in R^2 in terms of laminar pipe flow of so-called power-law fluids, and this interpretation motivates the name stream function for ψ. These aspects will be discussed elsewhere. Compare [9] and [18].

Some further analogies with complex analysis are derived in [5]. A very useful fact is that the complex gradient $\varphi_x - i\varphi_y$ of a p-harmonic function is a *quasi-regular* function of $z = x + iy$. In particular, there is a representation

$$\varphi_x - i\varphi_y = h \circ \chi \tag{4}$$

where h is analytic and χ is quasi-conformal. Concerning this, we refer to [5], [8] and [17].

The following result is important for the hodograph method. It is proved in [5].

Theorem 2. *Let φ be p-harmonic in a plane domain D. Assume that φ is not a linear function. Then there exists a set E of isolated points in D, such that φ is real-analytic and*

$$\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 \neq 0$$

in $D \setminus E$.

The proof uses the representation (4) and well-known properties of quasi-conformal mappings.

The reader, who wants to know more about the background of this work, is referred to [2], [5] and [14].

3. Hodograph transformation for the p-harmonic equation

The hodograph method is a classical technique, e.g. for linearizing potential equations in gas dynamics. The idea is roughly that $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$ are introduced as new independent variables. Good references for this method are [7], pp.13-14, and [6], pp. 130-132. The method is well suited for handling the p-harmonic equation in the plane, and a detailed exposition of the procedure is given in [4], pp. 90-93. For convenience, we agree that *Theorem 3 here means Theorem 3 in [4]*. It is further agreed that systems (5)-(8) here refer to systems (a)-(d), respectively, in [4].

It is now assumed that the reader is familiar with [4], pp. 90-93.

4. Derivation of the representation theorem

Let $\varphi(x,y)$ be p-harmonic in $\Omega \subset R^2$, and let $1 < p < \infty$, $p \neq 2$. Let $\nabla\varphi = 0$ at the origin and $\varphi \not\equiv$ constant. Theorem 4 gives a "hodographic" representation of φ and a stream function ψ near the origin. It will be derived using a step-by-step construction.

4.1 A. Construction of the mappings needed

We start from the representation (4):

$$\varphi_x - i\varphi_y = h \circ \chi,$$

with h analytic and χ quasi-conformal. Choose a neighbourhood U_0 of $z = 0$ such that $\bar{U}_0 \subset \Omega$ and $\nabla\varphi \neq 0$ in $U_0 \backslash \{0\}$. It is no restriction to assume that $\frac{\partial(\varphi_x,\varphi_y)}{\partial(x,y)} \neq 0$ in $U_0\backslash\{0\}$, as follows from Theorem 2. Let U_0 be simply connected, so that a stream function ψ exists in U_0.

Put $\xi = \chi(z)$ and $\varsigma = \varphi_x + i\varphi_y = (\bar{h} \circ \chi)(z)$. Let $\chi(0) = 0$ and let N be the order of the first nonvanishing derivative of $h(\xi)$ at $\xi = 0$. Thus $N \geq 1$. Put $B_1(\delta) = \{\xi : |\xi| < \delta\}$. There is a $\delta > 0$ so that $h(\xi) = (G(\xi))^N$ for all $\xi \in B_1(\delta)$, where $G(\xi)$ is analytic and univalent. (See [13], p 148). Assume that $\chi(U_0) \supset B_1(\delta)$. Put $\varsigma' = \overline{G(\xi)}$ and $B'(\rho') = \{\varsigma' : |\varsigma'| < \rho'\}$. Choose $\rho' > 0$ so that $G(B_1(\delta)) \supset \overline{B'(\rho')}$.

Put $U = (\chi^{-1} \circ G^{-1})(B'(\rho'))$. Then $U \subset U_0$ and $(\varphi_x + i\varphi_y)(z) = \overline{(G(\chi(z)))}^N$ for $z \in U$. Thus $\varsigma = (\varsigma')^N$, and $B'(\rho')$ is mapped onto $B(\rho) = \{\varsigma : |\varsigma| < \rho\}$, where $\rho = (\rho')^N$.

Clearly, φ and ψ are well-defined, continuous functions of $\varsigma' \in \overline{B'(\rho')}$.

Introduce polar coordinates in the ς- and ς'-planes : $q = |\varsigma|$, $\theta = \arg\varsigma$, $q' = |\varsigma'|$ and $\theta' = \arg\varsigma'$. Finally, put $\mu = -\log q$ and $\mu' = -\log q'$. It is no restriction to assume that $\varphi(0) = \psi(0) = 0$.

Definition. *The number N is called the* **order** *of the critical point. It is clearly independent of the choice of the representation $h \circ \chi$ for the complex gradient.*

Figure 1 should make the situation clear.

4.2 B. Modified hodograph equations

By construction, $z = z(\varsigma')$, $\varphi = \varphi(\varsigma')$ and $\psi = \psi(\varsigma')$ are well-defined and continuous for $\varsigma' \in \overline{B'(\rho')}$. Further, $z = z(\varsigma')$ is one-to-one and the function $z(0) = 0$. For $0 < |\varsigma'| \leq \rho'$ we have $z \in U_0\backslash\{0\}$, and thus $\nabla\varphi \neq 0$ and $\frac{\partial(\varphi_x,\varphi_y)}{\partial(x,y)} \neq 0$. Fix any ς_0', $0 < |\varsigma_0'| \leq \rho'$. By restricting ς' and ς, one can define $\varsigma' = \sqrt[N]{\varsigma}$ uniquely, so that $z = z(\varsigma)$ becomes well-defined and smooth near $\varsigma_0 = (\varsigma_0')^N$. But this is the situation studied in Section 3 and in [4]. There it was shown that $\varphi = \varphi(\varsigma)$, $\psi = \psi(\varsigma)$ satisfy the "Chaplygin system" (6), using polar coordinates q, θ:

$$\begin{cases} \varphi_\theta = \dfrac{\psi_q}{q^{p-3}} \\ \varphi_q = -\dfrac{(p-1)\psi_\theta}{q^{p-1}}. \end{cases}$$

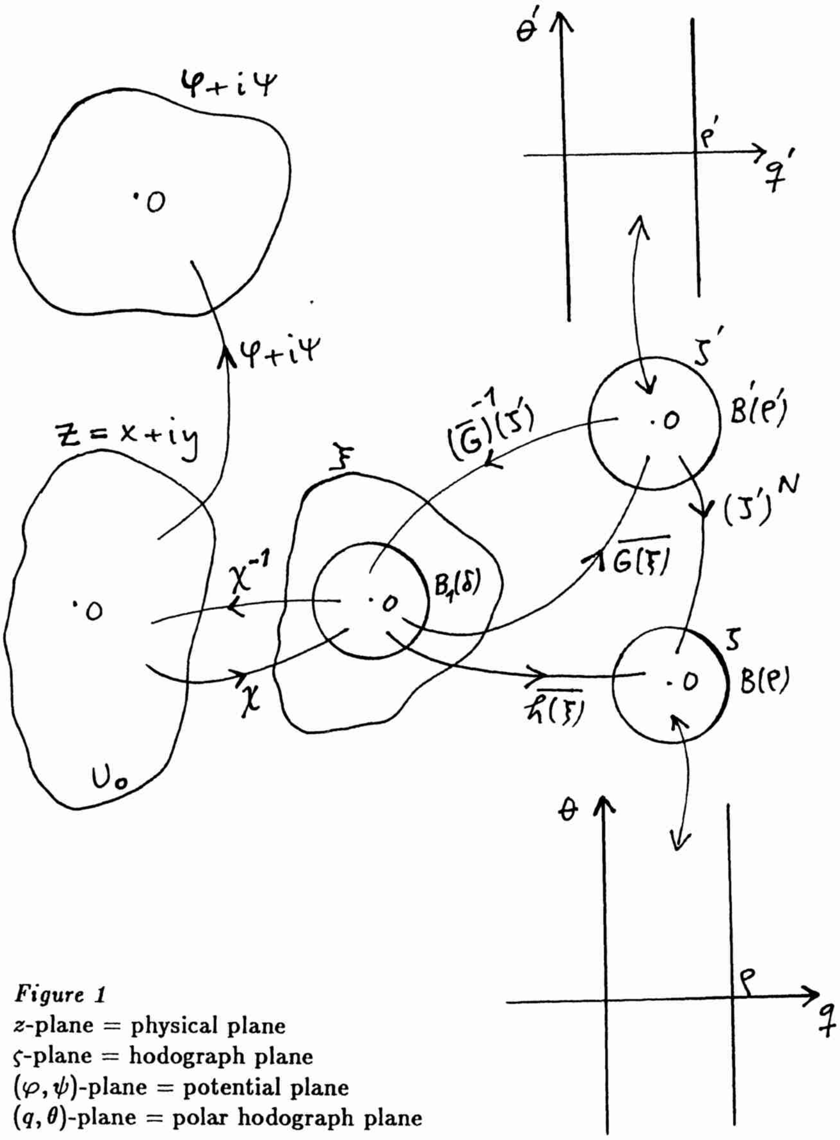

Figure 1
z-plane = physical plane
ς-plane = hodograph plane
(φ, ψ)-plane = potential plane
(q, θ)-plane = polar hodograph plane

With $\mu = -\log q$, the system takes the form

$$\begin{cases} \varphi_\mu = (p-1)e^{(p-2)\mu}\,\psi_\theta \\ \varphi_\theta = -e^{(p-2)\mu}\,\psi_\mu. \end{cases} \tag{9}$$

But $\mu = N\mu'$, $\theta = N\theta'$, and thus

$$\begin{cases} \varphi_{\mu'} = (p-1)e^{N(p-2)\mu'}\,\psi_{\theta'} \\ \varphi_{\theta'} = -e^{N(p-2)\mu'}\,\psi_{\mu'}. \end{cases} \tag{10}$$

Cross-differentiation and elimination gives:

$$\begin{cases} \varphi_{\theta'\theta'} + \dfrac{1}{p-1}\varphi_{\mu'\mu'} - N\dfrac{p-2}{p-1}\varphi_{\mu'} = 0 \\ \psi_{\theta'\theta'} + \dfrac{1}{p-1}\psi_{\mu'\mu'} + N\dfrac{p-2}{p-1}\psi_{\mu'} = 0. \end{cases} \tag{11}$$

Clearly, these equations are valid for $0 < |\zeta'| \le \rho'$, in spite of the "local" derivation. Note that the last two equations have constant coefficients and are formal adjoints of each other.

4.3 C. Finding a solution base

A sufficient class of solutions to (10) will be needed, along with expressions for $z(\mu', \theta')$, or rather $z(q', \theta')$. Put $\varphi = F(\theta')G(\mu')$ and insert into the first equation (11). This gives

$$\begin{cases} F'' + \lambda' F = 0 \\ G'' - N(p-2)G' - (p-1)\lambda' G = 0 \end{cases}$$

where λ' is at our disposal.

The characteristic equation for G is

$$\alpha^2 - N(p-2)\alpha - (p-1)\lambda' = 0.$$

It turns out to be sufficient to consider the cases $\lambda' \ge 0$ and $\alpha < 0$.

Assume first that $\lambda' > 0$. Put $\lambda = \frac{\lambda'}{N^2} > 0$ and $\beta = \frac{-\alpha}{N} > 0$. Thus,

$$\beta^2 + (p-2)\beta - \lambda(p-1) = 0, \tag{12}$$

and

$$\beta = \frac{1}{2}\left(\sqrt{(p-2)^2 + 4\lambda(p-1)} - p + 2\right).$$

The following observations will be useful.

Lemma 1. *Let p be fixed $(1 < p < \infty)$ and consider the function $\beta = \beta(\lambda)$, for $\lambda > 0$. It is positive and strictly increasing, $\beta(1) = 1$, and for $\lambda > 1$ one has $1 < \beta(\lambda) < \lambda$.*

Proof: The last statement follows by verifying that $\frac{d\beta}{d\lambda} < 1$ for $\lambda > 1$, and the rest is trivial.

The equation for φ is clearly satisfied by $\varphi = e^{-N\beta\mu'} \sin\sqrt{\lambda'}(\theta' - \theta_0)$, for any θ_0. A corresponding ψ is easily found from (10):

$$\psi = Ke^{-\mu' \cdot \frac{\sqrt{\lambda'}}{K}} \cdot \cos\sqrt{\lambda'}(\theta' - \theta_0),$$

where $K = \frac{N\beta}{(p-1)\sqrt{\lambda'}} = K(\lambda')$.

This pair (φ, ψ) satisfies (10) for *any* $\lambda' > 0$ *and any* θ_0. In terms of (q', θ'), this is

$$\begin{cases} \varphi = (q')^{N\beta} \sin\sqrt{\lambda'}(\theta' - \theta_0) \\ \psi = K(q')^{\frac{\sqrt{\lambda'}}{K}} \cos\sqrt{\lambda'}(\theta' - \theta_0). \end{cases} \tag{13}$$

Consider then the case $\lambda' = 0$. Here, $F(\theta')$ is linear and $G(\mu') = A + Be^{N(p-2)\mu'} = A + B(q')^{N(2-p)}$. (Recall that $p \neq 2$). Only the two pairs

$$\begin{cases} \varphi = (q')^{N(2-p)} \\ \psi = N\dfrac{p-2}{p-1}\theta' \end{cases} \tag{14}$$

and

$$\begin{cases} \varphi = N(p-2)\theta' \\ \psi = (q')^{N(p-2)} \end{cases} \tag{15}$$

will be needed, besides (13).

Formulas for $z = z(q', \theta')$ connected to (13) will be needed later.

The relations $\frac{\partial z}{\partial \varphi} = \frac{e^{i\theta}}{q}$ and $\frac{\partial z}{\partial \psi} = \frac{ie^{i\theta}}{q^{p-1}}$ were found in [4], p. 90. Thus, the formulas $\frac{\partial z}{\partial \varphi} = \frac{e^{iN\theta'}}{(q')^N}$ and $\frac{\partial z}{\partial \psi} = \frac{ie^{iN\theta'}}{(q')^{N(p-1)}}$ hold here, at least locally.

For the solution couple (13), the chain rule easily gives

$$\frac{\partial z}{\partial q'} = i\sqrt{\lambda'}\, e^{iN\theta'}(q')^{N(\beta-1)-1}\left(\cos\sqrt{\lambda'}(\theta' - \theta_0) - i\frac{N\beta}{\sqrt{\lambda'}}\sin\sqrt{\lambda'}(\theta' - \theta_0)\right)$$

and

$$\frac{\partial z}{\partial \theta'} = \sqrt{\lambda'}e^{iN\theta'}(q')^{N(\beta-1)}\cdot$$
$$\cdot\left(\cos\sqrt{\lambda'}(\theta' - \theta_0) - i\frac{N\beta}{(p-1)\sqrt{\lambda'}}\sin\sqrt{\lambda'}(\theta' - \theta_0)\right)$$

for any $\lambda' > 0$. The formula for $\frac{\partial z}{\partial q'}$ suggests that

$$z = \frac{i\sqrt{\lambda'}}{N(\beta-1)}e^{iN\theta'}(q')^{N(\beta-1)}\left(\cos\sqrt{\lambda'}(\theta' - \theta_0) - i\frac{N\beta}{\sqrt{\lambda'}}\sin\sqrt{\lambda'}(\theta' - \theta_0)\right) \tag{16}$$

provided $\lambda' > 0$, $\lambda' \neq N^2$. (Then $\beta \neq 1$.) Routine calculations, using (12), show that (16) also gives the correct expression for $\frac{\partial z}{\partial \theta'}$. If $\lambda' = N^2$, then $\beta = 1$ and

$$\frac{\partial z}{\partial q'} = \frac{1}{q'}iNe^{iN\theta_0}. \tag{17}$$

Further formulas will not be needed. The expressions (13) and (16), for varying λ', create "base vectors" to be used in Theorem 4.

Remark: The pair (14) is the "hodographic image" of the pair

$$\begin{cases} \varphi = \left(\dfrac{p-1}{p-2}\, r\right)^{\frac{p-2}{p-1}} \\ \psi = \dfrac{p-2}{p-1}\phi \end{cases}$$

for $p > 2$, and of the pair

$$\begin{cases} \varphi = \left(\dfrac{p-1}{2-p}\, r\right)^{\frac{p-2}{p-1}} \\ \psi = \dfrac{p-2}{p-1}(\phi + \pi) \end{cases}$$

for $1 < p < 2$. It is understood in this interpretation that $N\theta'$ is replaced by θ and $(q')^N$ is replaced by q. Analogously, the pair (15) is interpreted as the hodographic image of

$$\begin{cases} \varphi = (p-2)(\phi + \dfrac{\pi}{2}) \\ \psi = (p-2)^{p-2} r^{2-p} \end{cases}$$

for $p > 2$, and of the pair

$$\begin{cases} \varphi = (p-2)(\phi - \dfrac{\pi}{2}) \\ \psi = (2-p)^{p-2} r^{2-p} \end{cases}$$

for $1 < p < 2$. All this is easily verified. Clearly, these pairs $(\varphi(z),\ \psi(z))$ can be seen as nonlinear analogs of $\varphi = \operatorname{Re}\log z$, $\psi = \operatorname{Im}\log z$, $p = 2$.

Finally, the case $\lambda' = N^2$, i.e. with $\beta = 1$ and the relation (17), is related to p-harmonic functions of the form $\varphi = e^x f(y)$, constructed by T. Wolff in [23]. We omit the details.

4.4 D. Preliminary determination of $\varphi(q', \theta')$ and $\psi(q', \theta')$.

By construction, $\varphi(q', \theta')$ and $\psi(q', \theta')$ are 2π-periodic in the variable θ', and belong to C^∞ for $0 < q' \leq \rho'$. Put

$$\begin{cases} \varphi(\rho', \theta') = A_0 + \displaystyle\sum_{m=1}^{\infty} A_m \sin m(\theta' - \theta_m) \\ \psi(\rho', \theta') = B_0 + \displaystyle\sum_{m=1}^{\infty} B_m \cos m(\theta' - \theta'_m). \end{cases}$$

We must show that $A_0 = B_0 = 0$.

Assume first that $1 < p < 2$.

In the first formula (13), put $\lambda' = m^2$ for $m = 1, 2, 3, \ldots$,

$$\beta = \beta_m = \frac{1}{2}\left(\sqrt{(p-2)^2 + 4\frac{m^2}{N^2}(p-1)} - p + 2\right)$$

and $\theta_0 = \theta_m$. Consider the function

$$H(q', \theta') = A_0 \left(\frac{q'}{\rho'}\right)^{N(2-p)} + \sum_{m=1}^{\infty} A_m \left(\frac{q'}{\rho'}\right)^{N\beta_m} \sin m(\theta' - \theta_m).$$

Now (11) implies that

$$L(\varphi) = \varphi_{\theta'\theta'} + \frac{(q')^2}{p-1}\varphi_{q'q'} + \frac{N(p-2)+1}{p-1} q' \varphi_{q'} = 0$$

in $\Omega' = \{(q', \theta') \mid 0 < q' < \rho'\}$ and $\varphi \in C(\overline{\Omega'})$. Clearly, the same holds for $H(q', \theta')$. Thus, $L(\varphi - H) = 0$ in Ω' and $\varphi - H = 0$ on $\partial\Omega'$. Further, $\varphi - H$ is periodic with respect to θ'. If $\varphi - H \not\equiv 0$, then $\varphi - H$ must have an interior maximum or minimum, which is impossible, because of $L(\varphi - H) = 0$. See [19], p. 61. Thus $\varphi \equiv H$ in Ω'.

Now $\psi(\rho', \theta^*) = \psi(\rho', \theta^* + 2\pi)$ for any θ^*, and it follows from (10) that $\int_{\theta^*}^{\theta^*+2\pi} \varphi_{\mu'}(\cdots) d\theta' = 0$. Thus, $\int_{\theta^*}^{\theta^*+2\pi} \varphi_{q'}(\rho', \theta') d\theta' = 0$, which gives $A_0 = 0$. Finally, ψ is determined from (10) and the condition $\psi(0, \theta') \equiv 0$.

This clearly gives

$$\psi(q', \theta') = \sum_{m=1}^{\infty} \frac{A_m}{(\rho')^{N\beta_m}} \cdot K_m (q')^{\frac{m}{K_m}} \cos m(\theta' - \theta_m),$$

where $K_m = \frac{N\beta m}{(p-1)m}$. Using (12), one easily verifies that $\frac{m}{\beta_m} = N(p + \beta_m - 2)$. This gives the result

$$\begin{cases} \varphi(q', \theta') = \sum_{m=1}^{\infty} A_m \left(\frac{q'}{\rho'}\right)^{N\beta_m} \sin m(\theta' - \theta_m) \\ \psi(q', \theta') = \frac{N(q')^{N(p-2)}}{p-1} \cdot \sum_{m=1}^{\infty} A_m \frac{\beta_m}{m} \left(\frac{q'}{\rho'}\right)^{N\beta_m} \cos m(\theta' - \theta_m). \end{cases} \tag{18}$$

Note that $\lim_{m\to\infty} \frac{\beta_m}{m} = \frac{\sqrt{p-1}}{N}$. Further, $N(p + \beta_m - 2) > 0$ for all m. Clearly, both series converge uniformly for $0 \le q' \le \rho'$.

Assume then that $2 < p < \infty$.

In this case one determines ψ first, and then φ, in a similar way. The pair (15) is used instead of (14). The result is a representation of $\varphi(q', \theta')$ and $\psi(q', \theta')$ of the same form as above.

Remark. Since $\varphi(\rho', \cdot) \in C^\infty$ and φ is periodic in θ', the Fourier coefficients $A_m = O(m^{-k})$ for any integer $k \ge 1$. Also, the growth of $\{\beta_m\}_1^\infty$ is quite regular; $\frac{\beta_m}{m} \to \frac{\sqrt{p-1}}{N}$. Because of this, convergence questions related to (18) will not offer any difficulties.

4.5 E. Completion of the proof

Put $\lambda' = m^2$ and $\theta_0 = \theta_m$ for $m = 1,2,3,\ldots,\ m \neq N$, in (16) and get

$$Z_m(q',\theta') = \frac{im}{N(\beta_m - 1)} e^{iN\theta'} (q')^{N(\beta_m-1)} \Big(\cos m(\theta' - \theta_m) - \\ - \frac{iN\beta_m}{m} \sin m(\theta' - \theta_m)\Big). \quad (19)$$

Also recall equation (17) for the case $m = N$, $\lambda' = N^2$. It is clear from subsections C and D that the sequence $\{\beta_m\}_1^\infty$ is positive and strictly increasing. Further, $\beta_N = 1$. Thus, the exponents $N(\beta_m - 1)$ in (19) are different and negative for $m = 1,2,\ldots,\ N-1$.

Take δ_1 and δ_2 such that $0 < \delta_1 < \delta_2 < \rho'$ and consider

$$z(\delta_2,\theta') - z(\delta_1,\theta') = \int_{\delta_1}^{\delta_2} \frac{\partial z}{\partial q'}(q',\theta')dq'.$$

Equations (18) have the form

$$\varphi(q',\theta') = \sum_{m=1}^{\infty} \varphi_m(q',\theta'),\ \psi(q',\theta') = \sum_{m=1}^{\infty} \psi_m(q',\theta').$$

Further, $\frac{\partial z}{\partial q'} = \frac{\partial z}{\partial \varphi}\frac{\partial \varphi}{\partial q'} + \frac{\partial z}{\partial \psi}\frac{\partial \psi}{\partial q'}$, $\frac{\partial \varphi}{\partial q'} = \sum_{m=1}^{\infty} \frac{\partial \varphi_m}{\partial q'}$ and $\frac{\partial \psi}{\partial q'} = \sum_{m=1}^{\infty} \frac{\partial \psi_m}{\partial q'}$ for $\delta_1 \leq q' \leq \delta_2$, with uniform convergence. Thus,

$$\frac{\partial z}{\partial q'} = \sum_{m=1}^{\infty} \Big(\frac{\partial z}{\partial \varphi}\frac{\partial \varphi_m}{\partial q'} + \frac{\partial z}{\partial \psi}\frac{\partial \psi_m}{\partial q'}\Big) = \sum_{m=1}^{\infty} \Big(\frac{e^{iN\theta'}}{(q')^N}\frac{\partial \varphi_m}{\partial q'} + \frac{ie^{iN\theta'}}{(q')^{N(p-1)}}\frac{\partial \psi_m}{\partial q'}\Big)$$

for $\delta_1 \leq q' \leq \delta_2$. Using (16), (17) and (19) this gives

$$\frac{\partial z}{\partial q'} = \sum_{\substack{m=1\\ m\neq N}}^{\infty} \frac{A_m}{(\rho')^{N\beta_m}} \cdot \frac{\partial}{\partial q'}(Z_m(q',\theta')) + \frac{A_N}{(\rho')^N} \cdot \frac{iN}{q'} e^{iN\theta_N},$$

i.e.

$$z(\delta_2,\theta') - z(\delta_1,\theta') = \\ = \sum_{\substack{m=1\\ m\neq N}}^{\infty} \frac{A_m}{(\rho')^{N\beta_m}} (Z_m(\delta_2,\theta') - Z_m(\delta_1,\theta')) + \frac{A_N}{(\rho')^N} iN \log\frac{\delta_2}{\delta_1} e^{iN\theta_N}.$$

Write this relation as

$$z(\delta_2,\theta') - z(\delta_1,\theta') = \sum_{m=1}^{N-1} \cdots + \frac{A_N iN}{(\rho')^N} \log\frac{\delta_2}{\delta_1} e^{iN\theta_N} + \sum_{m=N+1}^{\infty} \cdots$$

Here, the last sum contains positive powers of δ_1 and δ_2, and is clearly bounded when $\delta_1 \to +0$. The first sum contains negative powers of δ_1, δ_2 *with different exponents.* The left-hand side is bounded when $\delta_1 \to +0$. It follows immediately that $A_m = 0$ for $m = 1, 2, \ldots, N$. (Compare (19).) Let $\delta_1 \to +0$ and conclude that

$$z(\delta_2, \theta') = \sum_{m=N+1}^{\infty} \frac{A_m}{(\rho')^{N\beta_m}} Z_m(\delta_2, \theta'). \tag{20}$$

This is clearly true for $0 \le \delta_2 \le \rho'$ and all θ'.

It only remains to show that $A_{N+1} \neq 0$. Let A_M be the first non-zero element in the sequence $\{A_m\}_{N+1}^{\infty}$. Choose θ_M so that $A_M > 0$. Then (19) gives, for the leading term in (20),

$$\arg\Big(\frac{A_M}{(\rho')^{N\beta_M}} Z_M(\delta_2, \theta')\Big) = \frac{\pi}{2} + N\theta' - \arg\big[\cos M(\theta' - \theta_M) + \\ + \frac{iN\beta_M}{M} \sin M(\theta' - \theta_M)\big] \equiv \gamma(\theta').$$

Let θ' increase from θ_M to $\theta_M + 2\pi$. Evidently, $\gamma(\theta')$ changes by $N \cdot 2\pi - M \cdot 2\pi$. Trivial estimates show that

$$\Big| \sum_{M+1}^{\infty} \frac{A_m}{(\rho')^{N\beta_m}} Z_m(\delta_2, \theta') \Big| \le C \cdot \sum_{M+1}^{\infty} |A_m| \Big(\frac{\delta_2}{\rho'}\Big)^{N(\beta_m - 1)},$$

where C is independent of δ_2 and θ'. But this quantity is clearly dominated by half the leading term of (20) if $\delta_2 = \delta_0$, small enough. Therefore, also $\arg z(\delta_0, \theta')$ changes by $(N - M)2\pi$, when θ' increases by 2π. Now the relation $\varsigma' = \overline{(G \circ \chi)(z)}$ gives $(N - M)2\pi = -2\pi$. (Recall from subsection A that G is univalent and χ quasi-conformal.) Thus $M = N + 1$. The results are collected in the basic theorem of this paper.

4.6 The representation theorem (Th. 4):

Consider a p-harmonic function φ in a domain $\Omega \subset R^2$ containing the origin. Let $1 < p < \infty$, $p \neq 2$, and $\varphi \not\equiv$ constant. Suppose $\nabla\varphi = 0$ at $z = 0$ and denote by N the order of the critical point $z = 0$. Let $\omega(\tau)$ be that component of the open set $\{z \in \Omega : |\nabla\varphi(z)| < \tau^N\}$, which contains the origin. A stream function ψ exists in $\omega(\tau)$ if $0 < \tau \le \tau_0$. Assume that $\varphi(0) = \psi(0) = 0$. Then there exists a ρ', $0 < \rho' \le \tau_0$, such that φ and ψ, considered on $\omega(\rho')$, admit a parametric representation as follows:

a) There exist scalar sequences $\{A_m\}_{N+1}^{\infty}$ and $\{\theta_m\}_{N+1}^{\infty}$ satisfying $A_{N+1} > 0$ and $A_m = O(m^{-k})$ for any $k > 0$, such that the formulas

$$\begin{cases} \varsigma' = q' e^{i\theta'} \quad (q' \ge 0) \\ z = \displaystyle\sum_{m=N+1}^{\infty} \frac{A_m}{(\rho')^{N\beta_m}} Z_m(q', \theta') \end{cases}$$

define a one-to-one bicontinuous mapping from $\{\varsigma' : |\varsigma'| \le \rho'\}$ to $\overline{\omega(\rho')}$. Here, as before

$$Z_m(q', \theta') = \frac{im}{N(\beta_m - 1)} e^{iN\theta'} (q')^{N(\beta_m - 1)} \big[\cos m(\theta' - \theta_m) - \\ - \frac{iN\beta_m}{m} \sin m(\theta' - \theta_m)\big]$$

and

$$\beta_m = \frac{1}{2}\Big(\sqrt{(p-2)^2 + 4\frac{m^2}{N^2}(p-1)} - p + 2\Big).$$

The point $\varsigma' = 0$ is mapped on $z = 0$.

b) The values of φ and ψ at $z = z(\varsigma')$ are given by

$$\begin{cases} \varphi = \displaystyle\sum_{m=N+1}^{\infty} A_m \left(\frac{q'}{\rho'}\right)^{N\beta_m} \sin m(\theta' - \theta_m) \\ \psi = \displaystyle\frac{N}{p-1}(\rho')^{N(p-2)} \sum_{m=N+1}^{\infty} A_m \frac{\beta_m}{m} \left(\frac{q'}{\rho'}\right)^{N(p+\beta_m-2)} \cos m(\theta' - \theta_m). \end{cases}$$

The three series above converge uniformly for $|\varsigma'| = q' \leq \rho'$. Further, the value of $\nabla\varphi$ at $z(\varsigma')$ is $(\varsigma')^N = (q')^N e^{iN\theta'}$. Finally, observe that $\sum_{N+1}^{\infty} A_m \sin m(\theta' - \theta_m)$ is the Fourier series expansion of φ as a function of θ' on the level curve $|\nabla\varphi| = (\rho')^N$, i.e. on $\partial\omega(\rho')$.

Remark 1: For any $N = 1, 2, 3, \ldots$ and any $A_{N+1} > 0$ there certainly exists a p-harmonic function φ, defined in $\Omega = R^2$, corresponding to the given A_{N+1} and $A_m = 0$ for all $m \geq N + 2$. Such a function φ and the corresponding ψ *have been specified* and studied in some detail. They are of the form $\varphi = r^k f(\phi)$, $\psi = r^l g(\phi)$. We refer to [2], pp 138–150, [4] and [5]. Observe the graphical presentations in [2], pp 154–156. (The integer m there is the same as $M = N + 1$ here.) More information is given in Lemma 2, Section 5C.

Remark 2: Clearly, the theorem is valid and simplifies greatly for $p = 2$, although this case was avoided at some step in the proof (subsection 4C). In this case, $\varphi + i\psi = f(z)$ is holomorphic and z is a holomorphic function of $\overline{\varsigma'} = \sqrt[N]{f'(z)} = \sqrt[N]{\varphi_x - i\varphi_y}$ near some point z_0, if z_0 is a zero of $f'(z)$ of the N-th order. Therefore, $\varphi + i\psi$ is also a holomorphic function of $\overline{\varsigma'}$ near $\varsigma' = 0$ and can be expanded in a power series, just like z. The theorem produces the correct formulas for this.

5. Some consequences of the representation theorem

Certain implications of Theorem 4 deserve to be explained in some detail.

5.7 A. Behaviour of $|\nabla\varphi|$ near a critical point. Optimal exponents in Alessandrini's estimate.

Consider again the expansion $z(q',\theta') = \sum_{N+1}^{\infty}\cdots$. The modulus of the leading term is

$$\frac{A_{N+1}}{(\rho')^{N\beta_{N+1}}}\cdot\frac{N+1}{N(\beta_{N+1}-1)}(q')^{N(\beta_{N+1}-1)}h(\theta'-\theta_{N+1}),$$

where

$$h(\vartheta) = |\cos(N+1)\vartheta - \frac{iN\beta_{N+1}}{N+1}\sin(N+1)\vartheta|. \tag{22}$$

The expansion of $z(q',\theta')$ is uniformly convergent and the exponent $N(\beta_m - 1)$ is positive and strictly increasing with m. One easily finds (see the remark in subsection 4D) that

$$\lim_{q'\to+0}\frac{|z(q',\theta')|}{(q')^{N(\beta_{N+1}-1)}} = \frac{C}{(\rho')^{N\beta_{N+1}}}\cdot h(\theta'-\theta_{N+1}),$$

where $C = \frac{A_{N+1}(N+1)}{(\beta_{N+1}-1)N} > 0$ and the convergence is uniform in θ'. But $(q')^N = q = |\nabla\varphi|$ and $(\rho')^N = \rho$, so that

$$\lim_{q'\to+0}\frac{|z(q',\theta')|}{|\nabla\varphi|^{\beta_{N+1}-1}} = \frac{C}{\rho^{\beta_{N+1}}}h(\theta'-\theta_{N+1}). \tag{23}$$

Corollary 1 of Theorem 4.

$$C_1|z|^{\frac{1}{\beta_{N+1}-1}} \le |\nabla\varphi| \le C_2|z|^{\frac{1}{\beta_{N+1}-1}} \tag{24}$$

holds in a neighbourhood of the origin for suitable positive constants C_1 and C_2. Further,

$$\beta_{N+1} - 1 = \frac{1}{2}\left[\sqrt{(p-2)^2 + 4(1+\frac{1}{N})^2(p-1)} - p\right]. \tag{25}$$

Recall that $N(\ge 1)$ is the order of the critical point and $1 < p < \infty$. (Obviously, the corollary is correct also for $p = 2$.)

G. Alessandrini [1] has proved interesting lower bounds for the gradient of a p-harmonic φ in a domain $\Omega \subset R^2$, too lengthy to be stated here. The bounds refer to a compact subset of Ω, and each critical point z_i contributes a factor $|z - z_i|^{Cm_i}$, where $C > 1$ is a constant and $m_i \ge 1$ an integer. It follows from (24) that *the best possible (i.e. smallest) exponent* attached to a critical point of order N is $\frac{1}{\beta_{N+1}-1}$.

The following facts follow easily from (25): $\beta_{N+1}(p)$ is a strictly increasing function of p, for any fixed N, and strictly decreasing as a function of N, for any fixed p. (It is understood that $p > 1$ and $N \ge 1$.)

Further, $\beta_{N+1}(p) = 2$ for $N = 1$, $p = 2$ and for $N = 2$, $p = 9$. Finally, $\lim_{p\to\infty}\beta_4(p) = \frac{16}{9}$. It follows that $\beta_{N+1}(p) > 2$ if and only if $N = 1$, $p > 2$ *or* $N = 2$, $p > 9$.

The above observations, combined with (24), lead to the following:

5.8 Corollary 2.

Let $p > 2$. Then, in a neighbourhood of a critical point of order $N = 1$, grad φ can not satisfy a Hölder condition with an exponent greater than

$$\gamma(1,p) = \frac{2}{\sqrt{p^2 + 12(p-1)} - p} < 1.$$

Further, let $p > 9$. Then, near a critical point of order $N = 2$, grad φ can not satisfy a Hölder condition with an exponent greater than

$$\gamma(2,p) = \frac{2}{\sqrt{p^2 + 5(p-1)} - p} < 1.$$

5.9 B. Optimal Hölder continuity of the gradient of φ.

Consider again the function φ in Theorem 4, and recall from Section 2 that $\varphi \in C^{1,\alpha}$. The question is: for which α is $\nabla\varphi$ α-Höldercontinuous near $z = 0$? Clearly one wants α as large as possible. Because of Theorem 4, the question is now reformulated: for which $\alpha > 0$ are there constants C and $\rho^* > 0$ such that

$$|(q_1')^N e^{iN\theta_1'} - (q_2')^N e^{iN\theta_2'}| \leq C|z(q_1',\theta_1') - z(q_2',\theta_2')|^\alpha \tag{26}$$

for all (q_1',θ_1'), (q_2',θ_2'), satisfying $q_1' \leq \rho^*$, $q_2' \leq \rho^*$? The case where q_1' or q_2' is zero is covered by the previous subsection; see equation (24). It is no restriction here to assume that $0 < q_1' \leq q_2'$. Further, it is no restriction to assume $\rho' = 1$, so that

$$z(q',\theta') = \sum_{m=N+1}^{\infty} A_m Z_m(q',\theta').$$

The leading term here will first be analysed.

Apart from a constant factor $\neq 0$, it is

$$e^{iN\theta'}(q')^s[\cos(N+1)(\theta' - \theta_{N+1}) - i\sigma\sin(N+1)(\theta' - \theta_{N+1})],$$

where $s = N(\beta_{N+1} - 1)$ and $\sigma = \frac{N\beta_{N+1}}{N+1}$ are positive. Clearly, it is no restriction to assume $\theta_{N+1} = 0$. (A change of θ_{N+1} simply means a twist in the z-plane). It will be sufficient to study the mapping $z_{N+1}(q',\theta') =$ $= (q')^s g(\theta')$, where

$$g(\theta') = e^{iN\theta'}[\cos(N+1)\theta' - i\sigma\sin(N+1)\theta'].$$

Define mappings $F_i : C \to C$, $i = 1,2,3$, by the formulas

$$\begin{cases} F_1(re^{i\vartheta}) = r^s e^{i\vartheta} \\ F_2(re^{i\vartheta}) = r|g(\vartheta)|e^{i\vartheta} \\ F_3(re^{i\vartheta}) = r\dfrac{g(\vartheta)}{|g(\vartheta)|}. \end{cases}$$

Then $z_{N+1}(q',\theta') \equiv (F_3 \circ F_2 \circ F_1)(q'e^{i\theta'})$.

Clearly, $\left|\frac{\partial F_2}{\partial r}\right|$ and $\frac{1}{r}\left|\frac{\partial F_2}{\partial \vartheta}\right|$ are bounded, uniformly in r and ϑ. A similar statement holds for the inverse mapping, multiplication by $\frac{1}{|g(\vartheta)|}$. Thus, F_2 is bi-Lipschitzian, i.e. there are positive constants K_1, K_2 such that

$$K_1|z_1 - z_2| \leq |F_2(z_1) - F_2(z_2)| \leq K_2|z_1 - z_2|$$

for all z_1, z_2.

Next, consider $F_3(re^{i\vartheta})$. It is trivial to verify that

$$\frac{d}{d\vartheta}(\arg g(\vartheta)) = N - \frac{1}{\frac{1}{\sigma(N+1)}\cos^2(N+1)\vartheta + \frac{\sigma}{N+1}\sin^2(N+1)\vartheta}.$$

But $\sigma = \frac{N\beta_{N+1}}{N+1}$, and $1 < \beta_{N+1} < \frac{(N+1)^2}{N^2}$, according to Lemma 1. Therefore, $\frac{1}{\sigma(N+1)} = \frac{1}{N\beta_{N+1}} < \frac{1}{N}$ and $\frac{\sigma}{N+1} = \frac{N\beta_{N+1}}{(N+1)^2} < \frac{1}{N}$. Thus $A = \max\left(\frac{1}{\sigma(N+1)}, \frac{\sigma}{N+1}\right) < \frac{1}{N}$, and $\frac{d}{d\vartheta}(\arg g(\vartheta)) \leq N - \frac{1}{A} < 0$. Let $\tau(\vartheta)$ denote the continuous branch of $\arg g(\vartheta)$ determined from $\tau(0) = 0$. Clearly, $\tau(\vartheta)$ is an odd function and $\tau'(\vartheta)$ is periodic, with period $\frac{\pi}{N+1}$. Further,

$$\tau(\vartheta) - N\vartheta = -\arg[\cos(N+1)\vartheta + i\sigma\sin(N+1)\vartheta].$$

Let ϑ increase from 0 to $\frac{\pi}{N+1}$ and conclude that $\tau(\frac{\pi}{N+1}) - \frac{N}{N+1}\pi = -\pi$, i.e. $\tau(\frac{\pi}{N+1}) = -\frac{\pi}{N+1}$. Hence, $\tau(\ell \cdot \frac{\pi}{N+1}) = -\ell\frac{\pi}{N+1}$ for any integer ℓ. In particular, $\tau(2\pi) = -2\pi$. The function $\tau(\vartheta)$ is a bi-Lipschitzian mapping of R^1 onto itself, and obviously $e^{i\vartheta} \to e^{i\tau(\vartheta)}$ defines a bi-Lipschitzian mapping of the unit circle onto itself. It follows that $F_3(re^{i\vartheta}) = re^{i\tau(\vartheta)}$ is a bi-Lipschitzian mapping, and consequently $F_3 \circ F_2$ also has this property. (Compare the argument for F_2.)

In studying F_1, we will use the following

Notation: $A(x,y,z,...) \approx B(x,y,z,...)$ means that there are positive constants C_1, C_2 such that

$$C_1B(x,y,z,...) \leq A(...) \leq C_2B(...)$$

for all $x, y, z, ...$ in question.

The elementary inequalities

$$\frac{1}{3}\left(\left|\frac{z}{|z|} - 1\right| + 1 - |z|\right) \leq |z-1| \leq \left|\frac{z}{|z|} - 1\right| + 1 - |z| \tag{27}$$

hold for $0 < |z| \leq 1$. Therefore, $|z-1| \approx |\frac{z}{|z|} - 1| + 1 - |z|$.

Consider $F_1(q'_je^{i\theta'_j}) = (q'_j)^s e^{i\theta'_j}$ for $j = 1,2$ and put $R = \frac{q'_1}{q'_2}$. Then $0 < R \leq 1$ and (27) gives

$$\begin{aligned}|F_1(q'_1e^{i\theta'_1}) - F_1(q'_2e^{i\theta'_2})| &= (q'_2)^s|R^se^{i(\theta'_1-\theta'_2)} - 1| \\ &\approx (q'_2)^s(|e^{i(\theta'_1-\theta'_2)} - 1| + 1 - R^s) \\ &\approx (q'_2)^s(|e^{i(\theta'_1-\theta'_2)} - 1| + 1 - R).\end{aligned}$$

Since $F_3 \circ F_2$ is bi-Lipschitzian, it now follows that

$$|z_{N+1}(q_1', \theta_1') - z_{N+1}(q_2', \theta_2')| \approx (q_2')^s (|e^{i(\theta_1' - \theta_2')} - 1| + 1 - R). \qquad (28)$$

(Recall that $z_{N+1}(\cdot) = F_3 \circ F_2 \circ F_1(\cdot)$.)

Similarly,

$$|(q_1')^N e^{iN\theta_1'} - (q_2')^N e^{iN\theta_2'}| = (q_2')^N |(Re^{i(\theta_1'-\theta_2')})^N - 1|$$
$$\leq N(q_2')^N |Re^{i(\theta_1'-\theta_2')} - 1| \approx (q_2')^N (|e^{i(\theta_1'-\theta_2')} - 1| + 1 - R).$$

Note that the inequality here comes close to an equality, when $R \to 1$ and $(\theta_1' - \theta_2') \to 0$. A crucial question, coming from (26), is now: for which $\alpha > 0$ does there exist a $C > 0$ such that

$$C(q_2')^{s\alpha} (|e^{i(\theta_1'-\theta_2')} - 1| + 1 - R)^\alpha \geq (q_2')^N (|e^{i(\theta_1'-\theta_2')} - 1| + 1 - R)$$

holds for all θ_1', θ_2' and for all sufficiently small q_1', q_2' such that $0 < R = \frac{q_1'}{q_2'} \leq 1$? The obvious answer is that C exists if and only if $s\alpha \leq N$ and $\alpha \leq 1$. Therefore, in the particular case where $A_m = 0$ for $m \geq N+2$, it follows that $\alpha_{\max} = \min\left(\frac{1}{\beta_{N+1} - 1}, 1\right)$. (See the remark of Th. 4.)

It remains to show that the result is the same in the general case. Recall that $z(q', \theta') = \sum_{N+1}^{\infty} A_m Z_m(q', \theta')$, and $A_{N+1} > 0$. Clearly, a relation like (28) also holds for $Z_{N+1}(q', \theta')$. Put $\Delta_m = Z_m(q_1', \theta_1') - Z_m(q_2', \theta_2')$ for any m. Next, $|\Delta_m|$ will be estimated from above, for $m \geq N+2$. Put $s_m = N(\beta_m - 1)$ and $g_m(\theta') = e^{iN\theta'}\left(\cos m(\theta' - \theta_m) - i\frac{N\beta_m}{m}\sin m(\theta' - \theta_m)\right)$, so that $Z_m(q', \theta') = \frac{im}{N(\beta_m - 1)}(q')^{s_m} g_m(\theta')$.

From now on, C denotes a positive constant independent of m, q_1', q_2', θ_1' and θ_2', not necessarily the same every time it occurs. Since $\lim_{m\to\infty} \frac{\beta_m}{m} = \frac{\sqrt{p-1}}{N}$, it follows that

$$\left|\frac{im}{N(\beta_m - 1)}\right| \leq C, \quad \left|\frac{N\beta_m}{m}\right| \leq C \quad \text{and} \quad \max_{\theta'} |g_m(\theta')| \leq C.$$

Further,

$$|(q_1')^{s_m} g_m(\theta_1') - (q_2')^{s_m} g_m(\theta_2')| \leq (q_1')^{s_m} |g_m(\theta_1') - g_m(\theta_2')| +$$
$$+ |g_m(\theta_2')| |(q_1')^{s_m} - (q_2')^{s_m}| \leq (q_1')^{s_m} |g_m(\theta_1') - g_m(\theta_2')| + C(q_2')^{s_m}(1 - R^{s_m}).$$

Elementary estimates give

$$|g_m(\theta_1') - g_m(\theta_2')| \leq C\, m |e^{i(\theta_1'-\theta_2')} - 1|$$

and $0 \leq 1 - R^{s_m} \leq C\, m(1 - R)$. Thus,

$$|(q_1')^{s_m} g_m(\theta_1') - (q_2')^{s_m} g_m(\theta_2')| \leq C\, m (q_2')^{s_m} (|e^{i(\theta_1'-\theta_2')} - 1| + 1 - R)$$

and

$$\sum_{N+2}^{\infty} |A_m \Delta_m| \leq C(|e^{i(\theta_1'-\theta_2')} - 1| + 1 - R) \cdot \sum_{N+2}^{\infty} m |A_m| (q_2')^{s_m}. \qquad (29)$$

It is clear from above (see (28)) that

$$A_{N+1}|\Delta_{N+1}| \geq C_0(q_2')^{s_{N+1}}(|e^{i(\theta_1'-\theta_2')}-1|+1-R) \tag{30}$$

for some $C_0 > 0$. It is also evident that $S_m - S_{N+2} \geq \alpha m$ for $m \geq N+3$ and for some $\alpha > 0$. Since $m|A_m| \leq C$, it is obvious that

$$\sum_{N+2}^{\infty} m|A_m|(q')^{s_m - s_{N+2}}$$

is bounded for (e.g.) $0 \leq q' \leq \frac{1}{2}$, and so

$$\sum_{N+2}^{\infty} m|A_m|(q_2')^{s_m} \leq C(q_2')^{s_{N+2}}.$$

Now (29) gives

$$\sum_{N+2}^{\infty} |A_m\Delta_m| \leq C(q_2')^{S_{N+2}}(|e^{i(\theta_1'-\theta_2')}-1|+1-R),$$

and a comparison with (30) shows that

$$\sum_{N+2}^{\infty} |A_m\Delta_m| \leq \frac{1}{2}A_{N+1}|\Delta_{N+1}|$$

for $0 \leq q_1' \leq q_2' \leq q^*$ and for some $q^* > 0$. It then follows that

$$|\sum_{N+1}^{\infty} A_m\Delta_m| \approx A_{N+1}|\Delta_{N+1}|$$

and therefore the best Hölder exponent for $\nabla\varphi$ is the same as in the previous case. The results will now be summarized in a basic theorem.

Theorem 5. *Let $1 < p < \infty$. For $N = 1, 2, 3, \dots$ define the quantity $\gamma(N,p)$ by*

$$\gamma(N,p) = \begin{cases} \dfrac{2}{\sqrt{p^2+12(p-1)}-p}, & \text{if } N=1 \text{ and } p>2 \\ \dfrac{2}{\sqrt{p^2+5(p-1)}-p}, & \text{if } N=2 \text{ and } p>9 \\ 1, & \text{in all other cases.} \end{cases}$$

Let $\varphi \not\equiv$ const. be p-harmonic in a neighbourhood of z_0 (two dimensions), and let z_0 be a critical point of order N. Then $\nabla\varphi$ satisfies a Hölder condition with the exponent $\gamma(N,p)$ in some neighbourhood of z_0. Moreover, $\nabla\varphi$ can not satisfy a Hölder condition with a greater exponent than $\gamma(N,p)$ in any neighbourhood of z_0.

Remark: $\gamma(N,p) < 1$ if and only if $N = 1$, $p > 2$ or $N = 2$, $p > 9$.

Corollary 1. *The best Hölder exponent for $\nabla\varphi$, valid for any p-harmonic φ is*

$$\alpha(p) = \begin{cases} \dfrac{2}{\sqrt{p^2+12(p-1)}-p}, & \text{for } p>2 \\ 1, & \text{for } 1<p\leq 2. \end{cases}$$

Further, $\alpha(p)$ is strictly decreasing for $p \geq 2$ and $\lim_{p\to\infty} \alpha(p) = \frac{1}{3}$.

5.10 C. A converse of Theorem 4.

Now that the estimates (28) – (30) are available, one can easily obtain a converse of Theorem 4. Given $\rho' > 0$, $N \geq 1$ and real sequences $\{A_m\}_{N+1}^{\infty}$, $\{\theta_m\}_{N+1}^{\infty}$, does there exist a corresponding p-harmonic function φ? Let $\{A_m\}_{N+1}^{\infty}$ be a bounded sequence such that $A_{N+1} > 0$. First, define $z = \sum_{N+1}^{\infty} A_m\, Z_m(q',\theta')$, $\varphi = \sum_{N+1}^{\infty} \cdots$, $\psi = \sum_{N+1}^{\infty} \cdots$, exactly as in Th. 4, but with ρ' replaced by 1. Clearly, all three series converge uniformly for q' small enough. It is crucial to verify that the mapping $q'e^{i\theta'} \rightarrow z$ is one-to-one. The mapping properties of the leading term $A_{N+1}Z_{N+1}(q',\theta')$ are clear from the derivation of Theorem 5, and the estimates (28), (29), (30) are still valid. It is further evident that

$$\sum_{N+2}^{\infty} m|A_m|(q')^{s_m - s_{N+2}}$$

is bounded for $0 \leq q' \leq q_0'$, for some $q_0' > 0$. (Recall that $s_m - s_{N+2} \geq \alpha m$ for $m \geq N+3$ and some $\alpha > 0$.) Thus, one still has

$$|\sum_{N+1}^{\infty} A_m\, \Delta_m| \geq \frac{1}{2} A_{N+1}|\Delta_{N+1}|$$

for $0 \leq q_1' \leq q_2' \leq q^*$, for some $q^* > 0$. (Notation as before.) It follows that $q'e^{i\theta'} \rightarrow z$ is an injective mapping for q' small enough, and the image clearly covers a neighbourhood of $z = 0$. The series for $z(q',\theta')$, $\varphi(q',\theta')$ and $\psi(q',\theta')$ stem from (13) and (16) in Subsection 4C and the systems (6) - (11) are therefore satisfied by construction. It is clear from the series for φ that $(q'\varphi_{q'})^2 + \varphi_{\theta'}^2 > 0$ for q' positive and small enough. (Again, domination by the leading term.) Next, Theorem 3 is applied, locally for small $q' > 0$. (The transition from (q',θ') to (q,θ) is trivial.) It follows that $\varphi = \varphi(z)$, $\psi = \psi(z)$ are p-harmonic and p'-harmonic, respectively, for $z \neq 0$ and small enough. The "singularity" at $z = 0$ is clearly removable. Thus, the desired converse follows if $\rho' = 1$.

Consider then the case $\rho' \neq 1$. The relations

$$\begin{cases} z^* = \displaystyle\sum_{N+1}^{\infty} A_m\, Z_m(q'',\theta') \\ \varphi^* = \displaystyle\sum_{N+1}^{\infty} A_m (q'')^{N\beta_m} \sin m(\theta' - \theta_m) \\ \psi^* = \dfrac{N}{p-1} \displaystyle\sum_{N+1}^{\infty} A_m \frac{\beta_m}{m} (q'')^{N(p+\beta_m-2)} \cos m(\theta' - \theta_m) \end{cases}$$

represent a pair $(\varphi^*(z^*), \psi^*(z^*))$ of the desired type, as shown above, for q'' properly restricted. Now put $q'' = \frac{q'}{\rho'}$ and

$$\begin{cases} z = \dfrac{1}{(\rho')^N} \displaystyle\sum_{N+1}^{\infty} A_m\, Z_m(q'',\theta') = \dfrac{1}{(\rho')^N} z^* \\ \varphi = \varphi^* \\ \psi = (\rho')^{N(p-2)} \psi^*. \end{cases}$$

These formulas for z, φ and ψ agree with the formulas in Th. 4. On the other hand, $\varphi(z)$ is clearly obtained from $\varphi^*(z^*)$ by a homotethy, and $\psi(z)$ is obtained from $\psi^*(z^*)$ by homotethy + multiplication by constant. It follows that $(\varphi(z), \psi(z))$ is a pair as desired. Further details are left to the reader.

One should observe that the sequence $\{A_m\}_{N+1}^{\infty}$ in Th. 4 is $= O(m^{-k})$ for any $k > 0$, whereas in the converse argument $\{A_m\}$ was only assumed to be bounded. The difference is due to the fact that the representation in Th. 4 holds up to $q' = \rho'$, whereas this need not be the case here. The following result has been proved:

Let $\rho' > 0$ and let $N \geq 1$ be an integer. Let $\{A_m\}_{N+1}^{\infty}$ and $\{\theta_m\}_{N+1}^{\infty}$ be bounded real sequences and let $A_{N+1} > 0$. Then the representation formulas in Theorem 4 define a p-harmonic function φ and a p'-harmonic stream function ψ, in some neighbourhood of $z = 0$. Further, $\nabla\varphi = \nabla\psi = 0$ at $z = 0$. Consequently, Theorem 4 contains all information about the local behaviour of φ and ψ near a critical point.

5.11 D. Singular expansions for φ and ψ near a critical point.

Let φ and ψ be as in Theorem 4. Using that theorem and estimates from subsections 5A–B, we shall derive "singular expansions" for φ and ψ at $z = 0$, of the form

$$\begin{cases} \varphi(re^{i\phi}) = r^k f(\phi) + O(r^{k+\delta}) \\ \psi(re^{i\phi}) = r^\ell g(\phi) + O(r^{\ell+\delta}), \end{cases}$$

for some $\delta > 0$. Here, $r^k f(\phi)$ is p-harmonic with $r^\ell g(\phi)$ a corresponding stream function, and thus p'-harmonic.

Choose r_0 such that Th. 4 is applicable for $|z| = r \leq r_0$. Then also (24) holds. It is no essential restriction to assume that $\rho' = 1$.

For any $Q' = q'e^{i\theta'}$, with $q' \leq 1$, put $F(Q') = \sum_{N+1}^{\infty} A_m Z_m(q', \theta')$ and $G(Q') = A_{N+1}Z_{N+1}(q', \theta')$. Let z_1 be a variable point such that $0 < |z_1| \leq r_0$. By Th. 4 there is a Q'_1 with $q'_1 \leq 1$ such that $F(Q'_1) = z_1$. It is also clear from subsection 5B (the analysis of the leading term) that there is a $Q'_2 = q'_2 e^{i\theta'_2}$ such that $G(Q'_2) = z_1$. It is further true (see (24)) that

$$C|z_1|^{1/s} \leq q'_i \leq C'|z_1|^{1/s}$$

for $i = 1, 2$ and for some positive constants C, C'. Here, as before, $s = N(\beta_{N+1} - 1) = s_{N+1}$. Trivial estimates give

$$|F(Q'_1) - G(Q'_1)| \leq \sum_{N+2}^{\infty} |A_m Z_m(Q'_1)| \leq C(q'_1)^{S_{N+2}}$$

and therefore

$$|G(Q'_2) - G(Q'_1)| = |F(Q'_1) - G(Q'_1)| \leq C(q'_1)^{S_{N+2}}.$$

But (28) implies that

$$|G(Q'_2) - G(Q'_1)| \geq C(\max_{i=1,2} q'_i)^{S_{N+1}}(|e^{i(\theta'_1 - \theta'_2)} - 1| + 1 - R),$$

where $R = \frac{\min q_i'}{\max q_i'}$.

Consequently,

$$|e^{i(\theta_1'-\theta_2')} - 1| + 1 - R \le C(q_1')^{S_{N+2}-S_{N+1}}. \tag{31}$$

For any $Q' = q'e^{i\theta'}$, put

$$H(Q') = A_{N+1}(q')^{N\beta_{N+1}} \sin(N+1)(\theta' - \theta_{N+1}),$$

and

$$I(Q') = \frac{N}{p-1} A_{N+1} \frac{\beta_{N+1}}{N+1} (q')^{N(\beta_{N+1}+p-2)} \cos(N+1)(\theta' - \theta_{N+1}).$$

These are the leading terms in the expansions of φ and ψ, and Th. 4 gives

$$|\varphi(z_1) - H(Q_1')| \le C(q_1')^{N\beta_{N+2}} \tag{32}$$

$$|\psi(z_1) - I(Q_1')| \le C(q_1')^{N(\beta_{N+2}+p-2)} \tag{33}$$

Further (compare subsection 5B),

$$|H(Q_1') - H(Q_2')| \le C(\max q_i')^{N\beta_{N+1}}(|e^{i(\theta_1'-\theta_2')} - 1| + 1 - R)$$

and

$$|I(Q_1') - I(Q_2')| \le C(\max q_i')^{N(\beta_{N+1}+p-2)}(|e^{i(\theta_1'-\theta_2')} - 1| + 1 - R).$$

Combination with (31) gives

$$|H(Q_1') - H(Q_2')| \le C(\max q_i')^{N\beta_{N+2}}$$

and

$$|I(Q_1') - I(Q_2')| \le C(\max q_i')^{N(\beta_{N+2}+p-2)}.$$

Finally, the last two estimates will be combined with (32) and (33). Write $Q_2' = G^{-1}(z_1)$ and recall that $q_1' \approx q_2'$. We thus find

$$\begin{cases} |\varphi(z_1) - (H \circ G^{-1})(z_1)| \le C(q_1')^{N\beta_{N+2}} & (34) \\ |\psi(z_1) - (I \circ G^{-1})(z_1)| \le C(q_1')^{N(\beta_{N+2}+p-2)} & (35) \end{cases}$$

or, equivalently,

$$\begin{cases} |\varphi(z_1) - (H \circ G^{-1})(z_1)| \le C|z_1|^{\frac{\beta_{N+2}}{\beta_{N+1}-1}} \\ |\psi(z_1) - (I \circ G^{-1})(z_1)| \le C|z_1|^{\frac{\beta_{N+2}+p-2}{\beta_{N+1}-1}}. \end{cases}$$

It only remains to interpret the results obtained.

The following lemma reviews the basic facts concerning $G(Q')$, $H(Q')$ and $I(Q')$.

Lemma 2. *Consider the mapping $G : C \to C$, given by*

$$G(q'e^{i\theta'}) = A_{N+1}Z_{N+1}(q',\theta'),$$

with $Z_{N+1}(\ldots)$ as in Theorem 4, and functions ($C \to R^1$) given by

$$H(q'e^{i\theta'}) = A_{N+1}(q')^{N\beta_{N+1}} \sin(N+1)(\theta' - \theta_{N+1}),$$

and

$$I(q'e^{i\theta'}) = A_{N+1}\frac{N\beta_{N+1}}{(p-1)(N+1)}(q')^{N(\beta_{N+1}+p-2)} \cos(N+1)(\theta' - \theta_{N+1}).$$

Then G is a homeomorphism $C \leftrightarrow C$ such that the function $\varphi_0(z) = (H \circ G^{-1})(z)$ is p-harmonic on C, the function $\psi_0(z) = (I \circ G^{-1})(z)$ is p'-harmonic on C, and ψ_0 is a stream function of φ_0. Further, $\varphi_0(z)$ and $\psi_0(z)$ are of the form

$$\begin{cases} \varphi_0(re^{i\phi}) = r^k f(\phi) \\ \psi_0(re^{i\phi}) = r^\ell g(\phi), \end{cases}$$

where $k > 1$, $\ell > 1$. Moreover, the functions $f(\phi)$ and $g(\phi)$ are real analytic and have parametric integral representations. Finally, $(\varphi_x + i\varphi_y)(z) = (G^{-1}(z))^N$ for all $z \in C$ and $z = 0$ is a critical point of order N for φ and ψ.

Proof: It is clear from section 5B (in particular the analysis of the leading term) that G is a homeomorphism $C \leftrightarrow C$. Thus, φ_0 and ψ_0 are defined and continuous on the whole z-plane. For $q' \neq 0$, i.e. for $z \neq 0$, H and I are locally well-defined functions of $qe^{i\theta} = (q'e^{i\theta'})^N$. The expressions for G, H, I are in complete agreement with [4], Th. 4, or [5], Th. 8, provided one chooses $\lambda = \frac{(N+1)^2}{N^2}$ and $\beta = \beta_{N+1}$ there. We are thus allowed to conclude that $\varphi_0(z)$ is p-harmonic, that $\psi_0(z)$ is p'-harmonic, and that ψ_0 is a stream function for φ_0, provided $z \neq 0$. The relation $(\varphi_x + i\varphi_y)(z) = qe^{i\theta}$ also follows from [4] or [5], and so

$$(\varphi_x + i\varphi_y)(z) = (q'e^{i\theta'})^N = (G^{-1}(z))^N.$$

Clearly, the "singularity" at $z = 0$ is removable, so $\Delta_p(\varphi_0) = \Delta_{p'}(\psi_0) = 0$ on C. It follows from $\varphi_x + i\varphi_y = (G^{-1}(z))^N$ that $z = 0$ is a critical point of the order N. Finally, it follows from [4], Lemma 5, that φ_0 and ψ_0 have the form stated above. Concerning the parametric representations of $f(\phi)$ and $g(\phi)$ we refer to [4], Th. 2. (The assumption $p > 2$ made there is no essential restriction at this moment). See also [2], Lemma 2. This completes the proof.

The next theorem is the final result of this paper.

Theorem 6. *Let $\varphi \not\equiv const.$ be p-harmonic ($1 < p < \infty$) in a neighbourhood of $z = 0$. Let ψ be a stream function, and suppose that $\varphi(0) = \psi(0) = 0$. Suppose further that $z = 0$ is a critical point of order $N \geq 1$. Then there are "singular expansions", valid near $z = 0$:*

$$\begin{cases} \varphi(re^{i\phi}) = r^k f(\phi) + O(r^{k+\delta}) \\ \psi(re^{i\phi}) = r^\ell g(\phi) + O(r^{\ell+\delta}) \end{cases}$$

where $k > 1$, $\ell > 1$ and $\delta > 0$. Further, $\varphi_0 = r^k f(\phi)$ is p-harmonic and $\psi_0 = r^\ell g(\phi)$ is p'-harmonic on C. Finally, ψ_0 is a stream function of φ_0.

Proof: Combine Lemma 2 with the estimates (34) and (35). The theorem follows.

Remark 1: It is evident that $k = \frac{\beta_{N+1}}{\beta_{N+1}-1}$ and $\ell = \frac{\beta_{N+1}+p-2}{\beta_{N+1}-1}$. Moreover, one can choose $\delta = \frac{\beta_{N+2}-\beta_{N+1}}{\beta_{N+1}-1}$. Also, using (31) one easily shows that $\nabla\varphi(re^{i\phi}) = \nabla\varphi_0(re^{i\phi}) + O(r^{k-1+\delta})$, with the same δ. Observe also that $k-1 = \frac{1}{\beta_{N+1}-1}$, i.e. (see subsection 5B) $\alpha_{\max} = \min(k-1, 1)$, as expected.

Remark 2: Singular expansions of a p-harmonic function φ near a corner of its domain of definition were obtained by P. Tolksdorf [20], [21] and M. Dobrowolski [11]. These results are valid in R^n. On the other hand, conditions on the sign of φ near the corner are needed. See also [4], Section 4 and Theorem 5, in particular Remark 1. Clearly, that theorem gives a simultaneous singular expansion of φ and ψ, as in the above theorem.

During the preparation of this report the author was informed about recent work by T. Iwaniec and J. Manfredi: "Best exponents for p-harmonic functions on the plane". The work by Iwaniec and Manfredi contains a corollary, similar to but not identical with Corollary 1 of Theorem 5. (It does not follow, if $p > 2$, from the work of I. and M. that $\alpha(p)$ itself is a valid Hölder exponent for $\nabla\varphi$.) I. and M. obtain very interesting results on the integrability of second-order derivatives of φ, by using the hodograph method in a form quite different from ours. They do not derive a representation for φ, and the corollary is the only overlap with the present work.

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